Paper I

1. Let $\alpha = 4+\sqrt{17}$ and $\beta = 4-\sqrt{17}$.

Then α and β are the roots of the equation $x^2 - 8x - 1 = 0$,

such that $\alpha^2 = 8\alpha + 1$, $\beta^2 = 8\beta + 1$, --- <*>

$\alpha + \beta = 8$ and $\alpha\beta = -1$.

For $n = 1$, $\alpha + \beta = 8 = a_1$

For $n = 2$, $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 8^2 - 2(1) = 66 = a_2$

$\therefore$ The statement is true for $n = 1$, 2.

Assume $a_k = \alpha^k + \beta^k$ and $a_{k+1} = \alpha^{k+1} + \beta^{k+1}$ for some positive integer *k*.

Consider $n = k+1$,

$$\begin{aligned} a_{k+2} &= 8a_{k+1} + a_k && \text{(by definition)} \\ &= 8(\alpha^{k+1} + \beta^{k+1}) + (\alpha^k + \beta^k) \\ &= \alpha^k (8\alpha + 1) + \beta^k (8\beta + 1) \\ &= \alpha^k \alpha^2 + \beta^k \beta^2 && \text{(by <*>)} \\ &= \alpha^{k+2} + \beta^{k+2} \end{aligned}$$

$\therefore$ The statement is also true for $n = k+1$.

i.e. By induction, the statement is true for all positive integers *n*.

2. (a) $(3+x)^n = \sum_{k=0}^{n} a_k x^k$

Put $x = 1$, $\sum_{k=0}^{n} a_k = (3+1)^n = 4^n$

(b) $\int_0^1 (3+x)^n dx = \int_0^1 (\sum_{k=0}^{n} a_k x^k) dx$

$$\left. \frac{(3+x)^{n+1}}{n+1} \right|_0^1 = \sum_{k=0}^{n} a_k \int_0^1 x^k dx$$

$$\sum_{k=0}^{n} \frac{a_k}{k+1} = \frac{4^{n+1} - 3^{n+1}}{n+1}$$

(c)
$$\begin{aligned} \sum_{k=0}^{n} \frac{k}{k+1} a_k &= \sum_{k=0}^{n} (1 - \frac{1}{k+1}) a_k \\ &= \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} \frac{a_k}{k+1} \\ &= 4^n - \frac{4^{n+1} - 3^{n+1}}{n+1} \\ &= \frac{4^n (n-3) + 3^{n+1}}{n+1} \end{aligned}$$

3. (a) Let $\dfrac{1}{x(x+2)(x+4)}=\dfrac{A}{x}+\dfrac{B}{x+2}+\dfrac{C}{x+4}$.

Then $1=A(x+2)(x+4)+Bx(x+4)+Cx(x+2)$

Put $x=0$, -2 and -4, we have

$$A=\frac{1}{8},\qquad B=-\frac{1}{4},\qquad C=\frac{1}{8}$$

$$\therefore\quad \frac{1}{x(x+2)(x+4)}=\frac{1}{8x}-\frac{1}{4(x+2)}+\frac{1}{8(x+4)}$$

(b) (i) $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+2)(k+4)}$

$$=\sum_{k=1}^{n}\left(\frac{1}{8k}-\frac{1}{4(k+2)}+\frac{1}{8(k+4)}\right)$$

$$=\frac{1}{8}\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{4}\sum_{k=3}^{n+2}\frac{1}{k}+\frac{1}{8}\sum_{k=5}^{n+4}\frac{1}{k}$$

$$=\frac{1}{8}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})-\frac{1}{4}(\frac{1}{3}+\frac{1}{4}+\frac{1}{n+1}+\frac{1}{n+2})$$

$$+\frac{1}{8}(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4})$$

$$=\frac{11}{96}-\frac{1/8}{n+1}-\frac{1/8}{n+2}+\frac{1/8}{n+3}+\frac{1/8}{n+4}$$

(ii) $\displaystyle\sum_{k=n+1}^{N}\frac{1}{k(k+2)(k+4)}$

$$=\sum_{k=1}^{N}\frac{1}{k(k+2)(k+4)}-\sum_{k=1}^{n}\frac{1}{k(k+2)(k+4)}$$

$$=\left(\frac{11}{96}-\frac{1/8}{N+1}-\frac{1/8}{N+2}+\frac{1/8}{N+3}+\frac{1/8}{N+4}\right)$$

$$-\left(\frac{11}{96}-\frac{1/8}{n+1}-\frac{1/8}{n+2}+\frac{1/8}{n+3}+\frac{1/8}{n+4}\right)$$

$$\to\frac{11}{96}-\left(\frac{11}{96}-\frac{1/8}{n+1}-\frac{1/8}{n+2}+\frac{1/8}{n+3}+\frac{1/8}{n+4}\right)\quad\text{as } N\to\infty$$

$$=\frac{1}{8(n+1)}+\frac{1}{8(n+2)}-\frac{1}{8(n+3)}-\frac{1}{8(n+4)}$$

4. (a) Method I

$$\cos 3\theta = \cos(2\theta+\theta)$$
$$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$= (2\cos^2\theta-1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$
$$= (2\cos^2\theta-1)\cos\theta - 2\cos\theta(1-\cos^2\theta)$$
$$= 4\cos^3\theta - \cos\theta$$

Method II

$$\cos 3\theta = \mathrm{Re}(cis3\theta) = \mathrm{Re}[(cis\theta)^3]$$
$$= \cos^3\theta + 3\cos\theta\sin^2\theta$$
$$= \cos^3\theta + 3\cos\theta(1-\cos^2\theta)$$
$$= 4\cos^3\theta - \cos\theta$$

(b) $0 = x^3 - 3x + 1 = 8\cos^3\theta - 6\cos\theta + 1$

$$= 2(4\cos^3\theta - \cos\theta) + 1$$
$$= 2\cos 3\theta + 1$$

$$\cos 3\theta = -\frac{1}{2}.$$

$$3\theta = 2n\pi \pm \frac{2\pi}{3} \quad \text{for } n \in \mathbf{Z}$$

$$\theta = \frac{(6n\pm 2)\pi}{9} \quad \text{for } n \in \mathbf{Z}$$

$$x = 2\cos\theta$$
$$= 2\cos\frac{2\pi}{9},\quad 2\cos\frac{4\pi}{9},\quad 2\cos\frac{8\pi}{9}$$

5. (a) $\begin{pmatrix}\sqrt{3}k & k\\ -k & \sqrt{3}k\end{pmatrix} = P = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ where $0<\theta<\pi$

$$\begin{cases}-k = \sin\theta\\ \sqrt{3}k = \cos\theta\end{cases}, \quad \text{where } 0<\theta<\pi$$

$$1 = \sin^2\theta + \cos^2\theta = k^2 + 3k^2 = 4k^2$$

$k = \dfrac{-1}{2}$ only (since $-k = \sin\theta > 0$)

(b) (i) Put $\alpha = \dfrac{\pi}{3}$, $Q = \begin{pmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{-1}{2}\end{pmatrix}$

(ii) $PQ = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{pmatrix}$

$$= \begin{pmatrix}\cos\theta\cos\alpha - \sin\theta\sin\alpha & \cos\theta\sin\alpha + \sin\theta\cos\alpha\\ \cos\theta\sin\alpha + \sin\theta\cos\alpha & -(\cos\theta\cos\alpha - \sin\theta\sin\alpha)\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\theta+\alpha) & \sin(\theta+\alpha)\\ \sin(\theta+\alpha) & -\cos(\theta+\alpha)\end{pmatrix}$$

which represents a reflection (in the line $y = x\tan(\dfrac{\theta+\alpha}{2})$).

6. (a) (i) $\sum_{k=1}^{n+1}\frac{A-a_k}{A}=\sum_{k=1}^{n+1}(1-\frac{a_k}{A})=\sum_{k=1}^{n+1}1-\frac{1}{A}\sum_{k=1}^{n+1}a_k=(n+1)-\frac{1}{A}A=n$

(ii) $$(\sum_{k=1}^{n+1}\frac{A-a_k}{A})(\sum_{k=1}^{n+1}\frac{A}{A-a_k})\geq\left(\sum_{k=1}^{n+1}\sqrt{\frac{A-a_k}{A}}\cdot\sqrt{\frac{A}{A-a_k}}\right)^2$$

(note: $A>a_k>0$ for $k=1,2,\ldots,n+1$)

$$n\cdot(\sum_{k=1}^{n+1}\frac{A}{A-a_k})\geq\left(\sum_{k=1}^{n+1}1\right)^2=(n+1)^2$$

i.e. $\sum_{k=1}^{n+1}\frac{A}{A-a_k}\geq\frac{(n+1)^2}{n}$

(b) Put $a_k=2k$ for $k=1,2,\ldots,n+1$

such that $a_k>0$ and $A=\sum_{k=1}^{n+1}a_k=2[1+2+3+\ldots+(n+1)]=(n+1)(n+2)$.

By (a)(ii), $$\frac{(n+1)^2}{n}\leq\sum_{k=1}^{n+1}\frac{A}{A-a_k}$$
$$=A\sum_{k=1}^{n+1}\frac{1}{A-a_k}$$
$$=(n+1)(n+2)\sum_{k=1}^{n+1}\frac{1}{(n+1)(n+2)-2k}$$

i.e. $\sum_{k=1}^{n+1}\frac{1}{(n+1)(n+2)-2k}\geq\frac{n+1}{n(n+2)}$ as $n>0$.

7. (a) (i) coefficient determinant D

$$=\begin{vmatrix}0 & 1 & \lambda+1\\ \lambda & 2 & 2\\ 1 & -\lambda & -4\end{vmatrix}$$
$$=0-(-4\lambda-2)+(\lambda+1)(-\lambda^2-2)$$
$$=-(\lambda^3+\lambda^2-2\lambda)$$

(1) Homogenous system (S) has non-trivial solutions

$\Leftrightarrow$ $D=0$

$\Leftrightarrow$ $\lambda^3+\lambda^2-2\lambda=0$

(2) When $\lambda=1$, $\mu=0$

$$(S):\begin{cases}y+2z=0\\ x+2y+2z=0\\ x-y-4z=0\end{cases}.$$

Solving any 2 of the above equations,

$x=2z$, $y=-2z$

Solution set $=\{(2t,-2t,t): t\in\Re\}$

(b) (1) Non-homogenous system (S) has a unique solution

$\Leftrightarrow$ $0\neq D=-\lambda(\lambda^2+\lambda-2)=-\lambda(\lambda-1)(\lambda+2)$

$\Leftrightarrow$ $\lambda\neq 0$, 1 and -2

$\Leftrightarrow$ $\lambda<-2$ or $-2<\lambda<0$ or $0<\lambda<1$ or $\lambda>1$

(2) $$D_x=\begin{vmatrix}0 & 1 & \lambda+1\\ \mu & 2 & 2\\ \mu^2 & -\lambda & -4\end{vmatrix}=-\mu[\lambda^2+(2\mu+1)\lambda-4]$$

$$D_y=\begin{vmatrix}0 & 0 & \lambda+1\\ \lambda & \mu & 2\\ 1 & \mu^2 & -4\end{vmatrix}=\mu(\lambda+1)(\lambda\mu-1)$$

$$D_z = \begin{vmatrix} 0 & 1 & 0 \\ \lambda & 2 & \mu \\ 1 & -\lambda & \mu^2 \end{vmatrix} = -\mu(\lambda\mu - 1)$$

In case of unique solution,

$$x = \frac{D_x}{D} = \frac{\mu[\lambda^2 + (2\mu+1)\lambda - 4]}{\lambda(\lambda-1)(\lambda+2)}$$

$$y = \frac{D_y}{D} = -\frac{\mu(\lambda+1)(\lambda\mu-1)}{\lambda(\lambda-1)(\lambda+2)}$$

$$z = \frac{D_z}{D} = \frac{\mu(\lambda\mu-1)}{\lambda(\lambda-1)(\lambda+2)}$$

(3) Case (i) $\lambda = 0$: (S) becomes $\begin{cases} y + z = 0 \\ y + z = \dfrac{\mu}{2} \\ x - 4z = \mu^2 \end{cases}$

which has no solution as $\mu \neq 0$

Case (ii) $\lambda = 1$: (S) becomes $\begin{cases} y + 2z = 0 \\ x + 2y + 2z = \mu \\ x - y - 4z = \mu^2 \end{cases}$

First two equations give

$x = \mu + 2z$, $y = -2z$

For consistence,

$(\mu + 2z) - (-2z) - 4z = \mu^2$

$0 = \mu^2 - \mu = \mu(\mu - 1)$

i.e. $\mu = 1$ (and $\lambda = 1$)

Case (iii) $\lambda = -2$: (S) becomes $\begin{cases} y - z = 0 \\ -2x + 2y + 2z = \mu \\ x + 2y - 4z = \mu^2 \end{cases}$

First two equations give

$x = 2z - \dfrac{\mu}{2}$, $y = z$

For consistence,

$(2z - \dfrac{\mu}{2}) + 2(z) - 4z = \mu^2$

$0 = 2\mu^2 + \mu = \mu(2\mu + 1)$

i.e. $\mu = \dfrac{-1}{2}$ (and $\lambda = -2$)

(b) The given system equals to (S) by putting $\mu = 1$ and $\lambda = 1$

Solutions are $x = 1 + 2z$, $y = -2z$ (Case (iii) of (a)(ii)(3))

For consistence, $1 = 3x^2 + 2y^2 + z^2 = (1 + 2z)^2 + (-2z)^2 + z^2$

$= 9z^2 + 4z + 1$

$\therefore$ $z = 0$ or $\dfrac{-4}{9}$

i.e. $x = 1$, $y = 0$, $z = 0$

or $x = \dfrac{1}{9}$, $y = \dfrac{8}{9}$, $z = \dfrac{-4}{9}$

8. (a) (i) Suppose the converse that P is singular.

Then $0=|P|=\begin{vmatrix} a & -1 \\ b & 1 \end{vmatrix}=a+b$, $\quad a=-b$

But $0<ab=-b^2$ which leads to a contradiction.

(ii) $P^{-1}=\dfrac{1}{a+b}\begin{pmatrix} 1 & 1 \\ -b & a \end{pmatrix}$

$$P^{-1}AP=\frac{1}{a+b}\begin{pmatrix} 1 & 1 \\ -b & a \end{pmatrix}\begin{pmatrix} 4-b & a \\ b & 4-a \end{pmatrix}\begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}$$

$$=\frac{1}{a+b}\begin{pmatrix} 1 & 1 \\ -b & a \end{pmatrix}\begin{pmatrix} 4a & a+b-4 \\ 4b & -(a+b-4) \end{pmatrix}$$

$$=\frac{1}{a+b}\begin{pmatrix} 4(a+b) & 0 \\ 0 & -(a+b)(a+b-4) \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 0 \\ 0 & -(a+b-4) \end{pmatrix}$$

(iii) $P^{-1}AP=\begin{pmatrix} 4 & 0 \\ 0 & -(a+b-4) \end{pmatrix}$, $\quad (P^{-1}AP)^n=\begin{pmatrix} 4 & 0 \\ 0 & -(a+b-4) \end{pmatrix}^n$

$P^{-1}A^nP=\begin{pmatrix} 4^n & 0 \\ 0 & (4-a-b)^n \end{pmatrix}$, $\quad A^n=P\begin{pmatrix} 4^n & 0 \\ 0 & (4-a-b)^n \end{pmatrix}P^{-1}$

i.e. $d_1=4^n$, $\quad d_2=(4-a-b)^n$

(b) Put $a=4$, $b=1$ such that $ab=4>0$

By (a), $B^k=P\begin{pmatrix} 4^k & 0 \\ 0 & (4-4-1)^k \end{pmatrix}P^{-1}=P\begin{pmatrix} 4^k & 0 \\ 0 & (-1)^k \end{pmatrix}P^{-1}$

where $P=\begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}$, $\quad P^{-1}=\dfrac{1}{5}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$

Now, $\displaystyle\sum_{k=1}^{n}B^{2k-1}=\sum_{k=1}^{n}P\begin{pmatrix} 4^{2k-1} & 0 \\ 0 & (-1)^{2k-1} \end{pmatrix}P^{-1}$

$$=P\left(\sum_{k=1}^{n}\begin{pmatrix} 4^{2k-1} & 0 \\ 0 & -1 \end{pmatrix}\right)P^{-1}$$

$$=P\begin{pmatrix} \sum_{k=1}^{n}4^{2k-1} & 0 \\ 0 & \sum_{k=1}^{n}(-1) \end{pmatrix}P^{-1}$$

$$=P\begin{pmatrix} 4\cdot\dfrac{16^n-1}{16-1} & 0 \\ 0 & -n \end{pmatrix}P^{-1}$$

$$=\begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4\cdot\dfrac{16^n-1}{15} & 0 \\ 0 & -n \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix} \dfrac{16(16^n-1)}{15} & n \\ \dfrac{4(16^n-1)}{15} & -n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix} \dfrac{16(16^n-1)}{15}-n & \dfrac{16(16^n-1)}{15}+4n \\ \dfrac{4(16^n-1)}{15}+n & \dfrac{4(16^n-1)}{15}-4n \end{pmatrix}$$

$$=\frac{1}{75}\begin{pmatrix} 16(16^n-1)-15n & 16(16^n-1)+60n \\ 4(16^n-1)+15n & 4(16^n-1)-60n \end{pmatrix}$$

9. (a) (i) (1) Claim: $\alpha_n \ge \beta_n > 0$ for all $n \in \mathbf{Z}^+$.

Proof: $n = 1$ is trivial.

Assume $\alpha_k \ge \beta_k > 0$ for some integer $k \ge 1$

* $\beta_{k+1} = \sqrt{\alpha_k \beta_k} > 0 \quad (\because \alpha_k, \beta_k > 0)$

* $$\alpha_{k+1}{}^2 - \beta_{k+1}{}^2 = (\alpha_k{}^2 - \alpha_k\beta_k + \beta_k{}^2) - \alpha_k\beta_k = (\alpha_k - \beta_k)^2 \ge 0$$

$\therefore \quad \alpha_{k+1} \ge \beta_{k+1} \quad (\because \alpha_k, \beta_k > 0)$

i.e. $\alpha_{k+1} \ge \beta_{k+1} > 0$

By induction, $\alpha_n \ge \beta_n > 0$ for all $n \in \mathbf{Z}^+$.

(2) $$\alpha_{n+1}{}^2 - \alpha_n{}^2 = (\alpha_n{}^2 - \alpha_n\beta_n + \beta_n{}^2) - \alpha_n{}^2 = -\beta_n(\alpha_n - \beta_n) \le 0 \quad (\because \alpha_n \ge \beta_n > 0)$$

i.e. $\alpha_{n+1} \le \alpha_n$

(3) $$\beta_{n+1}{}^2 - \beta_n{}^2 = \alpha_n\beta_n - \beta_n{}^2 = \beta_n(\alpha_n - \beta_n) \ge 0 \quad (\because \alpha_n \ge \beta_n > 0)$$

i.e. $\beta_{n+1} \ge \beta_n$

(ii) Now, $\beta_1 \le \beta_2 \le \ldots \le \beta_n \le \alpha_n \le \ldots \le \alpha_2 \le \alpha_1$

$\therefore$ $\{\alpha_n\}$ is decreasing and bounded below by β_1

$\{\beta_n\}$ is increasing and bounded above by α_1

$\therefore$ $\lim_{n\to\infty} \alpha_n = A$, $\lim_{n\to\infty} \beta_n = B$ both exist.

As $\beta_{n+1} = \sqrt{\alpha_n\beta_n}$, $\lim_{n\to\infty} \beta_{n+1} = \lim_{n\to\infty} \sqrt{\alpha_n\beta_n}$

$B^2 = AB$, $\quad B(A - B) = 0$

$A = B$ only $\quad$ (since $B = \lim_{n\to\infty} \beta_n \ge \lim_{n\to\infty} \beta_1 = \beta_1 > 0$)

(iii) $$\alpha_n{}^2 + \beta_n{}^2 = (\alpha_{n-1}{}^2 - \alpha_{n-1}\beta_{n-1} + \beta_{n-1}{}^2) - \alpha_{n-1}\beta_{n-1}$$
$$= \alpha_{n-1}{}^2 + \beta_{n-1}{}^2$$
$$= \alpha_{n-2}{}^2 + \beta_{n-2}{}^2$$
$$= \ldots = \alpha_1{}^2 + \beta_1{}^2$$

Take limit on both sides,

$$\alpha_1{}^2 + \beta_1{}^2 = \lim_{n\to\infty}(\alpha_n{}^2 + \beta_n{}^2)$$
$$= A^2 + B^2 = 2A^2$$

i.e. $\lim_{n\to\infty} \alpha_n = A = \sqrt{\dfrac{\alpha_1{}^2 + \beta_1{}^2}{2}}$ $\quad$ ($\alpha_n > 0$ gives $\lim_{n\to\infty} \alpha_n \ge 0$)

(b) Define $\alpha_n = \dfrac{1}{y_n}$ and $\beta_n = \dfrac{1}{x_n}$ for all positive integers n.

Then, (1) $\alpha_1 = \dfrac{1}{y_1} > \dfrac{1}{x_1} = \beta_1 > 0$ (as $x_1 > y_1 > 0$)

(2) For $n \ge 1$, $$\alpha_{n+1} = \frac{1}{y_{n+1}} = \frac{\sqrt{x_n{}^2 - x_n y_n + y_n{}^2}}{x_n y_n} = \sqrt{\frac{1}{y_n{}^2} - \frac{1}{x_n y_n} + \frac{1}{x_n{}^2}} = \sqrt{\alpha_n{}^2 - \alpha_n\beta_n + \beta_n{}^2}$$

$$\beta_{n+1} = \frac{1}{x_{n+1}} = \frac{1}{\sqrt{x_n y_n}} = \sqrt{\alpha_n\beta_n}$$

By (a), $\lim_{n\to\infty} \alpha_n$, $\lim_{n\to\infty} \beta_n$ both exist.

i.e. $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ both exist.

10. (a) Define $h(x) = p(x) - p(0)$.

Claim: $h(n) = 0$ for all positive integers n.

Proof: When $n = 1$, $h(1) = p(1) - p(0) = 0$ (given)

Assume $h(k) = 0$ for some positive integers k.

Consider $n = k + 1$,

$$h(k+1) = p(k+1) - p(0)$$
$$= p(k) - p(0)$$
$$= h(k) = 0 \quad \text{(by assumption)}$$

$\therefore$ The statement is also true for $n = k + 1$

i.e. By induction, $h(n) = 0$ for all positive integers n.

Furthermore, $h(x) = p(x) - p(0)$ is a polynomial with finite degree,

and $h(x) = 0$ has infinity many distinct roots.

$\therefore$ $h(x) \equiv 0$ i.e. $p(x) \equiv p(0)$

(b) (i) Put $x = k + 1$, $(k+1-\pi) f(k+1) = (k+1) f(k)$

$$= 0 \quad (\because f(k) = 0)$$

$\therefore$ $f(k+1) = 0$ $(\because k + 1 - \pi \neq 0$ for integer $k)$

(ii) Claim: $f(n) = 0$ for all integers $n \geq 0$.

Proof: Put $x = 0$, $-\pi f(0) = 0 \cdot f(-1) = 0$,

$\therefore f(0) = 0$

Assume $f(k) = 0$ for some integers $k \geq 0$.

By (b)(i), $f(k+1) = 0$

$\therefore$ The statement is also true for $n = k + 1$

i.e. By induction, $f(n) = 0$ for all integers $n \geq 0$.

Therefore, $f(n) = f(n-1)$ for all positive integers n.

i.e. $f(x) = 0$ for all $x \in \Re$ (by(a))

(c) (i) Put $x = 0$, $-3 \cdot g(0) = 0 \cdot g(-1)$, $\therefore g(0) = 0$

Put $x = 1$, $-2 \cdot g(1) = 1 \cdot g(0) = 0$, $\therefore g(1) = 0$

Put $x = 2$, $-1 \cdot g(2) = 2 \cdot g(1) = 0$, $\therefore g(2) = 0$

(ii) Let $g(x) = Q(x) \cdot x(x-1)(x-2)$ for some polynomial $Q(x)$.

$\because$ $xg(x-1) = (x-3)g(x)$

$\therefore$ $Q(x-1) \cdot x(x-1)(x-2)(x-3) = Q(x) \cdot x(x-1)(x-2)(x-3)$

$\therefore$ $Q(n) = Q(n-1)$ for integers $n \geq 4$

$\therefore$ $Q(x) = Q(3)$ for all $x \in \Re$ (by (a))

i.e. $g(x) = Q(3) \cdot x(x-1)(x-2) = Cx(x-1)(x-2)$

where $C = Q(3)$ is a constant.

11. (a) $0 = \text{Im}(z^2 + \bar{z}) = \text{Im}(cis2\theta + cis(-\theta)) = \sin 2\theta - \sin\theta = \sin\theta(2\cos\theta - 1)$

$\therefore \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = n\pi \quad \text{or} \quad 2n\pi \pm \frac{\pi}{3} \quad (\text{where } n \in \mathbf{Z})$

$\therefore \quad z = cis0,\ cis\pi,\ cis\frac{\pi}{3} \ \text{or}\ cis\frac{-\pi}{3}$

$= 0,\ -1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i \ \text{or}\ \frac{1}{2} - \frac{\sqrt{3}}{2}i$

(b) (i) $z_1 = \frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \omega = \frac{z_2}{z_1} = \frac{cis\frac{\pi}{3}}{cis(\frac{-\pi}{3})} = cis\frac{2\pi}{3}$

$\omega^3 = (cis\frac{2\pi}{3})^3 = cis2\pi = 1$

(ii) Let $n = 3k$ where k is a positive integer.

$S_n = S_{3k} = \sum_{r=1}^{3k} \omega^r = \omega \cdot \frac{1 - \omega^{3k}}{1 - \omega} \qquad (\text{note } \omega \neq 1)$

$= \omega \cdot \frac{1 - (1)^k}{1 - \omega} = 0$

(iii) For $n = 3k + 1$ where k is a positive integer.

$S_n = S_{3k+1} = \sum_{r=1}^{3k+1} \omega^r = \sum_{r=1}^{3k} \omega^r + \omega^{3k+1} = 0 + \omega = \omega$

For $n = 3k + 1$ where k is a positive integer.

$S_n = S_{3k+2} = \sum_{r=1}^{3k+2} \omega^r = \sum_{r=1}^{3k} \omega^r + \omega^{3k+1} + \omega^{3k+2}$

$= 0 + \omega + \omega^2 = (1 + \omega + \omega^2) - 1 = \frac{1 - \omega^3}{1 - \omega} - 1 = -1$

(iv) $2 = (S_{2009} + S_{2010} + S_{2011})^m = (S_{3\times 669+2} + S_{3\times 670} + S_{3\times 670+1})^m$

$= (-1 + 0 + \omega)^m = (\omega - 1)^m$

Consider $\quad 2 = |(\omega - 1)^m| = |\omega - 1|^m = |\frac{-3}{2} + \frac{\sqrt{3}}{2}i|^m$

$= (\sqrt{\frac{9}{4} + \frac{3}{4}})^m = 3^{\frac{m}{2}}$

Which has no solution for integer m.

(v) $\{ S_n, S_{n+1}, S_{n+2} \} = \{0,\ -1,\ \omega\}$

$\therefore \quad 2 = S_n{}^k + S_{n+1}{}^k + S_{n+2}{}^k$

$2 = 0^k + (-1)^k + \omega^k$

For k is even, $\quad 2 = 0^k + (-1)^k + \omega^k = 1 + \omega^k$

$\omega^k = 1$, $\quad k$ should be a multiple of 3.

i.e. $\quad k = 6N$ where N is a positive integer

For k is odd, $\quad 2 = 0^k + (-1)^k + \omega^k = -1 + \omega^k$

$\omega^k = 3$, $\quad k$ has no solution

since $|\omega^k| = |\omega|^k = 1 \neq 3$

i.e. $\quad k = 6N$ where N is a positive integer

--- End of Solutions of Paper I ---

Paper II

1. (a) $f(0)=\lim_{x\to 0^-} f(x)$ $\quad\therefore\quad$ $4\sin 0+5\cos 0=\lim_{x\to 0^-}(k-e^{3x})$

i.e. $k=6$

(b) $\lim_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\dfrac{(6-e^{3x})-5}{x}=\lim_{x\to 0^-}\dfrac{1-e^{3x}}{x}$ $\quad(\frac{0}{0})$

$=\lim_{x\to 0^-}\dfrac{-3e^{3x}}{1}=-3$

$\lim_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\dfrac{(4\sin x+5\cos x)-5}{x}$ $\quad(\frac{0}{0})$

$=\lim_{x\to 0^+}\dfrac{4\cos x-5\sin x}{1}=4$

As $\lim_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}\neq\lim_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}$

$f(x)$ is not differentiable at $x=0$

(c) For $x>0$, the function has no asymptote.

For $x<0$, obvious there is no vertical asymptote.

Let $y=mx+c$ be the non-vertical asymptote.

$m=\lim_{x\to -\infty}\dfrac{f(x)}{x}=\lim_{x\to -\infty}\dfrac{6-e^{3x}}{x}$ $\quad(\frac{\infty}{\infty})$

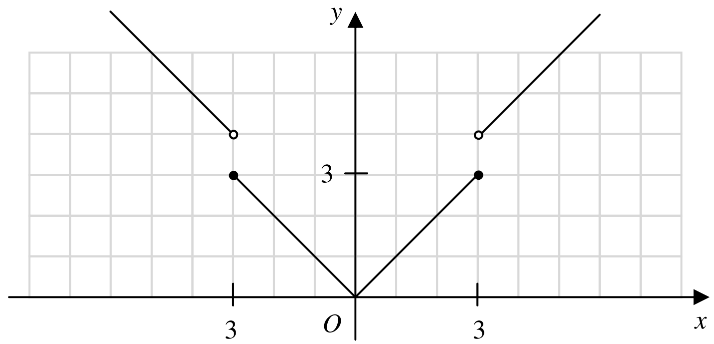
$=\lim_{x\to -\infty}\dfrac{-3e^{3x}}{1}=0$

$c=\lim_{x\to -\infty}[f(x)-mx]=\lim_{x\to -\infty}(6-e^{3x})=6$

i.e. the only asymptote is $y=6$.

2. (a) $f(-5)=f(5)=5+1=6$

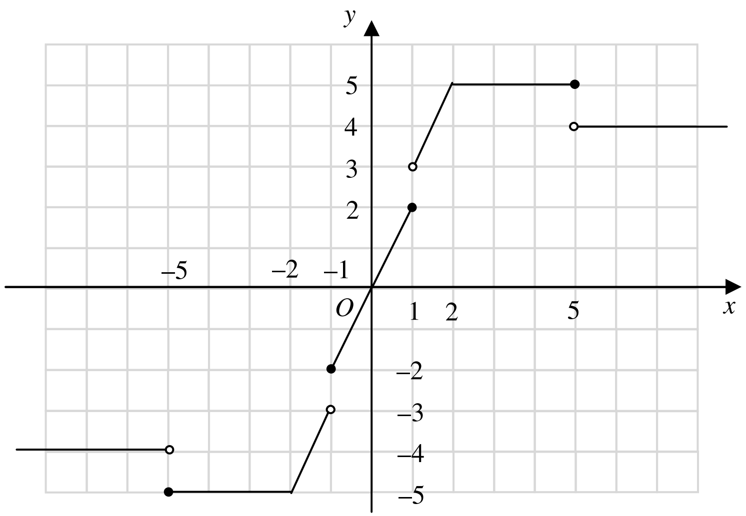
(b)



(c) (i) $g(-x)=f(-x+2)-f(-x-2)=f(-(x-2))-f(-(x+2))$

$=f(x-2)-f(x+2)=-g(x)$

(ii)



3. (a) $\ln f(x) = -\frac{1}{2}\ln(x^2+2x+5)$

$\therefore \quad \frac{f'(x)}{f(x)} = -\frac{1}{2}\times\frac{2x+2}{x^2+2x+5} = -\frac{x+1}{x^2+2x+5}$

i.e. $(x^2+2x+5)f'(x)+(x+1)f(x)=0$

<u>Method I</u>

$$0 = 0^{(n)} = [(x^2+2x+5)f'(x)]^{(n)} + [(x+1)f(x)]^{(n)}$$
$$= (x^2+2x+5)f^{(n+1)}(x) + {}_nC_1(2x+2)f^{(n)}(x) + {}_nC_2(2)f^{(n-1)}(x)$$
$$+ (x+1)f^{(n)}(x) + {}_nC_1(1)f^{(n-1)}(x)$$
$$= (x^2+2x+5)f^{(n+1)}(x) + (2n+1)(x+1)f^{(n)}(x) + n^2 f^{(n-1)}(x)$$

<u>Method II</u>

Show by induction.

(b) Put $x=0$, $f^{(n+1)}(-1) = -\frac{n^2}{4}f^{(n-1)}(-1)$ for $n \ge 1$.

Further, $f^{(0)}(-1) = f(-1) = \frac{1}{\sqrt{(-1)^2+2(-1)+5}} = \frac{1}{2}$ and

$[(-1)^2+2(-1)+5]f'(-1) + 0\cdot f(-1) = 0$

$\therefore \quad f'(-1) = 0$

Then, $f^{(6)}(-1) = -\frac{5^2}{4}f^{(4)}(-1)$

$$= (-1)^2\frac{5^2}{4}\times\frac{3^2}{4}f^{(2)}(-1)$$

$$= (-1)^3\frac{5^2}{4}\times\frac{3^2}{4}\times\frac{1^2}{4}f^{(0)}(-1)$$

$$= (-1)^3\frac{5^2}{4}\times\frac{3^2}{4}\times\frac{1^2}{4}\times\frac{1}{2} = -\frac{225}{128}$$

$$f^{(7)}(-1) = -\frac{6^2}{4}f^{(5)}(-1)$$

$$= (-1)^2\frac{6^2}{4}\times\frac{4^2}{4}f^{(3)}(-1)$$

$$= (-1)^3\frac{6^2}{4}\times\frac{4^2}{4}\times\frac{2^2}{4}f^{(1)}(-1)$$

$$= 0$$

4. (a) $\int x^2\sqrt{9-x^3}\,dx = \frac{1}{3}\int\sqrt{9-x^3}\,dx^3$

$$= \frac{-1}{3}\int\sqrt{9-x^3}\,d(9-x^3)$$

$$= \frac{-2}{9}(9-x^3)^{\frac{3}{2}} + C$$

(b) $\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{2n}k^2\sqrt{9-\frac{k^3}{n^3}}$

$$= \lim_{n\to\infty}\frac{2-0}{2n}\sum_{k=1}^{2n}[0+\frac{k}{2n}\times(2-0)]^2\sqrt{9-[0+\frac{k}{2n}\times(2-0)]^3}$$

$$= \int_0^2 x^2\sqrt{9-x^3}\,dx$$

$$= \left[\frac{-2}{9}(9-x^3)^{\frac{3}{2}}\right]_0^2 = \frac{52}{9}$$

5. (a) $x = 5 + 2\sin\theta$, $dx = 2\cos\theta d\theta$

$$\int \sqrt{(3-x)(x-7)}dx = \int \sqrt{-x^2+10x-21}dx$$

$$= \int \sqrt{2^2-(x-5)^2}dx$$

$$= \int 2\cos\theta(2\cos\theta d\theta)$$

$$= 2\int (1+\cos 2\theta)d\theta$$

$$= 2\theta + \sin 2\theta + C$$

$$= 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\frac{x-5}{2} + 2\times\frac{x-5}{2}\times\frac{\sqrt{2^2-(x-5)^2}}{2} + C$$

$$= 2\sin^{-1}\frac{x-5}{2} + \frac{(x-5)\sqrt{(x-3)(7-x)}}{2} + C$$

(b) Volume

$$= \pi\int_3^7 y^2 dx$$

$$= \pi\int_3^7 \sqrt{(3-x)(x-7)}dx$$

$$= \pi\left[2\sin^{-1}\frac{x-5}{2} + \frac{(x-5)\sqrt{(x-3)(7-x)}}{2}\right]_3^7$$

$$= 2\pi^2$$

6. (a) Put $x = 6\sec\theta$, $y = 12\tan\theta$

$$4x^2 - y^2 = 4(36\sec^2\theta) - 144\tan^2\theta = 144(\sec^2\theta - \tan^2\theta) = 144$$

$\therefore$ P lies on H.

(b) (i) $8x - 2y\dfrac{dy}{dx} = 0$, $\left.\dfrac{dy}{dx}\right|_P = \dfrac{4(6\sec\theta)}{12\tan\theta} = \dfrac{2}{\sin\theta}$ (as $0 < \theta < \dfrac{\pi}{2}$)

L: $y - 12\tan\theta = \dfrac{-\sin\theta}{2}(x - 6\sec\theta)$

Put $y = 0$, $x = 30\sec\theta$; put $x = 0$, $y = 15\tan\theta$

i.e. x-intercept $= 30\sec\theta$, y-intercept $= 15\tan\theta$

(ii) $150 = \text{Area} = \dfrac{1}{2}\left|30\sec\theta\times 15\tan\theta\right| = 15^2\sec\theta\tan\theta$ (as $0 < \theta < \dfrac{\pi}{2}$)

$$\frac{2}{3} = \sec\theta\tan\theta = \frac{\sin\theta}{\cos^2\theta}$$

$$0 = 2(1-\sin^2\theta) - 3\sin\theta = -(2\sin\theta - 1)(\sin\theta + 2)$$

$\sin\theta = \dfrac{1}{2}$, $\theta = \dfrac{\pi}{6}$

$\therefore$ $P = (6\sec\theta, 12\tan\theta) = (4\sqrt{3}, 4\sqrt{3})$

7. (a) $f'(x)=\dfrac{2(x-1)}{x^2-2x+10}$, $f''(x)=\dfrac{-2(x-4)(x+2)}{(x^2-2x+10)^2}$

(b) (i) $x\in\Re$

(ii) $x>1$

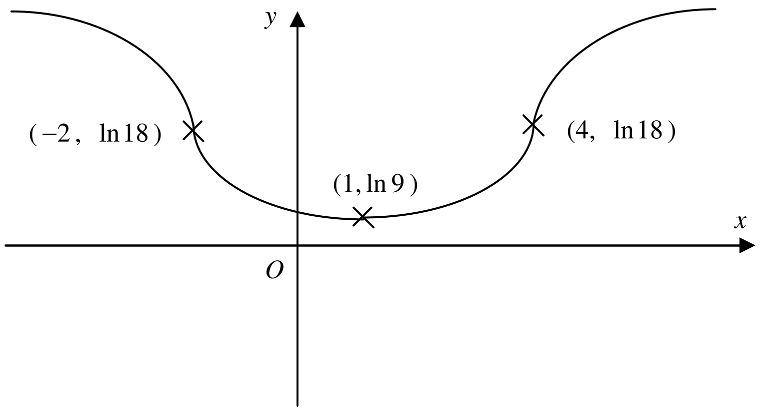
(iii) $-2<x<4$

(c) min pt. = $(1, f(1)) = (1, \ln 9)$

Point of inflexion = $(-2, f(-2))$ and $(4, f(4))$

= $(-2, \ln 18)$ and $(4, \ln 18)$

(d)



(e) $f(x)=\ln 18$ gives $x=-2$ or 4

Area = $6\times\ln 18-\int_{-2}^{4} f(x)dx = 6\ln 18-\int_{-2}^{4}\ln(x^2-2x+10)dx$

$=6\ln 18-\left[x\ln(x^2-2x+10)\right]_{-2}^{4}+\int_{-2}^{4}\dfrac{2(x^2-x)}{x^2-2x+10}dx$

$=2\int_{-2}^{4}\dfrac{(x^2-2x+10)+(x-10)}{x^2-2x+10}dx$

$=12+\int_{-2}^{4}\dfrac{2x-20}{x^2-2x+10}dx$

$=12+\int_{-2}^{4}\left(\dfrac{2x-2}{x^2-2x+10}-\dfrac{18}{x^2-2x+10}\right)dx$

$=12+\left[\ln(x^2-2x+10)\right]_{-2}^{4}-18\int_{-2}^{4}\dfrac{1}{(x-1)^2+3^2}dx$

$=12-18\times\left[\dfrac{1}{3}\tan^{-1}\dfrac{x-1}{3}\right]_{-2}^{4}$

$=12-3\pi$

8. (a) (i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2}+\sin^2\frac{x}{2}}dx$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{(\cos\frac{x}{2}+\sin\frac{x}{2})^2}dx = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{(1+\tan\frac{x}{2})^2}dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{(1+\tan\frac{x}{2})^2}d(1+\tan\frac{x}{2})$$

$$= 2\left[\frac{-1}{1+\tan\frac{x}{2}}\right]_0^{\frac{\pi}{2}} = 1$$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}dx = \int_0^{\frac{\pi}{2}} (1-\frac{1}{1+\sin x})dx = \frac{\pi}{2}-1$

(b) $\int_0^{\pi} f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)dx$

$$= \int_0^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{0} f(\pi-u)d(-u) \qquad \text{(put } u=\pi-x\text{)}$$

$$= \int_0^{\frac{\pi}{2}} f(x)dx + \int_0^{\frac{\pi}{2}} f(\pi-u)du$$

$$= 2\int_0^{\frac{\pi}{2}} f(x)dx \qquad (f(\pi-x)=f(x))$$

(c) $\int_0^{\pi} g(x)\ln(1+e^{\cos x})dx = \int_{\pi}^{0} g(\pi-u)\ln(1+e^{\cos(\pi-u)})d(-u)$

$$= \int_0^{\pi}(-g(x))\ln(1+\frac{1}{e^{\cos x}})dx$$

$$= \int_0^{\pi} g(x)[\ln e^{\cos x} - \ln(1+e^{\cos x})]dx$$

$$= \int_0^{\pi} g(x)\cos x dx - \int_0^{\pi} g(x)\ln(1+e^{\cos x})dx$$

$\therefore$ $2\int_0^{\pi} g(x)\ln(1+e^{\cos x})dx = \int_0^{\pi} g(x)\cos x dx$

i.e. $\int_0^{\pi} g(x)\ln(1+e^{\cos x})dx = \frac{1}{2}\int_0^{\pi} g(x)\cos x dx$

(d) $\int_0^{\pi} \frac{\cos x}{(1+\sin x)^2}\ln(1+e^{\cos x})dx$

$= \frac{1}{2}\int_0^{\pi} \frac{\cos x}{(1+\sin x)^2}\cos x dx$ (by (c), setting $g(x) = \frac{\cos x}{(1+\sin x)^2}$ such that $g(\pi-x) = -g(x)$)

$= \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{(1+\sin x)^2}dx$ (by (b), setting $f(x) = \frac{\cos^2 x}{(1+\sin x)^2}$ such that $f(\pi-x) = f(x)$)

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)^2}d(1+\sin x)$$

$$= -\int_0^{\frac{\pi}{2}} \cos x d \frac{1}{1+\sin x}$$

$$= -\left[\frac{\cos x}{1+\sin x}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x} dx = 1 - \frac{\pi}{2}$$

9. (a) (i) (1) $\frac{dF(x)}{dx} = \left(xf(x)g(x) + \int_0^x f(t)g(t)dt\right)$

$$-\left(g(x)\int_0^x f(t)dt + f(x)\int_0^x g(t)dt\right)$$

$$= \left(\int_0^x f(x)g(x)dt + \int_0^x f(t)g(t)dt\right)$$

$$-\left(\int_0^x f(t)g(x)dt + \int_0^x f(x)g(t)dt\right)$$

$$= \int_0^x [(f(x)g(x) - f(x)g(t)) + (f(t)g(t) - f(t)g(x))]dt$$

$$= \int_0^x [f(x)(g(x) - g(t)) - f(t)(g(x) - g(t))]dt$$

$$= \int_0^x (f(t) - f(x))(g(t) - g(x))dt$$

(2) Claim: $(f(t) - f(x))(g(t) - g(x)) \geq 0$

for $x \neq 0$ and t lies between 0 and x.

Proof: (I) For $0 < t < x$,

$f(t) \leq f(x)$ and $g(t) \leq g(x)$

($\because$ both are increasing)

$\therefore$ $(f(t) - f(x))(g(t) - g(x)) \geq 0$

(II) For $x < t < 0$,

$f(x) \leq f(t)$ and $g(x) \leq g(t)$

($\because$ both are increasing)

$\therefore$ $(f(t) - f(x))(g(t) - g(x)) \geq 0$

For $x > 0$, $\frac{dF(x)}{dx} = \int_0^x (f(t) - f(x))(g(t) - g(x))dt \geq 0$

For $x = 0$, $\frac{dF(x)}{dx} = \int_0^x (f(t) - f(x))(g(t) - g(x))dt = 0$

For $x < 0$, $\frac{dF(x)}{dx} = \int_0^x (f(t) - f(x))(g(t) - g(x))dt \leq 0$

Least of $F(x)$

$= F(0) = 0$

(ii) $F(1) \geq F(0) = 0$, i.e. $\left(\int_0^1 f(t)dt\right)\left(\int_0^1 g(t)dt\right) \leq \int_0^1 f(t)g(t)dt$

i.e. $\left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right) \leq \int_0^1 f(x)g(x)dx$

(iii) $n=1$, the statement is trivial.

Assume $\left(\int_0^1 f(x)dx\right)^k \le \int_0^1 (f(x))^k\,dx$ for some positive integer k.

$$\left(\int_0^1 f(x)dx\right)^{k+1} = \left(\int_0^1 f(x)dx\right)^k \left(\int_0^1 f(x)dx\right)$$

$$\le \left(\int_0^1 (f(x))^k\,dx\right)\left(\int_0^1 f(x)dx\right)$$

(by assumption and noting that both integrals are positive as $f(x)>0$)

$$\le \int_0^1 (f(x))^k \cdot f(x)dx$$ (by (a)(ii), $g(x)=(f(x))^k$ such that $g'(x)=k(f(x))^{k-1}f'(x)\ge 0$)

$$= \int_0^1 (f(x))^{k+1}dx$$

$\therefore$ By induction, the statement is true for all positive integers n.

(b) $$\left(\int_{-2}^3 h(x)dx\right)^{2011} = \left(\int_0^1 h(5u-2)d(5u)\right)^{2011}$$ (put $x=5u-2$)

$$= 5^{2011}\left(\int_0^1 h(5u-2)du\right)^{2011}$$

$$\le 5^{2011}\int_0^1 (h(5u-2))^{2011}du$$ (by (a) and noting that $h(x)$ is increasing continuous function and $h(x)>0$)

$$= 5^{2010}\int_0^1 (h(x))^{2011}dx$$ (put $x=5u-2$)

10. (a) Note that $b-u=b-(sa+tb)=(1-t)b-sa=sb-sa=s(b-a)>0$

and $u-a=(sa+tb)-a=tb-(1-s)a=tb-ta=t(b-a)>0$

(i) $$\frac{f(b)-f(u)}{s(b-a)}=\frac{f(b)-f(u)}{b-u}=f'(\xi_1) \quad \text{where } u<\xi_1<b$$

$$\le f'(u) \quad (\because f'(x) \text{ is decreasing})$$

$$\frac{f(u)-f(a)}{t(b-a)}=\frac{f(u)-f(a)}{u-a}=f'(\xi_2) \quad \text{where } a<\xi_2<u$$

$$\ge f'(u) \quad (\because f'(x) \text{ is decreasing})$$

i.e. $$\frac{f(b)-f(u)}{s(b-a)}\le f'(u)\le\frac{f(u)-f(a)}{t(b-a)}$$

(ii) $$\frac{f(b)-f(u)}{s(b-a)}\le\frac{f(u)-f(a)}{t(b-a)}$$ (by (a)(i))

$t[f(b)-f(u)]\le s[f(u)-f(a)]$ ($\because s,t,b-a>0$)

$s\,f(a)+t\,f(b)\le(s+t)f(u)=f(u)=f(sa+tb)$

(b) Case (1) $h=k$:

LS= $h^pk^q=h^{p+q}=h=(p+q)h=ph+qk$ =RS

Case (2) Without loss of generality, we assume $h<k$:

Set $f(x)=\ln x$ for all $x\in\Re$ such that $f(x)$ is twice differentiable and

$$f''(x)=\frac{-1}{x^2}\le 0 \text{ for all } x\in\Re$$

By (a)(ii), $p\,f(h)+q\,f(k)\le f(ph+qk)$

$p\ln h+q\ln k\le\ln(ph+qk)$

$\ln h^pk^q\le\ln(ph+qk)$

i.e. $h^pk^q\le ph+qk$ ($y=e^x$ is increasing)

(c) $n=1$, LS= $x_1^{\lambda_1} = x_1 = \lambda_1 x_1$ as $\lambda_1 = 1$.

Assume the statement is true for some positive integer *k*.

Consider $n = k+1$,

$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_{k-1}^{\lambda_{k-1}} x_k^{\lambda_k} x_{k+1}^{\lambda_{k+1}}$ (where $\lambda_1 + \lambda_2 + \dots + \lambda_{k+1} = 1$)

$$= x_1^{\lambda_1} x_2^{\lambda_2} \dots x_{k-1}^{\lambda_{k-1}} (x_k^{\frac{\lambda_k}{\lambda_k+\lambda_{k+1}}} x_{k+1}^{\frac{\lambda_{k+1}}{\lambda_k+\lambda_{k+1}}})^{\lambda_k+\lambda_{k+1}}$$

$$\le \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{k-1} x_{k-1} + (\lambda_k + \lambda_{k+1})(x_k^{\frac{\lambda_k}{\lambda_k+\lambda_{k+1}}} x_{k+1}^{\frac{\lambda_{k+1}}{\lambda_k+\lambda_{k+1}}})$$

(by assumption and notice that $\lambda_1 + \lambda_2 + \dots + \lambda_{k-1} + (\lambda_k + \lambda_{k+1}) = 1$)

$$\le \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{k-1} x_{k-1} + (\lambda_k + \lambda_{k+1})(\frac{\lambda_k}{\lambda_k + \lambda_{k+1}} x_k + \frac{\lambda_{k+1}}{\lambda_k + \lambda_{k+1}} x_{k+1})$$

(by (b) and notice that $\frac{\lambda_k}{\lambda_k + \lambda_{k+1}} + \frac{\lambda_{k+1}}{\lambda_k + \lambda_{k+1}} = 1$)

$$= \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{k-1} x_{k-1} + \lambda_k x_k + \lambda_{k+1} x_{k+1}$$

$\therefore$ The statement is also true for $n = k+1$

i.e. By induction, the statement is true for all positive integers *n*.

11. (a) $P_1: y^2 = 4x$, $2y\frac{dy}{dx} = 4$, $\left.\frac{dy}{dx}\right|_{S(s^2,2s)} = \frac{2}{2s} = \frac{1}{s}$ (note $s \ne 0$)

L: $x - sy = (s^2) - s(2s) = -s^2$, i.e. $x - sy + s^2 = 0$

(b) (i) $\begin{cases} L: \ x = sy - s^2 \\ P_2: y^2 = 8x \end{cases}$

$y^2 - (8s)y + 8s^2 = 0$ has roots 4α and 4β.

$\therefore$ $\alpha + \beta = \frac{1}{4}(4\alpha + 4\beta) = \frac{1}{4}(\text{sum of roots}) = \frac{1}{4}(8s) = 2s$

$\alpha\beta = \frac{1}{16}(4\alpha \cdot 4\beta) = \frac{1}{16}(\text{product of roots})$ $\frac{1}{16}(8s^2) = \frac{s^2}{2}$

(ii) Note: $\alpha\beta = \frac{s^2}{2} > 0$, $\therefore$ $\alpha, \beta \ne 0$

Let m_k be the slope of the line L_k.

$P_2: y^2 = 8x$, $y\frac{dy}{dx} = 4$

$m_1 = \left.\frac{dy}{dx}\right|_A = \frac{4}{4\alpha} = \frac{1}{\alpha}$, $m_2 = \left.\frac{dy}{dx}\right|_B = \frac{4}{4\beta} = \frac{1}{\beta}$

(1) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}\right|$

$$= \left|\frac{\sqrt{(\frac{\alpha+\beta}{\alpha\beta})^2 - \frac{4}{\alpha\beta}}}{1 + \frac{1}{\alpha\beta}}\right| = \left|\frac{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}{\alpha\beta + 1}\right|$$

$$= \left| \frac{\sqrt{4s^2 - 2s^2}}{\frac{s^2}{2} + 1} \right| = \frac{2\sqrt{2}s}{s^2 + 2} \qquad (\text{as } s > 0)$$

(2) $\tan\theta = \dfrac{\sqrt{2}s}{\frac{s^2+2}{2}} \le \dfrac{\sqrt{2}s}{\sqrt{s^2 \cdot 2}}$ (by A.M. ≥ G.M.)

$= 1$

$\therefore$ greatest value of θ is $\dfrac{\pi}{4}$

(iii) (1) $L_1: \ x - \alpha y = 2\alpha^2 - \alpha(4\alpha) = -2\alpha^2$

$L_2: \ x - \beta y = -2\beta^2$

$$\therefore \quad x = \frac{\begin{vmatrix} -2\alpha^2 & -\alpha \\ -2\beta^2 & -\beta \end{vmatrix}}{\begin{vmatrix} 1 & -\alpha \\ 1 & -\beta \end{vmatrix}} = \frac{2\alpha\beta(\alpha - \beta)}{\alpha - \beta} = 2\alpha\beta = s^2$$

$$y = \frac{\begin{vmatrix} 1 & -2\alpha^2 \\ 1 & -2\beta^2 \end{vmatrix}}{\begin{vmatrix} 1 & -\alpha \\ 1 & -\beta \end{vmatrix}} = \frac{2(\alpha^2 - \beta^2)}{\alpha - \beta} = 2(\alpha + \beta) = 4s$$

i.e. $C = (s^2, 4s)$

(2) Let $C = (x, y)$

Then, $\begin{cases} x = s^2 \\ y = 4s \end{cases}$. Eliminating s, we have $y^2 = 16x$

--- End of Solutions of Paper II ---