2011-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2011

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(C) answer book.

4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Let $a_1 = 8$, $a_2 = 66$ and $a_{n+2} - 8a_{n+1} - a_n = 0$ for all positive integers n . Using mathematical induction, prove that $a_n = (4+\sqrt{17})^n + (4-\sqrt{17})^n$ for any positive integer n .

(6 marks)

2. Let n be a positive integer. Denote the coefficient of x^k in the expansion of $(3+x)^n$ by a_k . Find

(a) $\displaystyle\sum_{k=0}^{n} a_k$,

(b) $\displaystyle\sum_{k=0}^{n} \frac{a_k}{k+1}$,

(c) $\displaystyle\sum_{k=1}^{n} \frac{k a_k}{k+1}$.

(7 marks)

3. (a) Resolve $\dfrac{1}{x(x+2)(x+4)}$ into partial fractions.

(b) Let n be a positive integer.

(i) Express $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+2)(k+4)}$ in the form $A + \dfrac{B}{n+1} + \dfrac{C}{n+2} + \dfrac{D}{n+3} + \dfrac{E}{n+4}$, where A , B , C , D and E are constants.

(ii) Find $\displaystyle\sum_{k=n+1}^{\infty} \frac{1}{k(k+2)(k+4)}$.

(7 marks)

4. (a) Express $\cos 3\theta$ in terms of $\cos\theta$.

(b) Using the substitution $x = 2\cos\theta$, solve the equation $x^3 - 3x + 1 = 0$.

(6 marks)

5. Suppose that the matrix $P = \begin{pmatrix} \sqrt{3}k & k \\ -k & \sqrt{3}k \end{pmatrix}$ represents the anticlockwise rotation about the origin by an angle θ in the Cartesian plane, where $0 < \theta < \pi$.

(a) Find k .

(b) It is known that the matrix $\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ represents the reflection in the straight line $y = \left(\tan\dfrac{\alpha}{2} \right) x$. Let Q be the matrix representing the reflection in the straight line $y = \left(\tan\dfrac{\pi}{6} \right) x$.

(i) Write down the matrix Q .

(ii) Does the matrix PQ represent a reflection? Explain your answer.

(7 marks)

6. Let n be a positive integer.

(a) Let $a_1, a_2, \ldots, a_{n+1}$ be positive real numbers. Define $A = \sum_{k=1}^{n+1} a_k$.

(i) Prove that $\sum_{k=1}^{n+1} \dfrac{A - a_k}{A} = n$.

(ii) Using Cauchy-Schwarz's inequality, or otherwise, prove that $\sum_{k=1}^{n+1} \dfrac{A}{A - a_k} \geq \dfrac{(n+1)^2}{n}$.

(b) Using (a)(ii), prove that $\sum_{k=1}^{n+1} \dfrac{1}{(n+1)(n+2) - 2k} \geq \dfrac{n+1}{n(n+2)}$.

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. (a) Consider the system of linear equations in x, y, z

$$(S): \begin{cases} y + (\lambda+1)z = 0 \\ \lambda x + 2y + 2z = \mu \\ x - \lambda y - 4z = \mu^2 \end{cases}, \text{ where } \lambda, \mu \in \mathbf{R} .$$

(i) Suppose that $\mu = 0$.

(1) Prove that (S) has non-trivial solutions if and only if $\lambda^3 + \lambda^2 - 2\lambda = 0$.

(2) Solve (S) when $\lambda = 1$.

(ii) Suppose that $\mu \neq 0$.

(1) Find the range of values of λ for which (S) has a unique solution.

(2) Solve (S) when (S) has a unique solution.

(3) Find λ and μ for which (S) has infinitely many solutions.

(11 marks)

(b) Is there a real solution (x, y, z) of the system of linear equations

$$\begin{cases} y + 2z = 0 \\ x + 2y + 2z = 1 \\ x - y - 4z = 1 \end{cases}$$

satisfying $3x^2 + 2y^2 + z^2 = 1$? Explain your answer.

(4 marks)

8. (a) Let $A = \begin{pmatrix} 4-b & a \\ b & 4-a \end{pmatrix}$ be a real matrix and $P = \begin{pmatrix} a & -1 \\ b & 1 \end{pmatrix}$, where $ab > 0$.

(i) Prove that P is a non-singular matrix.

(ii) Evaluate $P^{-1}AP$.

(iii) For any positive integer n , find d_1 and d_2 such that $A^n = P\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}P^{-1}$.

(9 marks)

(b) Let $B = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$. For any positive integer n , find $B + B^3 + B^5 + \cdots + B^{2n-1}$.

(6 marks)

9. (a) Let α_1 and β_1 be real numbers satisfying $\alpha_1 > \beta_1 > 0$.

For any positive integer n, define $\alpha_{n+1} = \sqrt{\alpha_n^{\,2} - \alpha_n \beta_n + \beta_n^{\,2}}$ and $\beta_{n+1} = \sqrt{\alpha_n \beta_n}$.

(i) Prove that

(1) $\alpha_n \geq \beta_n$,

(2) $\alpha_{n+1} \leq \alpha_n$,

(3) $\beta_{n+1} \geq \beta_n$.

(ii) Prove that the sequences $\{ \alpha_n \}$ and $\{ \beta_n \}$ converge to the same limit.

(iii) Prove that $\alpha_n^{\,2} + \beta_n^{\,2} = \alpha_1^{\,2} + \beta_1^{\,2}$.

Hence, or otherwise, express $\lim\limits_{n \to \infty} \alpha_n$ in terms of α_1 and β_1 .

(10 marks)

(b) Let x_1 and y_1 be real numbers such that $x_1 > y_1 > 0$.

For any positive integer n, define $x_{n+1} = \sqrt{x_n y_n}$ and $y_{n+1} = \dfrac{x_n y_n}{\sqrt{x_n^{\,2} - x_n y_n + y_n^{\,2}}}$.

Do $\lim\limits_{n \to \infty} x_n$ and $\lim\limits_{n \to \infty} y_n$ exist? Explain your answer.

(5 marks)

10. (a) Let $\mathrm{p}(x)$ be a polynomial with real coefficients such that $\mathrm{p}(n) = \mathrm{p}(n-1)$ for any positive integer n. Prove that $\mathrm{p}(x) = \mathrm{p}(0)$ for all $x \in \mathbf{R}$.

(3 marks)

(b) Let $\mathrm{f}(x)$ be a polynomial with real coefficients such that $x\,\mathrm{f}(x-1) = (x-\pi)\,\mathrm{f}(x)$. Prove that

(i) if there exists an integer k such that $\mathrm{f}(k) = 0$, then $\mathrm{f}(k+1) = 0$;

(ii) $\mathrm{f}(x) = 0$ for all $x \in \mathbf{R}$.

(5 marks)

(c) Let $\mathrm{g}(x)$ be a polynomial with real coefficients such that $x\,\mathrm{g}(x-1) = (x-3)\,\mathrm{g}(x)$. Prove that

(i) $\mathrm{g}(0) = \mathrm{g}(1) = \mathrm{g}(2) = 0$;

(ii) $\mathrm{g}(x) = Cx(x-1)(x-2)$, where C is a constant.

(7 marks)

11. (a) Let $z = \cos\theta + i\sin\theta$, where $\theta \in \mathbf{R}$. Find the four values of z such that $\operatorname{Im}(z^2 + \bar{z}) = 0$.

(4 marks)

(b) Let z_1 and z_2 be two of the values of z obtained in (a) such that $\operatorname{Im}(z_1) < 0 < \operatorname{Im}(z_2)$.

For any positive integer n , define $S_n = \sum_{r=1}^{n} \omega^r$, where $\omega = \frac{z_2}{z_1}$.

(i) Prove that $\omega^3 = 1$.

(ii) If n is a multiple of 3 , prove that $S_n = 0$.

(iii) If n is not a multiple of 3 , find S_n .

(iv) Does there exist an integer m such that $(S_{2009} + S_{2010} + S_{2011})^m = 2$? Explain your answer.

(v) Find all positive integers k such that $(S_n)^k + (S_{n+1})^k + (S_{n+2})^k = 2$ for any positive integer n .

(11 marks)

END OF PAPER

2011-AL
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PAPER 2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
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PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(C) answer book.

4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Let $k \in \mathbf{R}$ and $\mathrm{f}: \mathbf{R} \to \mathbf{R}$ be defined by

$$\mathrm{f}(x) = \begin{cases} k - e^{3x} & \text{when } x < 0 , \\ 4\sin x + 5\cos x & \text{when } x \geq 0 . \end{cases}$$

It is given that $\mathrm{f}(x)$ is continuous at $x = 0$.

(a) Find k .

(b) Is $\mathrm{f}(x)$ differentiable at $x = 0$? Explain your answer.

(c) Find the asymptote(s) of the graph of $y = \mathrm{f}(x)$.

(7 marks)

2. Let $\mathrm{f}: \mathbf{R} \to \mathbf{R}$ be an even function such that $\mathrm{f}(x) = \begin{cases} x & \text{when } 0 \leq x \leq 3 , \\ x+1 & \text{when } x > 3 . \end{cases}$

(a) Write down the value of $\mathrm{f}(-5)$.

(b) Sketch the graph of $y = \mathrm{f}(x)$.

(c) Let $\mathrm{g}: \mathbf{R} \to \mathbf{R}$ be defined by $\mathrm{g}(x) = \mathrm{f}(x+2) - \mathrm{f}(x-2)$.

(i) Prove that g is an odd function.

(ii) Sketch the graph of $y = \mathrm{g}(x)$.

(7 marks)

3. Let $\mathrm{f}: \mathbf{R} \to \mathbf{R}$ be defined by $\mathrm{f}(x) = \dfrac{1}{\sqrt{x^2 + 2x + 5}}$.

(a) Prove that $(x^2 + 2x + 5)\,\mathrm{f}'(x) + (x+1)\,\mathrm{f}(x) = 0$.

Hence prove that $(x^2 + 2x + 5)\,\mathrm{f}^{(n+1)}(x) + (2n+1)(x+1)\,\mathrm{f}^{(n)}(x) + n^2\,\mathrm{f}^{(n-1)}(x) = 0$ for any positive integer n , where $\mathrm{f}^{(0)} = \mathrm{f}$.

(b) Using (a), or otherwise, evaluate $\mathrm{f}^{(6)}(-1)$ and $\mathrm{f}^{(7)}(-1)$.

(6 marks)

4. (a) Find $\int x^2\sqrt{9-x^3}\,dx$.

(b) Evaluate $\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{2n}k^2\sqrt{9-\frac{k^3}{n^3}}$.

(6 marks)

5. (a) Using the substitution $x=5+2\sin\theta$, find $\int\sqrt{(x-3)(7-x)}\,dx$.

(b) Let D be the region bounded by the curve $y=\left((x-3)(7-x)\right)^{\frac{1}{4}}$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(7 marks)

6. The equation of the hyperbola H is $4x^2-y^2=144$. Let P be the point $(6\sec\theta, 12\tan\theta)$, where $0<\theta<\frac{\pi}{2}$.

(a) Prove that P lies on H .

(b) Let L be the normal to H at P.

(i) Find the x-intercept and the y-intercept of L .

(ii) If the area of the region bounded by L , the x-axis and the y-axis is 150 , find the coordinates of P .

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \ln(x^2 - 2x + 10)$.

(a) Find $f'(x)$ and $f''(x)$.

(2 marks)

(b) Solve each of the following inequalities:

(i) $f(x) > 0$,

(ii) $f'(x) > 0$,

(iii) $f''(x) > 0$.

(3 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y = f(x)$.

(3 marks)

(d) Sketch the graph of $y = f(x)$.

(2 marks)

(e) Find the area of the region bounded by the graph of $y = f(x)$ and the straight line $y = \ln 18$.

(5 marks)

8. (a) (i) Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = 1$.

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx$.

(5 marks)

(b) Let $f : [0, \pi] \to \mathbf{R}$ be a continuous function such that $f(\pi - x) = f(x)$ for all $x \in [0, \pi]$.

Using integration by substitution, prove that $\displaystyle\int_0^{\pi} f(x)\,dx = 2\int_0^{\frac{\pi}{2}} f(x)\,dx$.

(3 marks)

(c) Let $g : [0, \pi] \to \mathbf{R}$ be a continuous function such that $g(\pi - x) = -g(x)$ for all $x \in [0, \pi]$.

Using the substitution $u = \pi - x$, prove that $\displaystyle\int_0^{\pi} g(x)\ln(1+e^{\cos x})\,dx = \frac{1}{2}\int_0^{\pi} g(x)\cos x\,dx$.

(3 marks)

(d) Evaluate $\displaystyle\int_0^{\pi} \frac{\cos x \ln(1+e^{\cos x})}{(1+\sin x)^2}\,dx$.

(4 marks)

9. (a) Let $\mathrm{f}:\mathbf{R}\to\mathbf{R}$ and $\mathrm{g}:\mathbf{R}\to\mathbf{R}$ be increasing continuous functions.

(i) Let $\mathrm{F}:\mathbf{R}\to\mathbf{R}$ be defined by $\mathrm{F}(x)=x\int_0^x \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t-\left(\int_0^x \mathrm{f}(t)\,\mathrm{d}t\right)\left(\int_0^x \mathrm{g}(t)\,\mathrm{d}t\right)$.

(1) Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x)=\int_0^x (\mathrm{f}(t)-\mathrm{f}(x))(\mathrm{g}(t)-\mathrm{g}(x))\,\mathrm{d}t$.

(2) Find the least value of $\mathrm{F}(x)$.

(ii) Using (a)(i), prove that $\left(\int_0^1 \mathrm{f}(x)\,\mathrm{d}x\right)\left(\int_0^1 \mathrm{g}(x)\,\mathrm{d}x\right)\le\int_0^1 \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x$.

(iii) Furthermore, if $\mathrm{f}(x)>0$ for all $x\in\mathbf{R}$, prove that $\left(\int_0^1 \mathrm{f}(x)\,\mathrm{d}x\right)^n\le\int_0^1 (\mathrm{f}(x))^n\,\mathrm{d}x$ for any positive integer n .

(11 marks)

(b) Let $\mathrm{h}:\mathbf{R}\to\mathbf{R}^+$ be an increasing continuous function, where $\mathbf{R}^+$ is the set of positive real numbers. Prove that $\left(\int_{-2}^3 \mathrm{h}(x)\,\mathrm{d}x\right)^{2011}\le 5^{2010}\int_{-2}^3 (\mathrm{h}(x))^{2011}\,\mathrm{d}x$.

(4 marks)

10. Denote the set of positive real numbers by $\mathbf{R}^+$.

(a) It is given that $a,b,s,t\in\mathbf{R}^+$, where $a<b$ and $s+t=1$. Let $\mathrm{f}:\mathbf{R}^+\to\mathbf{R}$ be a twice differentiable function such that $\mathrm{f}''(x)\le 0$ for all $x\in\mathbf{R}^+$.

(i) Let $u=sa+tb$. Using Mean Value Theorem, prove that

$$\frac{\mathrm{f}(b)-\mathrm{f}(u)}{s(b-a)}\le\mathrm{f}'(u)\le\frac{\mathrm{f}(u)-\mathrm{f}(a)}{t(b-a)} .$$

(ii) Using (a)(i), prove that $s\,\mathrm{f}(a)+t\,\mathrm{f}(b)\le\mathrm{f}(sa+tb)$.

(6 marks)

(b) Let $h,k,p,q\in\mathbf{R}^+$, where $p+q=1$. Using (a), prove that $h^pk^q\le ph+qk$.

(3 marks)

(c) Let $x_1,x_2,\dots,x_n,\lambda_1,\lambda_2,\dots,\lambda_n\in\mathbf{R}^+$, where $\sum_{k=1}^n\lambda_k=1$. Using mathematical induction, prove that $\prod_{k=1}^n {x_k}^{\lambda_k}\le\sum_{k=1}^n\lambda_k x_k$.

(6 marks)

11. Consider the two parabolas $P_1 : y^2 = 4x$ and $P_2 : y^2 = 8x$. Let L be the tangent to P_1 at the point $S(s^2, 2s)$, where $s > 0$.

(a) Find the equation of L .

(2 marks)

(b) L cuts P_2 at the points $A(2\alpha^2, 4\alpha)$ and $B(2\beta^2, 4\beta)$. Let L_1 and L_2 be the tangents to P_2 at A and B respectively .

(i) Prove that $\alpha + \beta = 2s$ and $\alpha\beta = \frac{s^2}{2}$.

(ii) Let θ be the acute angle between L_1 and L_2 .

(1) Prove that $\tan\theta = \frac{2\sqrt{2}s}{s^2 + 2}$.

(2) Find the greatest value of θ .

(iii) It is given that L_1 and L_2 intersect at C .

(1) Express the coordinates of C in terms of s .

(2) Find the equation of the locus of C as s varies.

(13 marks)

END OF PAPER