2012-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2012

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(C) answer book.

4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Let $(2-x)^n = \sum_{k=0}^{n} a_k x^k$, where $n \in \mathbf{N}$.

   (a) Prove that

   (i) $\sum_{k=0}^{n} a_k = 1$,

   (ii) $\sum_{k=1}^{n} k a_k = -n$,

   (iii) $\sum_{k=1}^{n} k^2 a_k = n^2 - 2n$.

   (b) Find $\sum_{k=0}^{n} (n+k)^2 a_k$.

(7 marks)

2. (a) Resolve $\frac{2x-23}{(2x+1)(2x+5)(2x+7)}$ into partial fractions.

   (b) Express $\sum_{k=1}^{n} \frac{2k-23}{(2k+1)(2k+5)(2k+7)}$ in the form $A+\frac{B}{2n+3}+\frac{C}{2n+5}+\frac{D}{2n+7}$, where A , B , C and D are constants.

   (c) Evaluate $\sum_{k=4}^{\infty} \frac{2k-23}{(2k+1)(2k+5)(2k+7)}$.

(7 marks)

3. (a) Solve the equation $\sin \pi x = 1$, where $x \in \mathbf{R}$.

   (b) Let $\mathrm{f}(x)$ be a polynomial with real coefficients such that $\mathrm{f}(x) \sin \pi x = 0$ for all $x \in \mathbf{R}$. Prove that $\mathrm{f}(x) = 0$ for all $x \in \mathbf{R}$.

   (c) Let g be a real-valued function defined on $\mathbf{R}$ such that $\mathrm{g}(x) \sin \pi x = 0$ for all $x \in \mathbf{R}$. Someone claims that $\mathrm{g}(x) = 0$ for all $x \in \mathbf{R}$. Do you agree? Explain your answer.

(6 marks)

4. Let T be the transformation which transforms the point (x, y) to the point (x', y'), where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) If T transforms the point $(1, \sqrt{3})$ to the point P, find the coordinates of P.

(b) It is given that $\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $k > 0$ and $-\pi < \theta \le \pi$.

Find k and θ. Hence describe the geometric meaning of T.

(7 marks)

5. Let $a_1 = \frac{3}{2}$, $a_2 = \frac{7}{12}$ and $6\,a_{n+2} = 5\,a_{n+1} - a_n$ for all positive integers n.

(a) Using mathematical induction, prove that $a_n = \frac{1}{2^n} + \frac{1}{3^{n-1}}$ for any positive integer n.

(b) Does there exist a positive integer m such that $\sum_{k=1}^{m} a_k > 3$? Explain your answer.

(7 marks)

6. Let $S = \sum_{k=1}^{n} \left(1 + \frac{1}{k} \right)$, where $n \in \mathbf{N} \setminus \{1\}$.

(a) Using A.M. $\ge$ G.M., or otherwise, prove that $\frac{2n - S}{n - 1} \ge \left(\frac{1}{n} \right)^{\frac{1}{n-1}}$.

(b) Prove that $2n - (n-1)\, n^{\frac{1}{1-n}} \ge S \ge n(n+1)^{\frac{1}{n}}$.

(6 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. (a) Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x + 2y - z = 3 \\ 2x + 5y + (a-1)z = 4 \\ (a+2)x + y + (2a+1)z = b \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Prove that (E) has a unique solution if and only if $a \neq -1$ and $a \neq -3$. Solve (E) when (E) has a unique solution.

(ii) Suppose that $a = -3$. Find b for which (E) is consistent, and solve (E) when (E) is consistent.

(8 marks)

(b) Is the system of linear equations in real variables x, y, z

$$\begin{cases} x + 2y - z = 3 \\ 6x + 15y - 7z = 12 \\ 2x + 3y - 5z = -12 \\ 4x + 5y - 6z = 1 \end{cases}$$

consistent? Explain your answer.

(3 marks)

(c) Find the least value of $3x^2 - 7y^2 + 8z^2$, where x, y and z are real numbers satisfying

$$\begin{cases} x + 2y - z = 3 \\ 2x + 5y - 4z = 4 \\ x - y + 5z = 9 \end{cases}.$$

(4 marks)

8. Let $n \in \mathbf{N} \setminus \{1\}$.

(a) (i) Prove that $\sin\frac{\pi}{2n}\sum_{k=1}^{n-1}\sin\frac{k\pi}{n} = \sin\frac{(n-1)\pi}{2n}$.

(ii) Let $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Prove that $\sum_{k=1}^{n-1}|\alpha^k - 1| = 2\cot\frac{\pi}{2n}$.

(8 marks)

(b) Suppose that $\beta \in \mathbf{C}$ such that $\beta^n = 1$ and $\beta^k \neq 1$ for all $k = 1, 2, \ldots, n-1$.

(i) Prove that $1, \beta, \beta^2, \ldots, \beta^{n-1}$ are all distinct.

(ii) Find $\sum_{k=1}^{2n-1}|\beta^k - 1|$.

(7 marks)

9. (a) Let $\lambda > 1$. Prove that $(1+x)^{\lambda} > 1+\lambda x$ for any $x > 0$.

(3 marks)

(b) For any positive integer n , define $a_n = \left(1+\dfrac{1}{n}\right)^n$ and $b_n = \left(1+\dfrac{1}{n}\right)^{n+1}$.

(i) Using (a), or otherwise, prove that $a_{n+1} > a_n$.

(ii) Prove that

(1) $\dfrac{b_n}{b_{n+1}} = \left(1+\dfrac{1}{n(n+2)}\right)^{n+1}\left(\dfrac{n+1}{n+2}\right)$,

(2) $\dfrac{b_n}{b_{n+1}} > 1$.

(iii) Using (b)(i) and (b)(ii)(2), prove that both $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

(iv) Find $\prod\limits_{k=1}^{n} a_k$ and $\prod\limits_{k=1}^{n} b_k$.

Hence prove that $(n+1)^{n+1} > n!\,e^n > (n+1)^n$, where $e = \lim\limits_{n\to\infty} a_n$.

(12 marks)

10. Let $\mathrm{f}(x)$ and $\mathrm{g}(x)$ be polynomials with real coefficients such that $(\mathrm{f}(x))^2 - 1 = (x^2-1)(\mathrm{g}(x))^2$. It is given that the degree of $\mathrm{f}(x)$ is n , where $n \in \mathbf{N}$.

(a) Find the degree of $\mathrm{g}(x)$.

(2 marks)

(b) Prove that $\mathrm{f}(x)$ and $\mathrm{g}(x)$ have no non-constant common factors.

(3 marks)

(c) Prove that $\mathrm{f}'(x)$ is divisible by $\mathrm{g}(x)$.

(3 marks)

(d) Using (c), prove that $n^2\left((\mathrm{f}(x))^2 - 1\right) = (x^2-1)(\mathrm{f}'(x))^2$.

(4 marks)

(e) Denote the roots of the equation $\mathrm{f}(x) = 0$ by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Evaluate $\sum\limits_{k=1}^{n} \alpha_k$.

(3 marks)

11. (a) Let $\frac{1}{r}+\frac{1}{s}=1$, where $r>1$ and $s>1$.

(i) Let μ_1 and μ_2 be positive real numbers.

(1) By differentiating $f(x)=\mu_1 x+\mu_2(1-x^s)^{\frac{1}{s}}$, prove that

$\mu_1 x+\mu_2(1-x^s)^{\frac{1}{s}} \le (\mu_1{}^r+\mu_2{}^r)^{\frac{1}{r}}$ for all $x\in(0,1)$.

(2) Using (a)(i)(1), prove that $\mu_1\lambda_1+\mu_2\lambda_2 \le (\mu_1{}^r+\mu_2{}^r)^{\frac{1}{r}}(\lambda_1{}^s+\lambda_2{}^s)^{\frac{1}{s}}$ for any positive real numbers λ_1 and λ_2 .

(ii) Using (a)(i)(2), prove that $\sum_{k=1}^{n} a_k b_k \le \left(\sum_{k=1}^{n} a_k{}^r\right)^{\frac{1}{r}}\left(\sum_{k=1}^{n} b_k{}^s\right)^{\frac{1}{s}}$ for any positive real numbers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$.

(9 marks)

(b) Suppose that $0<\beta<1$. Using (a)(ii), prove that $\sum_{k=1}^{n} x_k{}^{1-\beta} \le n^{\beta}\left(\sum_{k=1}^{n} x_k\right)^{1-\beta}$ for any positive real numbers $x_1, x_2, \dots, x_n$.

(3 marks)

(c) Using (b), prove that $\sum_{k=1}^{1331}(2k-1)^{\frac{1}{3}} \le 14641$.

(3 marks)

END OF PAPER

2012-AL
P MATH
PAPER 2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
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PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(C) answer book.

4. Unless otherwise specified, all working must be clearly shown.

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FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

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$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. It is given that $f : \mathbf{R} \to \mathbf{R}$ is a differentiable function satisfying $f(\pi) = -1$ and $f'(\pi) = 3$.

   Let k be a real constant and $g : \mathbf{R} \to \mathbf{R}$ be defined by

$$g(x) = \begin{cases} f(x) + x + k & \text{when } x \le \pi , \\ \dfrac{\sin x}{x - \pi} & \text{when } x > \pi . \end{cases}$$

   Suppose that $g(x)$ is continuous at $x = \pi$.

   (a) Find k .

   (b) Is $g(x)$ differentiable at $x = \pi$? Explain your answer.

(6 marks)

2. (a) Prove that $\dfrac{d^n}{dx^n} \sin x = \sin\left(\dfrac{n\pi}{2} + x\right)$ for any positive integer n .

   (b) Let $f(x) = \dfrac{\sin x}{1 + 4x^2}$.

   (i) Prove that $f^{(n+2)}(0) = -4(n+2)(n+1)f^{(n)}(0) - \sin\dfrac{n\pi}{2}$ for any positive integer n .

   (ii) Evaluate $f^{(5)}(0)$.

(7 marks)

3. Let $I = \displaystyle\int \frac{\sin x}{3\sin x + 4\cos x} dx$ and $J = \displaystyle\int \frac{\cos x}{3\sin x + 4\cos x} dx$. Find

   (a) $3I + 4J$,

   (b) $4I - 3J$,

   (c) I .

(5 marks)

4. (a) Using the substitution $x = \frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}$, find $\int \frac{x+1}{(x^2+x+1)\sqrt{x^2+x+1}}\,dx$.

(b) Evaluate $\lim_{n\to\infty} \sum_{k=1}^{3n} \frac{n(k+n)}{(k^2+kn+n^2)\sqrt{k^2+kn+n^2}}$.

(7 marks)

5. (a) Find the derivative of $\sqrt{x}\,e^{2\sqrt{x}}$.

(b) Let D be the region bounded by the curve $y = e^{\sqrt{x}}$, the straight line $x = 4$, the straight line $x = 9$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(7 marks)

6. P is a point on the ellipse $E: \frac{x^2}{400} + \frac{y^2}{144} = 1$. The tangent to E at P passes through the points $A(h, 40)$ and $B(-h, 0)$, where $h > 0$.

(a) Find the coordinates of P and the value of h .

(b) L_1 is the straight line passing through P and perpendicular to the x-axis. L_2 is another straight line passing through P such that AB makes equal angles with L_1 and with L_2 .

(i) Find the equation of L_2 .

(ii) L_2 cuts the x-axis at Q . Is ΔAPQ a right-angled triangle? Explain your answer.

(8 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. Let $\mathrm{f}:\mathbf{R}\to\mathbf{R}$ be defined by $\mathrm{f}(x)=(6x^2+5x+6)e^{-x}$.

(a) Find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

(2 marks)

(b) Solve

(i) $\mathrm{f}(x)>0$,

(ii) $\mathrm{f}'(x)>0$,

(iii) $\mathrm{f}''(x)>0$.

(3 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y=\mathrm{f}(x)$.

(3 marks)

(d) Find the asymptote(s) of the graph of $y=\mathrm{f}(x)$.

(1 mark)

(e) Sketch the graph of $y=\mathrm{f}(x)$.

(3 marks)

(f) Let $\mathrm{n}(k)$ be the number of points of intersection of the graph of $y=\mathrm{f}(x)$ and the horizontal line $y=k$. Using the graph of $y=\mathrm{f}(x)$, find $\mathrm{n}(k)$ for any $k\in\mathbf{R}$.

(3 marks)

8. It is given that $\mathrm{f}:\mathbf{R}\to\mathbf{R}$ satisfies the following conditions:

(1) $\mathrm{f}(x+y)=\mathrm{f}(x)\,\mathrm{f}(y)-\mathrm{f}(x)-\mathrm{f}(y)+2$ for all $x, y\in\mathbf{R}$;

(2) there exists a unique real number r such that $\mathrm{f}(r)=2$.

(a) Prove that $\mathrm{f}(0)=2$.

(3 marks)

(b) Is f an injective function? Explain your answer.

(3 marks)

(c) Is f a surjective function? Explain your answer.

(3 marks)

(d) Suppose that $\lim_{h\to 0}\dfrac{\mathrm{f}(h)-2}{h}=12$.

(i) Prove that f is differentiable everywhere and $\mathrm{f}'(x)=12\,\mathrm{f}(x)-12$ for all $x\in\mathbf{R}$.

(ii) By differentiating $e^{-12x}\,\mathrm{f}(x)$, find $\mathrm{f}(x)$.

(6 marks)

9. (a) For each positive integer n, let $I_n = \int_0^{\pi} e^{-x}(\pi - x)^n \, dx$.

(i) Evaluate I_1 .

(ii) Express I_{n+1} in terms of I_n .

(iii) Prove that $\sum_{k=0}^{n} (-1)^k \frac{\pi^k}{k!} = (-1)^n \frac{I_n}{n!} + e^{-\pi}$.

(6 marks)

(b) For each positive integer n, let $a_n = \frac{\pi^n}{n!}$.

(i) Prove that $a_{n+1} < a_n$ for all $n \geq 3$.

(ii) Using (b)(i), or otherwise, prove that $\lim_{n\to\infty} a_n$ exists. Also evaluate $\lim_{n\to\infty} a_n$.

(5 marks)

(c) Using (a)(iii), evaluate $\sum_{k=0}^{\infty} (-1)^k \frac{\pi^k}{k!}$. (4 marks)

10. (a) Denote the closed interval $[1, 2]$ and the open interval $(1, 2)$ by $\mathbf{I}$ and $\mathbf{J}$ respectively.

(i) Assume that real-valued functions p and q are continuous on $\mathbf{I}$ and $q(x) > 0$ for all $x \in \mathbf{J}$. Define $h(x) = \int_1^2 q(t)\, dt \int_1^x p(t)q(t)\, dt - \int_1^2 p(t)q(t)\, dt \int_1^x q(t)\, dt$ for all $x \in \mathbf{I}$.

(1) Find $h'(x)$ for all $x \in \mathbf{J}$.

(2) Using the result of (a)(i)(1) and Mean Value Theorem, prove that there exists $\beta \in \mathbf{J}$ such that $\int_1^2 p(x)q(x)\, dx = p(\beta) \int_1^2 q(x)\, dx$.

(ii) Let f and g be real-valued functions such that f' and g' are continuous on $\mathbf{I}$ and $f'(x) > 0$ for all $x \in \mathbf{J}$. Prove that there exists $c \in \mathbf{J}$ such that

$$\int_1^2 f(x)\, g'(x)\, dx = f(2)\, g(2) - f(1)\, g(1) - g(c)\, (f(2) - f(1)) .$$

(9 marks)

(b) (i) Find $\frac{d}{dx} \cos x^{100}$.

(ii) Using (a)(ii), prove that $\left| \int_1^2 \sin x^{100}\, dx \right| \leq \frac{1}{50}$.

(6 marks)

11. $T\left(t, \frac{2}{t}\right)$ is a point on the hyperbola $H: xy = 2$, where $t > 0$. L is the straight line passing through the point $A(2, 2)$ and parallel to the normal to H at T.

(a) Find the equation of L.

(3 marks)

(b) Prove that the tangent to H at T cuts L at the point $\left(\frac{2t(t^3 - 2t + 4)}{t^4 + 4}, \frac{4(t^3 - t^2 + 2)}{t^4 + 4}\right)$.

(3 marks)

(c) A point M, other than A, lies on L such that $AT = MT$. Denote the point $(-2, -2)$ by B.

(i) Prove that the distance between B and M is independent of t.

(ii) Are B, M and T collinear? Explain your answer.

(9 marks)

END OF PAPER