HONG KONG ADVANCED LEVEL EXAMINATION 2012

PURE MATHEMATICS PAPER 1

SAMPLE SOLUTIONS

1. (a) (i) $\sum_{k=0}^{n} a_k x^k = (2-x)^n$ (1)

Substituting $x = 1$,

$$\sum_{k=0}^{n} a_k = 1$$

(ii) Differentiating (1) w.r.t. x,

$$\sum_{k=1}^{n} k a_k x^{k-1} = -n(2-x)^{n-1} \quad (2)$$

Substituting $x = 1$,

$$\sum_{k=1}^{n} k a_k = -n$$

(iii) Differentiating (2) w.r.t. x,

$$\sum_{k=2}^{n} k(k-1) a_k x^{k-2} = n(n-1)(2-x)^{n-2}$$

Substituting $x = 1$,

$$\begin{aligned} n^2 - n &= \sum_{k=2}^{n} (k^2 - k) a_k \\ &= \sum_{k=2}^{n} k^2 a_k - \sum_{k=1}^{n} k a_k + a_1 \\ &= \sum_{k=1}^{n} k^2 a_k + n \qquad \text{(By(a)(ii))} \\ n^2 - 2n &= \sum_{k=1}^{n} k^2 a_k \end{aligned}$$

(b)
$$\begin{aligned} \sum_{k=1}^{n} (n+k)^2 a_k &= n^2 a_0 + \sum_{k=1}^{n} (n^2 + 2nk + k^2) a_k \\ &= n^2 \sum_{k=0}^{n} a_k + 2n \sum_{k=1}^{n} k a_k + \sum_{k=1}^{n} k^2 a_k \qquad \text{(By (a))} \\ &= n^2 + 2n(-n) + (n^2 - 2n) \\ &= -2n \end{aligned}$$

2. (a) Let $\dfrac{2x-23}{(2x+1)(2x+5)(2x+7)} = \dfrac{A}{2x+1} + \dfrac{B}{2x+5} + \dfrac{C}{2x+7}$.

Comparing the numerator on both sides,

$2x - 23 = A(2x+5)(2x+7) + B(2x+1)(2x+7) + C(2x+1)(2x+5)$

Substituting $x = -\dfrac{1}{2}$, we have

$$2\left(-\frac{1}{2}\right) - 23 = A\left[2\left(-\frac{1}{2}\right)+5\right]\left[2\left(-\frac{1}{2}\right)+7\right]$$
$$A = -1$$

Substituting $x = -\dfrac{5}{2}$, we have

$$2\left(-\frac{5}{2}\right) - 23 = B\left[2\left(-\frac{5}{2}\right)+1\right]\left[2\left(-\frac{5}{2}\right)+7\right]$$
$$B = \frac{7}{2}$$

Substituting $x = -\dfrac{7}{2}$, we have

$$2\left(-\frac{7}{2}\right) - 23 = C\left[2\left(-\frac{7}{2}\right)+1\right]\left[2\left(-\frac{7}{2}\right)+5\right]$$
$$C = -\frac{5}{2}$$

$$\therefore \frac{2x-23}{(2x+1)(2x+5)(2x+7)} = -\frac{1}{2x+1} + \frac{7}{2}\cdot\frac{B}{2x+5} - \frac{5}{2}\cdot\frac{C}{2x+7}$$

(b)
$$\begin{aligned}
\sum_{k=1}^{n}\frac{2k-3}{(2k+1)(2k+5)(2k+7)} &= -\sum_{k=1}^{n}\frac{1}{2k+1} + \frac{7}{2}\sum_{k=1}^{n}\frac{1}{2k+5} - \frac{5}{2}\sum_{k=1}^{n}\frac{1}{2k+7} \qquad \text{(By (a))}\\
&= -\sum_{k=1}^{n}\frac{1}{2k+1} + \frac{7}{2}\sum_{k=3}^{n+2}\frac{1}{2k+1} - \frac{5}{2}\sum_{k=4}^{n+3}\frac{1}{2k+1}\\
&= -\frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{7}{2}\cdot\frac{1}{7} + \left(\frac{7}{2}-\frac{5}{2}\right)\left(\frac{1}{2n+3}\right)\\
&\quad + \left(\frac{7}{2}-\frac{5}{2}\right)\left(\frac{1}{2n+5}\right) - \frac{5}{2}\left(\frac{1}{2n+7}\right)\\
&= -\frac{37}{210} + \frac{1}{2n+3} + \frac{1}{2n+5} - \frac{5}{2}\left(\frac{1}{2n+7}\right)
\end{aligned}$$

(c)
$$\begin{aligned}
\sum_{k=4}^{\infty}\frac{2k-23}{(2k+1)(2k+5)(2k+7)} &= \lim_{n\to\infty}\sum_{k=4}^{n}\frac{2k-23}{(2k+1)(2k+5)(2k+7)}\\
&= \frac{21}{3\cdot7\cdot9} + \frac{19}{5\cdot9\cdot11} + \frac{17}{7\cdot11\cdot13} + \lim_{n\to\infty}\sum_{k=4}^{n}\frac{2k-23}{(2k+1)(2k+5)(2k+7)}\\
&= \frac{21}{3\cdot7\cdot9} + \frac{19}{5\cdot9\cdot11} + \frac{17}{7\cdot11\cdot13}\\
&\quad + \lim_{n\to\infty}\left[-\frac{37}{210} + \frac{1}{2n+3} + \frac{1}{2n+5} - \frac{5}{2}\left(\frac{1}{2n+7}\right)\right] \qquad \text{(By (b))}\\
&= -\frac{25}{2574}
\end{aligned}$$

3. (a) $\sin \pi x = 1$

$$\pi x = \frac{\pi}{2} + 2k\pi \quad \forall k \in \mathbf{Z}$$

$$x = \frac{1}{2} + 2k$$

(b)
$$\begin{aligned} & f(x)\sin \pi x = 0 \quad \forall x \in \mathbf{R} \qquad \text{and} \quad \sin \pi x = 0 \quad \forall x \in \mathbf{Z} \\ \Rightarrow\ & f(x) = 0 \qquad\qquad \forall x \in \mathbf{R}\backslash\mathbf{Z} \\ \Rightarrow\ & f(x) \text{ has infinitely many real roots} \\ \because\ & f(x) \text{ is a polynomial with real coefficients} \\ \therefore\ & f(x) = 0 \qquad\qquad \forall x \in \mathbf{R} \end{aligned}$$

(c) Let $g(x) = \begin{cases} 0 & \forall x \in \mathbf{R}\backslash\mathbf{Z} \\ 1 & \forall x \in \mathbf{Z} \end{cases}$.

$g(x)\sin \pi x = 0 \quad \forall x \in \mathbf{R}$

But $g(x) \neq 0 \quad \forall x \in \mathbf{Z}$

Hence, the claim is not true.

4. (a) $P = \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

(b)
$$\begin{aligned} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix} &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} k\cos\theta & k\sin\theta \\ k\sin\theta & -k\cos\theta \end{pmatrix} \end{aligned}$$

$$\Rightarrow \begin{cases} k\cos\theta = -\sqrt{3} \\ k\sin\theta = 1 \end{cases}$$

$$\Rightarrow \tan\theta = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = -\frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\Rightarrow \begin{cases} \text{when } \theta = -\frac{\pi}{6}, & k = -2 \quad (\text{rejected}, \because k > 0) \\ \text{when } \theta = \frac{5\pi}{6}, & k = 2 \end{cases}$$

T is a transformation that consists of a reflection about the x-axis, followed by a counter-clockwise rotation of $\frac{5\pi}{6}$, followed by a scale of 2.

5. (a) When $n = 1$, $a_1 = \frac{3}{2} = \frac{1}{2^1} + \frac{1}{3^0}$.

When $n = 2$, $a_2 = \frac{7}{12} = \frac{1}{2^2} + \frac{1}{3^1}$.

Assume, as the Inductive Hypothesis, that $a_m = \frac{1}{2^m} + \frac{1}{3^{m-1}}$ and $a_{m+1} = \frac{1}{2^{m+1}} + \frac{1}{3^m}$ for some positive integer m. Then, we have

$$\begin{aligned}
a_{m+2} &= \frac{1}{6}(5a_{m+1} - a_m) \\
&= \frac{1}{6}\left[5\left(\frac{1}{2^{m+1}} + \frac{1}{3^m}\right) - \left(\frac{1}{2^m} + \frac{1}{3^{m-1}}\right)\right] && \text{(By Inductive Hypothesis)} \\
&= \frac{1}{6}\left(\frac{5-2}{2^{m+1}} + \frac{5-3}{3^m}\right) \\
&= \frac{1}{2} \cdot \frac{1}{2^{m+1}} + \frac{1}{3} \cdot \frac{1}{3^m} \\
&= \frac{1}{2^{m+2}} + \frac{1}{3^{m+1}}
\end{aligned}$$

Thus, by the principle of mathematical induction, we have $a_n = \frac{1}{2^n} + \frac{1}{3^{n-1}}$ for all positive integer n.

(b)
$$\begin{aligned}
\sum_{k=1}^{m} a_k &= \sum_{k=1}^{m} \frac{1}{2^k} + \sum_{k=1}^{m} \frac{1}{3^{k-1}} && \text{(By (a))} \\
&= \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{m+1}}}{1 - \frac{1}{2}} + \frac{1 - \frac{1}{3^m}}{1 - \frac{1}{3}} \\
&= 1 + \frac{2}{3} - \frac{1}{2^{m+2}} - \frac{2}{3^{m+1}} \\
&< \frac{5}{3} \\
&< 3
\end{aligned}$$

Hence, there does not exist a positive integer m such that $\sum_{k=1}^{m} a_k > 3$.

6. (a) $2n - S = 2n - \sum_{k=1}^{n}\left(1+\frac{1}{k}\right)$

$$= \sum_{k=1}^{n}\left[2-\left(1+\frac{1}{k}\right)\right]$$

$$= \sum_{k=2}^{n}\left(1-\frac{1}{k}\right)$$

By A.M. $\geq$ G.M., we have

$$\frac{1}{n-1}\sum_{k=2}^{n}\left(1-\frac{1}{k}\right) \geq \left[\prod_{k=2}^{n}\left(1-\frac{1}{k}\right)\right]^{\frac{1}{n-1}}$$

$$= \left(\prod_{k=2}^{n}\frac{k-1}{k}\right)^{\frac{1}{n-1}}$$

$$= \left(1\cdot\frac{1}{n}\right)^{\frac{1}{n-1}}$$

$$\therefore \frac{2n-S}{n-1} \geq \left(\frac{1}{n}\right)^{\frac{1}{n-1}}$$

(b) $$\frac{2n-S}{n-1} \geq \left(\frac{1}{n}\right)^{\frac{1}{n-1}} \qquad \text{(By (a))}$$

$$2n - S \geq (n-1)\,n^{\frac{1}{1-n}}$$

$$S \leq 2n - (n-1)\,n^{\frac{1}{1-n}} \qquad (3)$$

$$S = \sum_{k=1}^{n}\left(1+\frac{1}{k}\right)$$

$$\geq n\left[\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\right]^{\frac{1}{n}} \qquad \text{(By A.M. } \geq \text{ G.M.)}$$

$$= n\left(\prod_{k=1}^{n}\frac{k+1}{k}\right)^{\frac{1}{n}}$$

$$= n(n+1)^{\frac{1}{n}} \qquad (4)$$

Combining (3) and (4), we have

$$2n - (n-1)n^{\frac{1}{1-n}} \geq S \geq n(n+1)^{\frac{1}{n}}$$

7. (a) (i) $\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & a-1 \\ a+2 & 1 & 2a+1 \end{vmatrix}$

$= 5(2a+1) + 2(a-1)(a+2) - 2 + 5(a+2) - 4(2a+1) - (a-1)$

$= 2a^2 + 8a + 6$

$= 2(a+1)(a+3)$

$\therefore (E)$ has a unique solution if and only if $a \neq -1$ and $a \neq -3$.

$\Delta_x = \begin{vmatrix} 3 & 2 & -1 \\ 4 & 5 & a-1 \\ b & 1 & 2a+1 \end{vmatrix}$

$= 15(2a+1) + 2b(a-1) - 4 + 5b - 8(2a+1) - 3(a-1)$

$= (2b+11)a + (3b+6)$

$\Delta_y = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 4 & a-1 \\ a+2 & b & 2a+1 \end{vmatrix}$

$= 4(2a+1) + 3(a-1)(a+2) - 2b + 4(a+2) - 6(2a+1) - b(a-1)$

$= 3a^2 + 3a - ba - b$

$= (a+1)(3a-b)$

$\Delta_z = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ a+2 & 1 & b \end{vmatrix}$

$= 5b + 8(a+2) + 6 - 15(a+2) - 4 - 4b$

$= -7a + b - 12$

Hence, the solution to (E) when (E) has a unique solution is

$$\begin{cases} x = \dfrac{(2b+11)a + (3b+6)}{2(a+1)(a+3)} \\ y = \dfrac{3a-b}{2(a+3)} \\ z = \dfrac{-7a+b-12}{2(a+1)(a+3)} \end{cases}$$

(ii) When $a = -4$, (E) becomes

$$\begin{cases} x + 2y - z = 3 & \dots (1) \\ 2x + 5y - 4z = 4 & \dots (2) \\ -x + y - 5z = b & \dots (3) \end{cases}$$

$4 \times (1) - (2) : 2x + 3y = 8$

$5 \times (1) - (3) : 6x + 9y = 15 - b$

In order to have (E) to be consistent,

$15 - b = 3 \cdot 8$

$b = -9$

When $b = -9$, let $x = t$, we have

$y = \dfrac{8-2t}{3}$

$z = t + 2 \cdot \dfrac{8-2t}{3} - 3 = \dfrac{7-t}{3}$

Hence, the solution to (E) when $a = -3$ and $b = -9$ is

$\left(t, \dfrac{8-2t}{3}, \dfrac{7-t}{3}\right) : t \in \mathbf{R}$

(b) When $a = -\frac{4}{3}$ and $b = -4$, (E) becomes the first 3 equations.
By (a), the solution to the first 3 equations is

$$\begin{cases} x = \dfrac{[2(-4)+11]\left(-\frac{4}{3}\right) + [3(-4)+6]}{2\left(-\frac{4}{3}+1\right)\left(-\frac{4}{3}+3\right)} = 9 \\ y = \dfrac{3\left(-\frac{4}{3}\right) - (-4)}{2\left(-\frac{4}{3}+3\right)} = 0 \\ z = \dfrac{-7\left(-\frac{4}{3}\right) + (-4) - 12}{2\left(-\frac{4}{3}+1\right)\left(-\frac{4}{3}+3\right)} = 6 \end{cases}$$

Substituting the solution into the forth equation, we have

$$L.S. = 4(9) + 5(0) - 6(6) = 0 \neq 1 = R.S.$$

Hence, the system of equations is inconsistent.

(c) When $a = -3$ and $b = -9$, (E) becomes the system of constraints.
By (b), the solution to the system of constraints is

$$\left(t, \frac{8-2t}{3}, \frac{7-t}{3}\right) : t \in \mathbf{R}$$

Substituting the solution into $3x^2 - 7y^2 + 8z^2$, we have

$$\begin{aligned} 3t^2 - 7\left(\frac{8-2t}{3}\right)^2 + 8\left(\frac{7-t}{3}\right)^2 &= \frac{27t^2 - 448 + 224t - 28t^2 + 392 - 112t + 8t^2}{9} \\ &= \frac{7t^2 + 112t - 56}{9} \\ &= \frac{7(t+8)^2}{9} - 56 \end{aligned}$$

Hence, the least value of $3x^2 - 7y^2 + 8z^2$ is -56.

8. (a) (i) $$\sin\frac{\pi}{2n}\sum_{k=1}^{n-1}\sin\frac{k\pi}{n} = \sum_{k=1}^{n-1}\sin\frac{\pi}{2n}\sin\frac{k\pi}{n}$$
$$= \frac{1}{2}\sum_{k=1}^{n-1}\left[\cos\left(k-\frac{1}{2}\right)\frac{\pi}{n} - \cos\left(k+\frac{1}{2}\right)\frac{\pi}{n}\right]$$
$$= \frac{1}{2}\left[\cos\frac{\pi}{2n} - \cos\left(n-\frac{1}{2}\right)\frac{\pi}{n}\right]$$
$$= \sin\frac{\pi}{2}\sin\frac{(n-1)\pi}{2n}$$
$$= \sin\frac{(n-1)\pi}{2n}$$

(ii) $$\sum_{k=1}^{n-1}\left|\alpha^k - 1\right| = \sum_{k=1}^{n-1}\left|\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} - 1\right| \quad \text{(By de Moivre's formula)}$$
$$= \sum_{k=1}^{n-1}\sqrt{\left(\cos\frac{2k\pi}{n} - 1\right)^2 + \sin^2\frac{2k\pi}{n}}$$
$$= \sum_{k=1}^{n-1}\sqrt{2\left(1-\cos\frac{2k\pi}{n}\right)}$$
$$= 2\sum_{k=1}^{n-1}\sin\frac{k\pi}{n}$$
$$= 2\left[\sin\frac{(n-1)\pi}{2n}\right]\left(\sin\frac{\pi}{2n}\right)^{-1} \quad \text{(By (a))}$$
$$= 2\left[\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\right]\left(\sin\frac{\pi}{2n}\right)^{-1}$$
$$= 2\cot\frac{\pi}{2n}$$

(b) (i) Suppose $1, \beta, \beta^2, \ldots, \beta^{n-1}$ are not all distinct.
There exists $0 \le i < j \le n-1$ such that $\beta^i = \beta^j$.
We have $\beta^{j-i} = 1$ and $0 < j - i < n - 1$.
But $\beta^k \neq 1$ for all $k = 1, 2, \ldots, n-1$.
Contradiction arises.
Hence, $1, \beta, \beta^2, \ldots, \beta^{n-1}$ are all distinct.

(ii) Since $\beta^n = 1$, β is the n-th unity root of 1.
$\beta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$$\sum_{k=1}^{2n-1}\left|\beta^k - 1\right| = \sum_{k=1}^{n-1}\left|\beta^k - 1\right| + \left|\beta^n - 1\right| + \sum_{k=n+1}^{2n-1}\left|\beta^k - 1\right|$$
$$= \sum_{k=1}^{n-1}\left|\beta^k - 1\right| + |1-1| + \sum_{k=1}^{n-1}\left|\beta^{n+k} - 1\right|$$
$$= \sum_{k=1}^{n-1}\left|\beta^k - 1\right| + 0 + \sum_{k=1}^{n-1}\left|1\cdot\beta^k - 1\right|$$
$$= 2\sum_{k=1}^{n-1}\left|\beta^k - 1\right|$$
$$= 4\cot\frac{\pi}{2n} \quad \text{(By (a)(ii))}$$

9. (a) Let $f(x) = (1+x)^\lambda - (1+\lambda x)$.

$$f'(x) = \lambda(1+x)^{\lambda-1} - \lambda$$

$$f''(x) = \lambda(\lambda-1)(1+x)^{\lambda-2}$$

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow \lambda(1+x)^{\lambda-1} - \lambda = 0 \\ &\Leftrightarrow (1+x)^{\lambda-1} = 1 \\ &\Leftrightarrow x = 0 \end{aligned}$$

$$\begin{aligned} f''(0) &= \lambda(\lambda-1) \\ &> 0 \qquad \text{for } \lambda > 1 \end{aligned}$$

Hence, $f(x)$ is minimum at $x = 0$, i.e. for any $x > 0$

$$f(x) > f(0)$$

$$(1+x)^\lambda - (1+\lambda x) > (1+0)^\lambda - (1+\lambda 0)$$

$$(1+x)^\lambda > 1 + \lambda x$$

(b) (i) $\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{n}} > 1 + \frac{n+1}{n}\cdot\frac{1}{n+1}$ (By (a))

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}$$

$$a_{n+1} > a_n$$

(ii) (1)

$$\begin{aligned} \frac{b_n}{b_{n+1}} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} \\ &= \left(\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \\ &= \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \\ &= \left(\frac{n(n+2)+1}{n(n+2)}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \\ &= \left(1+\frac{1}{n(n+2)}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \end{aligned}$$

(2) (By (b)(ii)(1))

$$\begin{aligned} \frac{b_n}{b_{n+1}} &= \left(1+\frac{1}{n(n+2)}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \\ &> \left(1+\frac{1}{n(n+2)+1}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \\ &= \left(1+\frac{1}{(n+2)^2}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \\ &= \left[C_0^{n+1} + \frac{C_1^{n+1}}{(n+1)^2} + \cdots\right]\left(\frac{n+1}{n+2}\right) \\ &> \left[1+\frac{n+1}{(n+1)^2}\right]\left(\frac{n+1}{n+2}\right) \\ &= \left(\frac{n+2}{n+1}\right)\left(\frac{n+1}{n+2}\right) \\ &= 1 \end{aligned}$$

(iii) $b_n = \left(1+\frac{1}{n}\right)^{n+1}$

$> \left(1+\frac{1}{n}\right)^n \qquad (\because 1+\frac{1}{n} > 1)$

$= a_n$

By (b)(i) and (b)(ii)(2), we have $a_n > a_{n-1} > \cdots > a_1$ and $b_n < b_{n-1} < \cdots < b_1$.
Hence, $a_1 < a_2 < \cdots < a_n < b_n < b_{n-1} < \cdots < b_1$.
Since a_n is strictly increasing and bounded above by b_1, $\lim_{n\to\infty} a_n$ exists.
Similarly, since b_n is strictly decreasing and bounded below by a_1, $\lim_{n\to\infty} b_n$ exists.

$$\begin{aligned}
b_n &= a_n\left(1+\frac{1}{n}\right) \\
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} a_n\left(1+\frac{1}{n}\right) \\
&= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} 1+\frac{1}{n}\right) \\
&= \left(\lim_{n\to\infty} a_n\right)(1+0) \\
&= \lim_{n\to\infty} a_n
\end{aligned}$$

(iv) $\prod_{k=1}^{n} a_k = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2 \cdots \left(\frac{n+1}{n}\right)^n$

$$\begin{aligned}
&= \frac{(n+1)^n}{n!} \\
\prod_{k=1}^{n} b_k &= \left(\frac{2}{1}\right)^2\left(\frac{3}{2}\right)^3 \cdots \left(\frac{n+1}{n}\right)^{n+1} \\
&= \frac{(n+1)^{n+1}}{n!}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} a_n &> a_k \\
\prod_{k=1}^{n} \lim_{n\to\infty} a_n &> \prod_{k=1}^{n} a_k \\
e^n &> \frac{(n+1)^n}{n!} \\
n!\, e^n &> (n+1)^n \\
\lim_{n\to\infty} b_n &< b_k \\
\prod_{k=1}^{n} \lim_{n\to\infty} b_n &< \prod_{k=1}^{n} b_k \\
e^n &< \frac{(n+1)^{n+1}}{n!} \\
n!\, e^n &< (n+1)^{n+1}
\end{aligned}$$

Hence, $(n+1)^{n+1} > n!\, e^n > (n+1)^n$.

10. (a) Let the degree of $g(x)$ be m.

From the equation $[f(x)]^2 - 1 = (x^2 - 1)[g(x)]^2$, we have

$2n = 2 + 2m$ Hence, the degree of $g(x)$ is $n - 1$.

$m = n - 1$

(b) Suppose $f(x)$ and $g(x)$ have non-constant common factor $x - \alpha$, then $f(\alpha) = 0$ and $g(\alpha) = 0$.

But we have

$$[f(\alpha)]^2 - 1 = (\alpha^2 - 1)[g(\alpha)]^2$$
$$0^2 - 1 = (\alpha^2 - 1)(0^2)$$
$$-1 = 0$$

which is impossible. Hence, $f(x)$ and $g(x)$ have no constant common factors.

(c) $[f(x)]^2 - 1 = (x^2 - 1)[g(x)]^2$

Differentiating both sides w.r.t. x, we have

$$2f(x)f'(x) = 2x[g(x)]^2 + 2(x^2 - 1)g(x)g'(x)$$
$$f(x)f'(x) = g(x)\left[x\, g(x) + (x^2 - 1)g'(x)\right]$$

which shows that $g(x)$ is a factor of $f(x)f'(x)$. But from (b), $f(x)$ and $g(x)$ have no non-constant common factors.

Therefore, $g(x)$ is a factor of $f'(x)$, i.e. $f'(x)$ is divisible by $g(x)$.

(d) From (c), $f'(x)$ is divisible by $g(x)$.

Let $g(x) = kf'(x)$ and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

$$g(x) = k\left[a_n n x^{n-1} + a_{n-1}(n-1)x^{n-1} + \cdots + a_1\right]$$
$$\left[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right]^2 - 1 = (x^2 - 1)k^2\left[a_n n x^{n-1} + a_{n-1}(n-1)x^{n-1} + \cdots + a_1\right]^2$$

Comparing the coefficient of the highest degree term on both sides, we have

$$a_n^2 = k^2 n^2 a_n^2$$
$$k^2 = \frac{1}{n^2}$$

Hence, $[f(x)]^2 - 1 = \frac{1}{n^2}(x^2 - 1)[f'(x)]^2$, i.e. $n^2\left\{[f(x)]^2 - 1\right\} = (x^2 - 1)[f'(x)]^2$.

(e)
$$n^2\left\{[f(x)]^2 - 1\right\} = (x^2 - 1)[f'(x)]^2 \quad \text{(By (d))}$$
$$n^2\left\{\left[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right]^2 - 1\right\} = (x^2 - 1)\left[a_n n x^{n-1} + a_{n-1}(n-1)x^{n-1} + \cdots + a_1\right]^2$$

Comparing the coefficient of the second highest degree term on both sides, we have

$$n^2(2a_n a_{n-1}) = 2(a_n n)[a_{n-1}(n-1)]$$
$$2n^2 a_n a_{n-1} = 2n^2 a_n a_{n-1} - 2n a_n a_{n-1}$$
$$a_n a_{n-1} = 0$$

Since the degree of $f(x)$ is n, $a_n \neq 0$. Then we have $a_{n-1} = 0$.

$$\sum_{k=1}^{n} a_k = -\frac{a_{n-1}}{a_n}$$
$$= -\frac{0}{a_n}$$
$$= 0$$

11. (a) (i) (1) Differentitating $f(x) = \mu_1 x + \mu_2(1-x^s)^{\frac{1}{s}}$, we have

$$\begin{aligned} f'(x) &= \mu_1 + \mu_2\left(\frac{1}{s}\right)(1-x^s)^{\frac{1}{s}-1}\left(-sx^{s-1}\right) \\ &= \mu_1 - \mu_2\left(x^{-s}-1\right)^{\frac{1}{s}-1} \\ f''(x) &= -\mu_2\left(\frac{1}{s}-1\right)\left(x^{-s}-1\right)^{\frac{1}{s}-2}\left(-sx^{-s-1}\right) \\ &= -\mu_2(s-1)\left(x^{-s}-1\right)^{\frac{1}{s}-2}\left[x^{-(s+1)}\right] \\ &< 0 \qquad\qquad \forall x\in(0,1) \end{aligned}$$

$$\begin{aligned} f'(x)=0 &\Leftrightarrow \quad \mu_1 - \mu_2\left(x^{-s}-1\right)^{\frac{1}{s}-1} = 0 \\ &\Leftrightarrow \quad (x^{-s}-1)^{-\frac{1}{r}} = \frac{\mu_1}{\mu_2} \\ &\Leftrightarrow \quad x = \left[\left(\frac{\mu_2}{\mu_1}\right)^r + 1\right]^{-\frac{1}{s}} \end{aligned}$$

Since $0 < \left[\left(\frac{\mu_2}{\mu_1}\right)^r + 1\right]^{-\frac{1}{s}} < 1$, $f''\left(\left[\left(\frac{\mu_2}{\mu_1}\right)^r + 1\right]^{-\frac{1}{s}}\right) < 0$.

$f(x)$ acheives its maximum at $x = \left[\left(\frac{\mu_2}{\mu_1}\right)^r + 1\right]^{-\frac{1}{s}}$.

Thus, $\forall x \in (0,1)$,

$$f(x) \le f\left(\left[\left(\frac{\mu_2}{\mu_1}\right)^r + 1\right]^{-\frac{1}{s}}\right)$$

$$\begin{aligned} \mu_1 x + \mu_2(1-x^s)^{\frac{1}{s}} &\le \mu_1\left[\left(\frac{\mu_2}{\mu_1}\right)^r + 1\right]^{-\frac{1}{s}} + \mu_2\left\{1 - \left[\left(\frac{\mu_2}{\mu_1}\right)^r + 1\right]^{-1}\right\}^{\frac{1}{s}} \\ &= \mu_1\left(\frac{\mu_1^r}{\mu_1^r+\mu_2^r}\right)^{\frac{1}{s}} + \mu_2\left(\frac{\mu_2^r}{\mu_1^r+\mu_2^r}\right)^{\frac{1}{s}} \\ &= \frac{\mu_1^r+\mu_2^r}{(\mu_1^r+\mu_2^r)^{\frac{1}{s}}} \qquad\qquad (1+\frac{r}{s} = \frac{r+s}{s} = \frac{rs}{s} = r) \\ &= (\mu_1^r+\mu_2^r)^{\frac{1}{r}} \end{aligned}$$

(2) Let $x = \dfrac{\lambda_1}{(\lambda_1^s+\lambda_2^s)^{\frac{1}{s}}}$.

$$\begin{aligned} \lambda_1^s &< \lambda_1^s + \lambda_2^s \\ \lambda_1 &< (\lambda_1^s+\lambda_2^s)^{\frac{1}{s}} \\ \frac{\lambda_1}{(\lambda_1^s+\lambda_2^s)^{\frac{1}{s}}} &< 1 \\ \lambda_1^s+\lambda_2^s &> 0 \\ (\lambda_1^s+\lambda_2^s)^{\frac{1}{s}} &> 0 \\ \frac{\lambda_1}{(\lambda_1^s+\lambda_2^s)^{\frac{1}{s}}} &> 0 \end{aligned}$$

From (a) (i) (1), we have

$$\begin{aligned} \mu_1\cdot\frac{\lambda_1}{(\lambda_1^s+\lambda_2^s)^{\frac{1}{s}}} + \mu_2\left[1-\frac{\lambda_1^s}{\lambda_1^s+\lambda_2^s}\right]^{\frac{1}{s}} &\le (\mu_1^r+\mu_2^r)^{\frac{1}{r}} \\ \mu_1\cdot\frac{\lambda_1}{(\lambda_1^s+\lambda_2^s)^{\frac{1}{s}}} + \mu_2\cdot\frac{\lambda_2}{(\lambda_1^s+\lambda_2^s)^{\frac{1}{s}}} &\le (\mu_1^r+\mu_2^r)^{\frac{1}{r}} \\ \mu_1\lambda_1 + \mu_2\lambda_2 &\le (\mu_1^r+\mu_2^r)^{\frac{1}{r}}(\lambda_1^s+\lambda_2^s)^{\frac{1}{s}} \end{aligned}$$

(ii) When $n = 2$, from (a) (i) (2), $\mu_1\lambda_1 + \mu_2\lambda_2 \leq (\mu_1^r + \mu_2^r)^{\frac{1}{r}} (\lambda_1^s + \lambda_2^s)^{\frac{1}{s}}$.

Assume, as the Inductive Hypothesis, that $\sum_{k=1}^{m} a_k b_k \leq \left(\sum_{k=1}^{m} a_k^r\right)^{\frac{1}{r}} \left(\sum_{k=1}^{m} b_k^s\right)^{\frac{1}{s}}$ for some positive integer m. Then, we have

$$\begin{aligned}
\sum_{k=1}^{m+1} a_k b_k &= \sum_{k=1}^{m} a_k b_k + a_{k+1}b_{k+1} \\
&\leq \left(\sum_{k=1}^{m} a_k^r\right)^{\frac{1}{r}} \left(\sum_{k=1}^{m} b_k^s\right)^{\frac{1}{s}} + a_{k+1}b_{k+1} && \text{(By Inductive Hypothesis)} \\
&\leq \left(\sum_{k=1}^{m} a_k^r + a_{k+1}^r\right)^{\frac{1}{r}} \left(\sum_{k=1}^{m} b_k^s + b_{k+1}^s\right)^{\frac{1}{s}} && \text{(By (a) (i) (2))} \\
&= \left(\sum_{k=1}^{m+1} a_k^r\right)^{\frac{1}{r}} \left(\sum_{k=1}^{m+1} b_k^s\right)^{\frac{1}{s}}
\end{aligned}$$

Thus, by the principle of mathematical induction, we have $\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^r\right)^{\frac{1}{r}} \left(\sum_{k=1}^{n} b_k^s\right)^{\frac{1}{s}}$ for all positive integer n.

(b) Let $a_k = 1$, $b_k = x_k^{1-\beta}$ and $\frac{1}{s} = 1 - \beta$.

$$\begin{aligned}
\beta &> 0 \\
1 - \beta &< 1 \\
s = \frac{1}{1-\beta} &> 1 \\
\frac{1}{r} &= 1 - \frac{1}{s} \\
&= 1 - (1 - \beta) \\
&= \beta \\
&< 1 \\
r &> 1
\end{aligned}$$

From (a) (ii) (2), we have

$$\begin{aligned}
\sum_{k=1}^{n} x_k^{1-\beta} &\leq \left(\sum_{k=1}^{n} 1\right)^{\beta} \left(\sum_{k=1}^{n} x_k\right)^{1-\beta} \\
&= n^{\beta} \left(\sum_{k=1}^{n} x_k\right)^{1-\beta}
\end{aligned}$$

(c) Put $x_k = 2k - 1$, $\beta = \frac{2}{3}$ and $n = 1331$. From (b), we have

$$\begin{aligned}
\sum_{k=1}^{1331} (2k-1)^{\frac{1}{3}} &\leq 1331^{\frac{2}{3}} \left(\sum_{k=1}^{1331} 2k - 1\right)^{\frac{1}{3}} \\
&= 121 \left(2\sum_{k=1}^{1331} k - 1331\right)^{\frac{1}{3}} \\
&= 121 \left[(1 + 1331)(1331) - 1331\right]^{\frac{1}{3}} \\
&= 121 \times 121 \\
&= 14641
\end{aligned}$$

HONG KONG ADVANCED LEVEL EXAMINATION 2012

PURE MATHEMATICS PAPER 2

SAMPLE SOLUTIONS

1. (a) Since $g(x)$ is continuous at $x = \pi$, we have

$$\begin{aligned}
g(\pi) &= \lim_{x \to \pi^+} g(x) \\
f(\pi) + \pi + k &= \lim_{x \to \pi^+} \frac{\sin x}{x - \pi} \\
-1 + \pi + k &= \lim_{x \to \pi^+} -\frac{\sin(x - \pi)}{x - \pi} \\
-1 + \pi + k &= -1 \\
k &= -\pi
\end{aligned}$$

(b)
$$\begin{aligned}
\lim_{n \to \pi^-} \frac{g(x) - g(\pi)}{x - \pi} &= \lim_{n \to \pi^-} \frac{f(x) + x - \pi - (-1)}{x - \pi} \\
&= \lim_{n \to \pi^-} \frac{f'(x) + 1}{1} && \text{(By l'Hôpital's rule)} \\
&= 4
\end{aligned}$$

$$\begin{aligned}
\lim_{n \to \pi^+} \frac{g(x) - g(\pi)}{x - \pi} &= \lim_{n \to \pi^+} \frac{\frac{\sin x}{x - \pi} - (-1)}{x - \pi} \\
&= \lim_{n \to \pi^+} \frac{\sin x + x - \pi}{(x - \pi)^2} \\
&= \lim_{n \to \pi^+} \frac{\cos x + 1}{2(x - \pi)} && \text{(By l'Hôpital's rule)} \\
&= \lim_{n \to \pi^+} \frac{-\sin x}{2} && \text{(By l'Hôpital's rule)} \\
&= 0
\end{aligned}$$

Since $\lim_{n \to \pi^-} \frac{g(x) - g(\pi)}{x - \pi} \neq \lim_{n \to \pi^+} \frac{g(x) - g(\pi)}{x - \pi}$, $g(x)$ is not differentiable at $x = \pi$.

2. (a) When $n = 1$, $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x = \sin\left(\frac{\pi}{2} + x\right)$.

Assume, as the Inductive Hypothesis, that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin x = \sin\left(\frac{k\pi}{2} + x\right)$. Then we have

$$\begin{aligned}
\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\sin x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin x\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{k\pi}{2} + x\right) && \text{(By Inductive Hypothesis)} \\
&= \cos\left(\frac{k\pi}{2} + x\right) \\
&= \sin\left(\frac{\pi}{2} + \frac{k\pi}{2} + x\right) \\
&= \sin\left[\frac{(k+1)\pi}{2} + x\right]
\end{aligned}$$

Hence, by the principle of Mathematical Induction, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\sin x = \sin\left(\frac{n\pi}{2} + x\right)$ for all positive integer n.

(b) (i) $$f(x) = \frac{\sin x}{1 + 4x^2}$$

$\left(1 + 4x^2\right) f(x) = \sin x$

Differentiating both side $n + 2$ times w.r.t. x, we have

$$\binom{n+2}{0}(1 + 4x^2)f^{(n+2)}(x) + \binom{n+2}{1}(4x)f^{(n+1)}(x) + \binom{n+2}{2}(4)f^{(n)}(x) = \sin\left[\frac{(n+2)\pi}{2} + x\right]$$

(By (a))

$$(1 + 4x^2)f^{(n+2)}(x) + 4(n+2)xf^{(n+1)}(x) + 4(n+2)(n+1)f^{(n)}(x) = \sin\left[\frac{(n+2)\pi}{2} + x\right]$$

Substituting $x = 0$, we have

$$f^{(n+2)}(x) + 4(n+2)(n+1)f^{(n)}(x) = \sin\frac{(n+2)\pi}{2}$$

$$f^{(n+2)}(x) = -4(n+2)(n+1)f^{(n)}(x) - \sin\frac{n\pi}{2}$$

(ii)
$$\begin{aligned}
f^{(5)}(0) &= -4(5)(4)f^{(3)}(0) - \sin\frac{3\pi}{2} && \text{(By (b) (i))} \\
&= -80\left[-4(3)(2)f^{(1)}(0) - \sin\frac{\pi}{2}\right] - (-1) && \text{(By (b) (i))} \\
&= -80\left\{-24\left[\frac{\cos x}{1 + 4x^2} - \frac{(\sin x)(8x)}{(1 + 4x^2)^2}\right]_{x=0} - 1\right\} + 1 \\
&= 2001
\end{aligned}$$

3. (a) $$3I + 4J = 3\int \frac{\sin x}{3\sin x + 4\cos x}\,\mathrm{d}x + 4\int \frac{\cos}{3\sin x + 4\cos x}\,\mathrm{d}x$$
$$= \int \frac{3\sin x + 4\cos x}{3\sin x + 4\cos x}\,\mathrm{d}x$$
$$= \int \mathrm{d}x$$
$$= x + C_1$$

(b) $$4I - 3J = 4\int \frac{\sin x}{3\sin x + 4\cos x}\,\mathrm{d}x - 3\int \frac{\cos}{3\sin x + 4\cos x}\,\mathrm{d}x$$
$$= \int \frac{4\sin x - 3\cos x}{3\sin x + 4\cos x}\,\mathrm{d}x$$
$$= \int \frac{\mathrm{d}(-4\cos x - 3\sin x)}{3\sin x + 4\cos x}$$
$$= -\ln(3\sin x + 4\cos x) + C_2$$

(c) From (a) and (b), we have

$$\begin{cases} 3I + 4J = x + C_1 & \ldots\ (1) \\ 4I - 3J = -\ln(3\sin x + 4\cos x) + C_2 & \ldots\ (2) \end{cases}$$

$3 \times (1) + 4 \times (2)$, we have

$$25I = 3x - 4\ln(3\sin x + 4\cos x) + 3C_1 + 4C_2$$
$$I = \frac{3x - 4\ln(3\sin x + 4\cos x)}{25} + C$$

where C is a constant.

4. (a) Let $x = \frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}$. Then $\mathrm{d}x = \frac{\sqrt{3}}{2}\sec^2\theta\,\mathrm{d}\theta$.

$$\sin\theta = \frac{x+\frac{1}{2}}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}} = \frac{2x+1}{\sqrt{(2x+1)^2+3}}, \quad \cos\theta = \frac{\frac{\sqrt{3}}{2}}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}} = \frac{\sqrt{3}}{\sqrt{(2x+1)^2+3}}$$

$$\begin{aligned} x^2+x+1 &= \left(x+\frac{1}{2}\right)^2+\frac{3}{4} \\ &= \frac{3}{4}\left(\tan^2\theta+1\right) \\ &= \frac{3}{4}\sec^2\theta \end{aligned}$$

$$\begin{aligned} \int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,\mathrm{d}x &= \int \frac{\left(\frac{\sqrt{3}}{2}\tan\theta+\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\sec^2\theta\right)}{\frac{3}{4}\sec^2\theta\sqrt{\frac{3}{4}\sec^2\theta}}\,\mathrm{d}\theta \\ &= \frac{2}{3}\int \sqrt{3}\sin\theta+\cos\theta\,\mathrm{d}\theta \\ &= \frac{2}{3}\left(-\sqrt{3}\cos\theta+\sin\theta\right)+C \\ &= \frac{4(x-1)}{3\sqrt{(2x+1)^2+3}}+C \end{aligned}$$

(b)

$$\begin{aligned} \lim_{n\to\infty}\sum_{k=1}^{3n}\frac{n(k+n)}{(k^2+kn+n^2)\sqrt{k^2+kn+n^2}} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{3n}\frac{\frac{k}{n}+1}{\left[\left(\frac{k}{n}\right)^2+\frac{k}{n}+1\right]\sqrt{\left(\frac{k}{n}\right)^2+\frac{k}{n}+1}} \\ &= \int_0^3 \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,\mathrm{d}x \\ &= \left[\frac{4(x-1)}{3\sqrt{(2x+1)^2+3}}\right]_0^3 \qquad \text{(By (a))} \\ &= \frac{8}{3\sqrt{52}}+\frac{2}{3} \\ &= \frac{4\sqrt{13}+26}{39} \end{aligned}$$

5. (a)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}e^{2\sqrt{x}} &= \frac{1}{2\sqrt{x}}e^{2\sqrt{x}}+\sqrt{x}\left(2\cdot\frac{1}{2\sqrt{x}}\right)e^{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}e^{2\sqrt{x}}+e^{2\sqrt{x}} \end{aligned}$$

(b) The volume of the solid is given by

$$\begin{aligned} \pi\int_4^9 e^{2\sqrt{x}}\,\mathrm{d}x &= \pi\left(\sqrt{x}e^{2\sqrt{x}}\Big|_4^9 - \int_4^9 \frac{1}{2\sqrt{x}}e^{2\sqrt{x}}\,\mathrm{d}x\right) \\ &= \pi\left(3e^6-2e^4-\int_4^9 e^{2\sqrt{x}}\,\mathrm{d}2\sqrt{x}\right) \\ &= \pi\left(3e^6-2e^4-e^{2\sqrt{x}}\Big|_4^9\right) \\ &= \pi\left(2e^6-e^4\right) \end{aligned}$$

6. (a) Let P be (x', y'). Then the equation of the tangent is

$$\frac{x'x}{400} + \frac{y'y}{144} = 1$$

Since the tangent passes through A and B, we have

$$\frac{x'(h)}{400} + \frac{y'(40)}{144} = 1 \qquad (1)$$

$$\frac{x'(-h)}{400} + \frac{y'(0)}{144} = 1 \qquad (2)$$

(1)+(2), we have

$$\frac{40y'}{144} = 2$$

$$y' = 7.2 \qquad (3)$$

Substituting (3) into (E), we have

$$\frac{x'^2}{400} + \frac{7.2^2}{144} = 1$$

$$x' = 16 \quad \text{(rejected)} \quad \text{or} \quad -16 \qquad (4)$$

Hence, the coordinates of P are $(-16, 7.2)$.

Substituting (4) into (2), we have

$$\frac{(-16)(-h)}{400} = 1$$

$$h = 25$$

(b) (i) Let the equation of L_2 be

$$y - 7.2 = m(x + 16)$$

$$mx - y + 7.2 + 16m = 0$$

Since AB is the angle bisector of L_1 and L_2, the equation of AB can be written as

$$\frac{mx - y + 7.2 - 16m}{\sqrt{m^2+1}} = -16 - x$$

$$\left(m + \sqrt{m^2+1}\right)x - y + 7.2 + 16\left(m + \sqrt{m^2+1}\right) = 0$$

By considering the points A and B, the equation of AB can also be written as

$$y = \frac{4}{5}(x + 25)$$

$$4x - 5y + 100 = 0$$

Comparing the two equation, we have

$$m + \sqrt{m^2+1} = \frac{4}{5}$$

$$m = -\frac{9}{40}$$

Hence, the equation of L_2 is

$$y - 7.2 = -\frac{9}{40}(x + 16)$$

$$9x + 40y - 432 = 0$$

(ii) Substituting $y = 0$ into the equation of L_2, we have the coordinates of Q to be $(48, 0)$.

The slope of AQ is $\dfrac{40}{25-48} = -\dfrac{40}{23}$.

The slope of PQ is $\dfrac{7.2}{-16-48} = \dfrac{9}{80}$.

The slope of AP is $\dfrac{40}{25+25} = \dfrac{4}{5}$.

Since none of the products of the slopes of any 2 sides is -1, $\triangle APQ$ is not a right-angled triangle.

7. (a) $f'(x) = (12x+5)e^{-x} - (6x^2+5x+6)e^{-x}$

$$= (-6x^2+7x-1)e^{-x}$$
$$= -(6x-1)(x-1)e^{-x}$$

$f''(x) = (-12x+7)e^{-x} - (-6x^2+7x-1)e^{-x}$

$$= (6x^2-19x+8)e^{-x}$$
$$= (3x-8)(2x-1)e^{-x}$$

(b) Note that $f(x) \neq 0 \quad \forall x \in \mathbf{R}$, $f'(x) = 0 \Leftrightarrow x = \frac{1}{6}$ or 1, $f''(x) = 0 \Leftrightarrow x = \frac{1}{2}$ or $\frac{8}{3}$.

x	$\left(-\infty, \frac{1}{6}\right)$	$\frac{1}{6}$	$\left(\frac{1}{6}, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, 1\right)$	1	$\left(1, \frac{8}{3}\right)$	$\frac{8}{3}$	$\left(\frac{8}{3}, +\infty\right)$
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$

(i) $f(x) > 0 \quad \Leftrightarrow \quad x \in \mathbf{R}$

(ii) $f'(x) > 0 \quad \Leftrightarrow \quad \frac{1}{6} < x < 1$

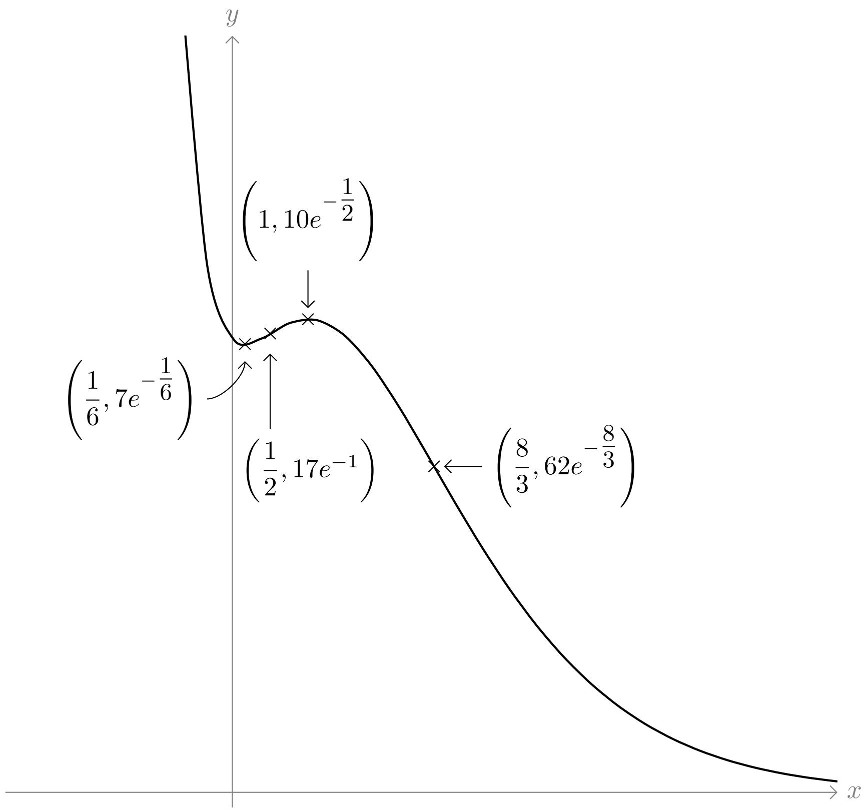
(iii) $f''(x) > 0 \quad \Leftrightarrow \quad x < \frac{1}{2}$ or $x > \frac{8}{3}$

(c) From the table in (b), the minimum point is $\left(\frac{1}{6}, 7e^{-\frac{1}{6}}\right)$, the maximum point is $\left(1, 17e^{-1}\right)$, the points of inflexion are $\left(\frac{1}{2}, 10e^{-\frac{1}{2}}\right)$ and $\left(\frac{8}{3}, 62e^{-\frac{8}{3}}\right)$.

(d) $\because \lim_{n\to+\infty} f(x) = \lim_{n\to+\infty} \left(6x^2+5x+6\right) e^{-x} = 0$

$\therefore$ the horizontal asymptote is $y = 0$.

(e)



(f) With the help of the graph of $f(x)$, we have

$$n(k) = \begin{cases} 0 & \text{when } k \le 0 \\ 1 & \text{when } 0 < k < 7e^{-\frac{1}{6}} \text{ or } k > 10e^{-\frac{1}{2}} \\ 2 & \text{when } k = 7e^{-\frac{1}{6}} \text{ or } k = 10e^{-\frac{1}{2}} \\ 3 & \text{when } 7e^{-\frac{1}{6}} < k < 10e^{-\frac{1}{2}} \end{cases}$$

8. (a) $f(r+0) = f(r)f(0) - f(r) - f(0) + 2$

$$2 = 2f(0) - 2 - f(0) + 2$$

$$f(0) = 2$$

(b) Suppose $f(x)$ is not injective.
There exists $x \neq y$ such that $f(x) = f(y)$. Then

$$\begin{aligned} f(x-y) &= f(x)f(y) - f(x) - f(y) + 2 \\ &= f(x)f(x) - f(x) - f(x) + 2 \\ &= f(x-x) \\ &= f(0) \\ &= 2 \end{aligned}$$

But from property (2), we know that there exists a unique real number r such that $f(r) = 2$.
From (a), we know that $r = 0$.
Since $x \neq y$, $x - y \neq 0$.
Contradiction arises.
$f(x)$ is injective.

(c) For any $x \neq 0$,

$$\begin{aligned} f(x) &= f\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right) + 2 \\ &= \left[f\left(\frac{x}{2}\right)\right]^2 - 2f\left(\frac{x}{2}\right) + 2 \\ &= \left[f\left(\frac{x}{2}\right) - 1\right]^2 \\ &> 0 \end{aligned}$$

Hence, there does not exist x' such that $f(x') = k$ for any $k < 0$.
$f(x)$ is not surjective.

(d) (i)
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{x+h-x} \\ &= \lim_{h\to 0} \frac{f(x)f(h) - f(x) - f(h) + 2 - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[f(x)-1][f(h)-2]}{h} \\ &= \left[\lim_{h\to 0} f(x) - 1\right]\left[\lim_{h\to 0} \frac{f(h)-2}{h}\right] \quad \left(\because \lim_{h\to 0} f(x) - 1 \text{ and } \lim_{h\to 0} \frac{f(h)-2}{h} \text{ exist}\right) \\ &= 12f(x) - 12 \end{aligned}$$

Hence, $f(x)$ is differentiable everywhere.

(ii)
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-12x} f(x)\right] &= -12e^{-12x} f(x) + e^{-12x} f'(x) \\ &= \left[-12f(x) + 12f(x) - 12\right] e^{-12x} \quad \text{(By (d) (i))} \\ &= -12e^{-12x} \\ e^{-12x} f(x) &= \int -12e^{-12x}\,\mathrm{d}x \\ &= e^{-12x} + C \end{aligned}$$

$$\begin{aligned} f(0) &= 1 + C \\ 2 &= 1 + C \quad \text{(By (a))} \\ C &= 1 \end{aligned}$$

Hence, $f(x) = 1 + e^{12x}$.

9. (a) (i) $I_1 = \int_0^\pi e^{-x}(\pi - x)\,\mathrm{d}x$

$$= -\int_0^\pi \pi - x\,\mathrm{d}e^{-x}$$

$$= -e^{-x}(\pi - x)\Big|_0^\pi + \int_0^\pi e^{-x}\,\mathrm{d}(\pi - x)$$

$$= \pi + e^{-x}\Big|_0^\pi$$

$$= \pi + e^{-\pi} - 1$$

(ii) $I_{n+1} = \int_0^\pi e^{-x}(\pi - x)^{n+1}\,\mathrm{d}x$

$$= -\int_0^\pi (\pi - x)^{n+1}\,\mathrm{d}e^{-x}$$

$$= -e^{-x}(\pi - x)^{n+1}\Big|_0^\pi + \int_0^\pi e^{-x}\,\mathrm{d}(\pi - x)^{n+1}$$

$$= \pi^{n+1} - (n+1)\int_0^\pi e^{-x}(\pi - x)^n\,\mathrm{d}x$$

$$= \pi^{n+1} - (n+1)I_n$$

(iii) By repeatedly applying the equation in (a) (ii), we have

$$\begin{aligned}
I_n &= \pi^n - nI_{n-1} \\
&= \pi^n - n\pi^{n-1} + n(n-1)I_{n-2} \\
&\vdots \\
&= \sum_{k=2}^{n} (-1)^{n-k}\frac{n!\pi^k}{k!} + (-1)^{n-1}n!I_1 \\
&= \sum_{k=2}^{n} (-1)^{n-k}\frac{n!\pi^k}{k!} + (-1)^{n-1}n!\left(\pi + e^{-\pi} - 1\right) && \text{(By (a) (i))} \\
&= \sum_{k=2}^{n} (-1)^{n-k}\frac{n!\pi^k}{k!} + (-1)^{n-1}\frac{n!\pi}{1!} + (-1)^{n-1}n!e^{-\pi} + (-1)^n\frac{n!\pi^0}{0!} \\
I_n + (-1)^n n!e^{-\pi} &= \sum_{k=0}^{n} (-1)^{n-k}\frac{n!\pi^k}{k!} \\
(-1)^n\frac{I_n}{n!} + e^{-\pi} &= \sum_{k=0}^{n} (-1)^k\frac{\pi^k}{k!}
\end{aligned}$$

(b) (i) For $n \geq 3$,

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\pi^{n+1}}{(n+1)!}\cdot\frac{n!}{\pi^n} \\
&= \frac{\pi}{n+1} \\
&< 1 \\
a_{n+1} &< a_n
\end{aligned}$$

(ii) From (b) (i), a_n is strictly decreasing.
a_n is bounded below because
$a_n = \dfrac{\pi^n}{n!} > 0.$
Hence, $\lim\limits_{n\to\infty} a_n$ exists.

$$0 < \lim_{n\to\infty} \frac{\pi^n}{n!} < \lim_{n\to\infty} \left(\frac{\pi}{n}\right)^n$$

$$0 < \lim_{n\to\infty} a_n < 0$$

By Sandwich Theorem, $\lim\limits_{n\to\infty} a_n = 0.$

(c)
$$\begin{aligned}
\frac{1}{n!}\int_0^\pi e^\pi(\pi - x)^n \, \mathrm{d}x &\le \frac{1}{n!}\int_0^\pi e^\pi(\pi - 0)^n \, \mathrm{d}x \\
\frac{I_n}{n!} &\le \int_0^\pi e^\pi \frac{\pi^n}{n!} \, \mathrm{d}x \\
\lim_{n\to\infty} \frac{I_n}{n!} &\le \lim_{n\to\infty} \int_0^\pi e^\pi \frac{\pi^n}{n!} \, \mathrm{d}x \\
\lim_{n\to\infty} \frac{I_n}{n!} &\le 0 \qquad \text{(By (b) (ii))}
\end{aligned}$$

In addition,

$$\begin{aligned}
\lim_{n\to\infty} \frac{I_n}{n!} &\ge \lim_{n\to\infty} \frac{1}{n!}\int_0^\pi e^\pi(\pi - \pi)^n \, \mathrm{d}x \\
&= 0
\end{aligned}$$

By Sandwich Theorem,

$$\lim_{n\to\infty} \frac{I_n}{n!} = 0$$

$$\begin{aligned}
\sum_{k=0}^{\infty} (-1)^k \frac{\pi^k}{k!} &= \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \frac{\pi^k}{k!} \\
&= \lim_{n\to\infty} (-1)^n \frac{I_n}{n!} + e^{-\pi} \qquad \text{(By (a) (iii))} \\
&= e^{-\pi}
\end{aligned}$$

10. (a) (i) (1) $h(x) = \int_1^2 q(t)\,\mathrm{d}t \int_1^x p(t)q(t)\,\mathrm{d}t - \int_1^2 p(t)q(t)\,\mathrm{d}t \int_1^x q(t)\,\mathrm{d}t$

$$h'(x) = p(x)q(x)\int_1^2 q(t)\,\mathrm{d}t - q(x)\int_1^2 p(t)q(t)\,\mathrm{d}t \qquad \forall x \in \mathbf{J}$$

(2) By Mean Value Theorem, there exists $\beta \in \mathbf{J}$ such that

$$h'(\beta) = \frac{h(2)-h(1)}{2-1}$$

$$p(\beta)q(\beta)\int_1^2 q(t)\,\mathrm{d}t - q(\beta)\int_1^2 p(t)q(t)\,\mathrm{d}t = 0 \qquad \text{(By (a) (i) (1))}$$

$$\int_1^2 p(t)q(t)\,\mathrm{d}t = p(\beta)\int_1^2 q(t)\,\mathrm{d}t \qquad (\because q(x) > 0 \quad \forall x \in \mathbf{J})$$

(ii) $$\int_1^2 f(x)g'(x)\,\mathrm{d}x = \int_1^2 f(x)\,\mathrm{d}g(x)$$
$$= \Big[f(x)g(x)\Big]_1^2 - \int_1^2 g(x)\,\mathrm{d}f(x)$$
$$= f(2)g(2) - f(1)g(1) - \int_1^2 f'(x)g(x)\,\mathrm{d}x$$

Since $f'(x) > 0$ for all $x \in \mathbf{J}$, by using (a) (i) (2), put $p = g$ and $q = f'$, there exists $c \in \mathbf{J}$ such that

$$\int_1^2 f'(x)g(x)\,\mathrm{d}x = g(c)\int_1^2 f'(x)\,\mathrm{d}x$$
$$= g(c)\,[f(2) - f(1)]$$

Hence,

$$\int_1^2 f(x)g'(x)\,\mathrm{d}x = f(2)g(2) - f(1)g(1) - g(c)\,[f(2) - f(1)]$$

(b) (i) $\frac{\mathrm{d}}{\mathrm{d}x}\cos x^{100} = -\left(\sin x^{100}\right)\left(100x^{99}\right)$

(ii) Let $f(x) = -\frac{1}{100x^{99}}$ and $g(x) = \cos x^{100}$.

$$f'(x) = -\frac{-99}{100x^{100}}$$
$$> 0 \qquad \forall x \in \mathbf{J}$$

By (a) (ii), there exists $c \in \mathbf{J}$ such that

$$\int_1^2 -\frac{1}{100x^{99}}\frac{\mathrm{d}^{\cos}}{\mathrm{d}x^{\cos}}x^{100}\,\mathrm{d}x = -\frac{\cos 2^{100}}{(100)(2^{99})} + \frac{\cos 1^{100}}{(100)(1^{99})} - \left(\cos c^{100}\right)\left[-\frac{1}{(100)(2^{99})} + \frac{1}{(100)(1^{99})}\right]$$

$$\left|\int_1^2 \sin x^{100}\,\mathrm{d}x\right| = \left|\int_1^2 -\frac{1}{100x^{99}}\frac{\mathrm{d}^{\cos}}{\mathrm{d}x^{\cos}}x^{100}\,\mathrm{d}x\right|$$
$$= \left|-\frac{\cos 2^{100}}{(100)(2^{99})} + \frac{\cos 1^{100}}{(100)(1^{99})} - \left(\cos c^{100}\right)\left[-\frac{1}{(100)(2^{99})} + \frac{1}{(100)(1^{99})}\right]\right|$$
$$\leq \left|\frac{1}{100}\right| + \left|-\cos c^{100}\right|\left|-\frac{1}{(100)(2^{99})} + \frac{1}{100}\right|$$
$$\leq \frac{1}{100} + (1)\left|\frac{1}{100}\right|$$
$$= \frac{1}{50}$$

11. (a) $xy = 2$

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$$

Slope of the normal to H at T is

$$\left.\frac{x}{y}\right|_{\left(t, \frac{2}{t}\right)} = \frac{t^2}{2}$$

Equation of L is

$$y - 2 = \left(\frac{t^2}{2}\right)(x - 2)$$

$$t^2x - 2y + 4 - 2t^2 = 0$$

(b) The parametric equation of the tangent to H at T is

$$\left(t + kt, \frac{2}{t} - k \cdot \frac{2}{t}\right) \tag{5}$$

Substituting (5) into the equation of L,

$$(t^2)(1 + k)(t) - 2(1 - k)\left(\frac{2}{t}\right) + 4 - 2t^2 = 0$$

$$t^3 + kt^3 - \frac{2}{t} + k \cdot \frac{4}{t} + 4 - 2t^2 = 0$$

$$k = -\frac{t^4 - 2t^3 + 4t - 4}{t^4 + 4}$$

Hence, the point of intersection is

$$\left(t\left(1 - \frac{t^4 - 2t^3 + 4t - 4}{t^4 + 4}\right), \frac{2}{t}\left(1 + \frac{t^4 - 2t^2 + 4t - 4}{t^4 + 4}\right)\right)$$

$$= \left(\frac{t(2t^3 - 4t + 8}{t^4 + 4}, \frac{2}{t} \cdot \frac{2t^4 - 2t^3 + 4t}{t^4 + 4}\right)$$

$$= \left(\frac{2t(t^3 - 2t + 4)}{t^4 + 4}, \frac{4(t^3 - t^2 + 2)}{t^4 + 4}\right)$$

(c) (i) The coordinates of M are

$$\left(2+2\left[\frac{2t(t^3-2t+4)}{t^4+4}-2\right], 2+2\left[\frac{4(t^3-t^2+2}{t^4+4}-2\right]\right)$$
$$=\left(\frac{4t(t^3-2t+4)}{t^4+4}-2, \frac{8(t^3-t^2+2)}{t^4+4}-2\right)$$

$$|BM| = \sqrt{\left[\frac{4t(t^3-2t+4)}{t^4+4}-2-(-2)\right]^2+\left[\frac{8(t^3-t^2+2)}{t^4+4}-2-(-2)\right]^2}$$
$$= \sqrt{\frac{16t^2(t^3-2t+4)^2}{(t^4+4)^2}+\frac{64(t^3-t^2+2)^2}{(t^4+4)^2}}$$
$$= 4\sqrt{\frac{(t^4-2t^2+4t)^2+4(t^3-t^2+2)^2}{(t^4+4)^2}}$$
$$= 4\sqrt{\frac{t^8-4t^6+8t^5+4t^4-16t^3+16t^2+4t^6-8t^5+16t^3+4t^4-16t^2+16}{(t^4+4)^2}}$$
$$= 4\sqrt{\frac{t^8+8t^4+16}{(t^4+4)^2}}$$
$$= 4$$

Hence, $|BM|$ is independent of t.

(ii) Slope of $BM = \dfrac{\dfrac{8(t^3-t^2+2}{t^4+4}-2-(-2)}{\dfrac{4t(t^3-2t+4)}{t^4+4}-2-(-2)}$

$$= \frac{8(t^3-t^2+2)}{4t(t^3-2t+4)}$$
$$= \frac{2(t+1)(t^2-2t+2)}{t(t+2)(t^2-2t+2)}$$
$$= \frac{2(t+1)}{t(t+2)}$$

Slope of $TB = \dfrac{\dfrac{2}{t}-(-2)}{t-(-2)}$

$$= \frac{2(1+t)}{t(t+2)}$$

Since slope of BM = slope of TB, B, M and T are collinear.