2010 AL Pure Mathematics

Marking Schemes

These documents were prepared for markers' reference. They should not be regarded as sets of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret their contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. Marks may be deducted for poor presentation (*pp*). The symbol (*pp-1*) should be used to denote 1 mark deducted for *pp*. At most deduct 1 mark from Section A and 1 mark from Section B for *pp*. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

7. In the marking scheme, 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

Paper 1

Solution	Marks
1. (a) Note that $(1+2x)^n = \sum_{k=0}^{n} a_k x^k$.	1M
Putting $x=1$, we have $\sum_{k=0}^{n} a_k = 3^n$.	1
(b) Differentiating both sides of $(1+2x)^n = \sum_{k=0}^{n} a_k x^k$ w.r.t. x ,	1M
we have $2n(1+2x)^{n-1} = \sum_{k=1}^{n} k a_k x^{k-1}$.	
Putting $x=1$, we have $\sum_{k=1}^{n} k a_k = 2n3^{n-1}$.	1
(c) $\sum_{k=0}^{n} (3k+1) a_k$	
$= \sum_{k=0}^{n} (3k a_k + a_k)$	
$= 3\sum_{k=0}^{n} k a_k + \sum_{k=0}^{n} a_k$	1M
$= 3\sum_{k=1}^{n} k a_k + \sum_{k=0}^{n} a_k$	
$= 3(2n3^{n-1}) + 3^n$ (by (b) and (a))	1M for using (a) and (b)
$= 2n3^n + 3^n$	
$= (2n+1)3^n$	1
	----------(7)

Solution	Marks
2. (a) Let $\dfrac{x}{(x^2-1)(x^2-4)}=\dfrac{C_1}{x+1}+\dfrac{C_2}{x-1}+\dfrac{C_3}{x+2}+\dfrac{C_4}{x-2}$.	1M
Solving, we have $C_1=\dfrac{-1}{6}$, $C_2=\dfrac{-1}{6}$, $C_3=\dfrac{1}{6}$ and $C_4=\dfrac{1}{6}$.	1A for all correct
Thus, we have $\dfrac{x}{(x^2-1)(x^2-4)}=\dfrac{1}{6}\left(\dfrac{-1}{x+1}+\dfrac{-1}{x-1}+\dfrac{1}{x+2}+\dfrac{1}{x-2}\right)$.	
(b) Note that $\dfrac{d}{dx}\left(\dfrac{x}{(x^2-1)(x^2-4)}\right)=\dfrac{-3x^4+5x^2+4}{(x^2-1)^2(x^2-4)^2}$.	1A or equivalent
By (a), we have $\dfrac{x}{(x^2-1)(x^2-4)}=\dfrac{-1}{6(x+1)}+\dfrac{-1}{6(x-1)}+\dfrac{1}{6(x+2)}+\dfrac{1}{6(x-2)}$.	
Differentiating both sides with respect to x , we have	
$\dfrac{-3x^4+5x^2+4}{(x^2-1)^2(x^2-4)^2}=\dfrac{1}{6(x+1)^2}+\dfrac{1}{6(x-1)^2}+\dfrac{-1}{6(x+2)^2}+\dfrac{-1}{6(x-2)^2}$	
$\dfrac{3x^4-5x^2-4}{(x^2-1)^2(x^2-4)^2}=\dfrac{-1}{6(x+1)^2}+\dfrac{-1}{6(x-1)^2}+\dfrac{1}{6(x+2)^2}+\dfrac{1}{6(x-2)^2}$	1A for correct partial fractions
Let $\dfrac{3x^4-5x^2-4}{(x^2-1)^2(x^2-4)^2}=\dfrac{D_1}{x+1}+\dfrac{D_2}{(x+1)^2}+\dfrac{D_3}{x-1}+\dfrac{D_4}{(x-1)^2}+\dfrac{D_5}{x+2}+\dfrac{D_6}{(x+2)^2}+\dfrac{D_7}{x-2}+\dfrac{D_8}{(x-2)^2}$.	1A or equivalent
So, we have $D_1=D_3=D_5=D_7=0$, $D_2=D_4=\dfrac{-1}{6}$ and $D_6=D_8=\dfrac{1}{6}$.	1A for all correct
Thus, we have $\dfrac{3x^4-5x^2-4}{(x^2-1)^2(x^2-4)^2}=\dfrac{-1}{6(x+1)^2}+\dfrac{-1}{6(x-1)^2}+\dfrac{1}{6(x+2)^2}+\dfrac{1}{6(x-2)^2}$.	
(c) $\displaystyle\sum_{k=3}^{n}\frac{3k^4-5k^2-4}{(k^2-1)^2(k^2-4)^2}$	
$\displaystyle=\frac{1}{6}\sum_{k=3}^{n}\left(\left(\frac{1}{(k+2)^2}-\frac{1}{(k+1)^2}\right)+\left(\frac{1}{(k-2)^2}-\frac{1}{(k-1)^2}\right)\right)$ (by (b))	
$\displaystyle=\frac{1}{6}\left(\sum_{k=3}^{n}\left(\frac{1}{(k+2)^2}-\frac{1}{(k+1)^2}\right)+\sum_{k=3}^{n}\left(\frac{1}{(k-2)^2}-\frac{1}{(k-1)^2}\right)\right)$	1M
$\displaystyle=\frac{1}{6}\left(-\frac{1}{16}+\frac{1}{(n+2)^2}+1-\frac{1}{(n-1)^2}\right)$	
$\displaystyle=\frac{5}{32}+\frac{1}{6(n+2)^2}-\frac{1}{6(n-1)^2}$	
$\displaystyle\sum_{k=3}^{\infty}\frac{3k^4-5k^2-4}{(k^2-1)^2(k^2-4)^2}$	
$\displaystyle=\lim_{n\to\infty}\left(\frac{5}{32}+\frac{1}{6(n+2)^2}-\frac{1}{6(n-1)^2}\right)$	
$=\dfrac{5}{32}$	1A
	----------(6)

	Solution	Marks
3.	(a) Note that $f(x)=(x-1)(x-4)r(x)-x+k$, where $r(x)$ is a polynomial.	1M (any one)
	Also note that $f(x)=(x-4)^2 s(x)+kx-10$, where $s(x)$ is a polynomial.	
	So, we have $f(4)=-4+k$ and $f(4)=4k-10$.	
	Hence, we have $-4+k=4k-10$.	1M
	Thus, we have $k=2$.	1A
	(b) As $f(x)$ is a cubic polynomial, we have $f(x)=(x-4)^2(x+A)+2x-10$, where A is a constant.	
	By (a), we have $f(1)=1$.	
	So, we have $9(1+A)-8=1$.	1M
	Solving, we have $A=0$.	
	$f(x)$ $=(x-4)^2x+2x-10$ $=x^3-8x^2+18x-10$	
	Thus, we have $g(x)=-8x^2+18x-10$.	1A
	(c) When $(g(x))^3$ is divided by $x+1$, the remainder is $(g(-1))^3$.	1M
	As $g(-1)=-36$, we have $(g(-1))^3=-46656$.	1A
	Thus, the required remainder is -46656 .	
		------------(7)

Solution	Marks
4. (a) The required matrix	
$= \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$	1A
$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	
(b) (i) Let (x, y) be the coordinates of Q.	
$\begin{pmatrix} x \\ y \end{pmatrix}$	
$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	
$= \begin{pmatrix} -3 \\ 1 \end{pmatrix}$	1A
Thus, the coordinates of Q are $(-3, 1)$.	
Let (x, y) be the coordinates of Q.	
Then, we have $\begin{cases} x^2 + y^2 = 10 \\ x = -3y \end{cases}$	
Solving, we have $\begin{cases} x = -3 \\ y = 1 \end{cases}$ or $\begin{cases} x = 3 \\ y = -1 \end{cases}$	
As Q lies in the second quadrant, the coordinates of Q are $(-3, 1)$.	1A
(ii) (1) $M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	1A
(2) Note that $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$ for any point (x, y) .	
So, T transforms any point (x, y) to the point $(-y, -x)$.	
Thus, T is the reflection in the straight line $x + y = 0$.	1A for reflection + 1A for $x + y = 0$
(3) The area of $\Delta O'P'Q'$	
= the area of ΔOPQ	1M provided T is a reflection
$= \frac{1}{2}\left(\sqrt{1^2 + 3^2}\right)^2$	
$= \frac{1}{2}(10)$	
$= 5$	1A

Solution	Marks
The coordinates of O' , P' and Q' are $(0,0)$, $(-3,-1)$ and $(-1,3)$ respectively.	1M
Note that $\Delta O'P'Q'$ is an isosceles triangle and $\angle P'O'Q' = \frac{\pi}{2}$.	
The area of $\Delta O'P'Q'$	
$= \frac{1}{2}\left(\sqrt{1^2+3^2}\right)^2$	
$= \frac{1}{2}(10)$	
$= 5$	1A
	----------(7)

Solution	Marks
5. (a) $z\bar{z}=(12+16i)z+(12-16i)\bar{z}-375$	
$z\bar{z}-(\overline{12-16i})z-(12-16i)\bar{z}=-375$	
$z\bar{z}-(\overline{12-16i})z-(12-16i)\bar{z}+(\overline{12-16i})(12-16i)=-375+(\overline{12-16i})(12-16i)$	1M
$(z-(12-16i))(\bar{z}-(\overline{12-16i}))=-375+400$	
$(z-(12-16i))(\overline{z-(12-16i)})=25$	
$\lvert z-(12-16i)\rvert^2=5^2$	1A
$\lvert z-(12-16i)\rvert=5$	
Let $z=x+iy$, where $x, y\in\mathbf{R}$.	
$x^2+y^2-(12+16i)(x+iy)-(12-16i)(x-iy)=-375$	
$x^2+y^2-(12x-16y+i(16x+12y))-(12x-16y+i(-16x-12y))=-375$	
$x^2+y^2-24x+32y+400=-375+400$	1M
$x^2-24x+144+y^2+32y+256=25$	
$(x-12)^2+(y+16)^2=5^2$	1A
Thus, the centre of the circle is $12-16i$.	1A accept $(12,-16)$
The radius of the circle is 5 .	1A
(b) The distance between the centre of the circle and the pole	
$=\sqrt{(12-0)^2+(-16-0)^2}$	
$=20$	
$\lvert z_1\rvert$	
$=20-5$	1M
$=15$	
z_1	
$=\frac{15}{20}(12-16i)$	1M
$=9-12i$	1A
The distance between the centre of the circle and the pole	
$=\sqrt{(12-0)^2+(-16-0)^2}$	
$=20$	
$\lvert z_1\rvert$	
$=20-5$	1M
$=15$	
Let $z_1=x+iy$, where $x, y\in\mathbf{R}$.	
So, we have $(x-12)^2+(y+16)^2=5^2$ and $x^2+y^2=15^2$.	1M
Solving, we have $x=9$ and $y=-12$.	
Thus, we have $z_1=9-12i$.	1A
	----------(7)

Solution	Marks
6. (a) By Cauchy-Schwarz's inequality, we have	
$((\sqrt{\alpha})^2+(\sqrt{\beta})^2+(\sqrt{\gamma})^2+(\sqrt{\delta})^2)\left(\left(\frac{1}{\sqrt{\alpha}}\right)^2+\left(\frac{1}{\sqrt{\beta}}\right)^2+\left(\frac{1}{\sqrt{\gamma}}\right)^2+\left(\frac{1}{\sqrt{\delta}}\right)^2\right)$	
$\geq\left(\sqrt{\alpha}\frac{1}{\sqrt{\alpha}}+\sqrt{\beta}\frac{1}{\sqrt{\beta}}+\sqrt{\gamma}\frac{1}{\sqrt{\gamma}}+\sqrt{\delta}\frac{1}{\sqrt{\delta}}\right)^2$	1M
$(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\geq(1+1+1+1)^2$	
$(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\geq 16$	1
By A.M. $\geq$ G.M., we have $\frac{\alpha+\beta+\gamma+\delta}{4}\geq\sqrt[4]{\alpha\beta\gamma\delta}$ and	1M (either one)
$\frac{1}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\geq\frac{1}{\sqrt[4]{\alpha\beta\gamma\delta}}$.	
Multiplying, we have $\frac{1}{16}(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\geq 1$.	
Thus, we have $(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)\geq 16$.	1
(b) Since α , β , γ and δ are positive real numbers,	
$\beta+\gamma+\delta$, $\gamma+\delta+\alpha$, $\delta+\alpha+\beta$ and $\alpha+\beta+\gamma$ are positive.	
$\left((\beta+\gamma+\delta)+(\gamma+\delta+\alpha)+(\delta+\alpha+\beta)+(\alpha+\beta+\gamma)\right)\left(\frac{1}{\beta+\gamma+\delta}+\frac{1}{\gamma+\delta+\alpha}+\frac{1}{\delta+\alpha+\beta}+\frac{1}{\alpha+\beta+\gamma}\right)\geq 16$	1M for using (a)
$3(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\beta+\gamma+\delta}+\frac{1}{\gamma+\delta+\alpha}+\frac{1}{\delta+\alpha+\beta}+\frac{1}{\alpha+\beta+\gamma}\right)\geq 16$	
$(\alpha+\beta+\gamma+\delta)\left(\frac{3}{\beta+\gamma+\delta}+\frac{3}{\gamma+\delta+\alpha}+\frac{3}{\delta+\alpha+\beta}+\frac{3}{\alpha+\beta+\gamma}\right)\geq 16$	
$\frac{3}{\beta+\gamma+\delta}+\frac{3}{\gamma+\delta+\alpha}+\frac{3}{\delta+\alpha+\beta}+\frac{3}{\alpha+\beta+\gamma}\geq\frac{16}{\alpha+\beta+\gamma+\delta}$	1
Since α , β , γ and δ are positive real numbers,	
$\frac{3}{\beta+\gamma+\delta}$, $\frac{3}{\gamma+\delta+\alpha}$, $\frac{3}{\delta+\alpha+\beta}$ and $\frac{3}{\alpha+\beta+\gamma}$ are positive.	
$\left(\frac{3}{\beta+\gamma+\delta}+\frac{3}{\gamma+\delta+\alpha}+\frac{3}{\delta+\alpha+\beta}+\frac{3}{\alpha+\beta+\gamma}\right)\left(\frac{\beta+\gamma+\delta}{3}+\frac{\gamma+\delta+\alpha}{3}+\frac{\delta+\alpha+\beta}{3}+\frac{\alpha+\beta+\gamma}{3}\right)\geq 16$	1M for using (a)
$\left(\frac{3}{\beta+\gamma+\delta}+\frac{3}{\gamma+\delta+\alpha}+\frac{3}{\delta+\alpha+\beta}+\frac{3}{\alpha+\beta+\gamma}\right)\left(\frac{3(\alpha+\beta+\gamma+\delta)}{3}\right)\geq 16$	
$\frac{3}{\beta+\gamma+\delta}+\frac{3}{\gamma+\delta+\alpha}+\frac{3}{\delta+\alpha+\beta}+\frac{3}{\alpha+\beta+\gamma}\geq\frac{16}{\alpha+\beta+\gamma+\delta}$	1

Solution	Marks
(c) $\frac{3}{\beta+\gamma+\delta}+\frac{3}{\gamma+\delta+\alpha}+\frac{3}{\delta+\alpha+\beta}+\frac{3}{\alpha+\beta+\gamma}\geq\frac{16}{\alpha+\beta+\gamma+\delta}$	
$\left(\frac{1}{\beta+\gamma+\delta}+\frac{1}{\gamma+\delta+\alpha}+\frac{1}{\delta+\alpha+\beta}+\frac{1}{\alpha+\beta+\gamma}\right)(\alpha+\beta+\gamma+\delta)\geq\frac{16}{3}$	
$\frac{\alpha+\beta+\gamma+\delta}{\beta+\gamma+\delta}+\frac{\alpha+\beta+\gamma+\delta}{\gamma+\delta+\alpha}+\frac{\alpha+\beta+\gamma+\delta}{\delta+\alpha+\beta}+\frac{\alpha+\beta+\gamma+\delta}{\alpha+\beta+\gamma}\geq\frac{16}{3}$	
$1+\frac{\alpha}{\beta+\gamma+\delta}+1+\frac{\beta}{\gamma+\delta+\alpha}+1+\frac{\gamma}{\delta+\alpha+\beta}+1+\frac{\delta}{\alpha+\beta+\gamma}\geq\frac{16}{3}$	1M
$\frac{\alpha}{\beta+\gamma+\delta}+\frac{\beta}{\gamma+\delta+\alpha}+\frac{\gamma}{\delta+\alpha+\beta}+\frac{\delta}{\alpha+\beta+\gamma}\geq\frac{16}{3}-4$	
$\frac{\alpha}{\beta+\gamma+\delta}+\frac{\beta}{\gamma+\delta+\alpha}+\frac{\gamma}{\delta+\alpha+\beta}+\frac{\delta}{\alpha+\beta+\gamma}\geq\frac{4}{3}$	1
$\frac{\alpha}{\beta+\gamma+\delta}+\frac{\beta}{\gamma+\delta+\alpha}+\frac{\gamma}{\delta+\alpha+\beta}+\frac{\delta}{\alpha+\beta+\gamma}$	
$=\left(1+\frac{\alpha}{\beta+\gamma+\delta}\right)+\left(1+\frac{\beta}{\gamma+\delta+\alpha}\right)+\left(1+\frac{\gamma}{\delta+\alpha+\beta}\right)+\left(1+\frac{\delta}{\alpha+\beta+\gamma}\right)-4$	
$=\frac{\alpha+\beta+\gamma+\delta}{\beta+\gamma+\delta}+\frac{\alpha+\beta+\gamma+\delta}{\gamma+\delta+\alpha}+\frac{\alpha+\beta+\gamma+\delta}{\delta+\alpha+\beta}+\frac{\alpha+\beta+\gamma+\delta}{\alpha+\beta+\gamma}-4$	1M
Since $\frac{\alpha+\beta+\gamma+\delta}{\beta+\gamma+\delta}+\frac{\alpha+\beta+\gamma+\delta}{\gamma+\delta+\alpha}+\frac{\alpha+\beta+\gamma+\delta}{\delta+\alpha+\beta}+\frac{\alpha+\beta+\gamma+\delta}{\alpha+\beta+\gamma}\geq\frac{16}{3}$,	
we have $\frac{\alpha}{\beta+\gamma+\delta}+\frac{\beta}{\gamma+\delta+\alpha}+\frac{\gamma}{\delta+\alpha+\beta}+\frac{\delta}{\alpha+\beta+\gamma}\geq\frac{16}{3}-4$.	
Thus, we have $\frac{\alpha}{\beta+\gamma+\delta}+\frac{\beta}{\gamma+\delta+\alpha}+\frac{\gamma}{\delta+\alpha+\beta}+\frac{\delta}{\alpha+\beta+\gamma}\geq\frac{4}{3}$.	1
	----------(6)

Solution	Marks
7. (a) (i) (E) has a unique solution	
$\Leftrightarrow \Delta \neq 0$	1M
$\Leftrightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & 0 & -4 \\ 3 & 4 & a+4 \end{vmatrix} \neq 0$	
$\Leftrightarrow -12+4a+16-a(a+4) \neq 0$	
$\Leftrightarrow 4-a^2 \neq 0$	1A
$\Leftrightarrow a^2 \neq 4$	
$\Leftrightarrow a \neq -2$ and $a \neq 2$	1A
$\Leftrightarrow a<-2$, $-2<a<2$ or $a>2$	
The augmented matrix of (E) is	
$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ a & 0 & -4 & 2 \\ 3 & 4 & a+4 & b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & -a & -4-a & 2-2a \\ 0 & 1 & a+1 & b-6 \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & a+1 & b-6 \\ 0 & 0 & a^2-4 & ab-8a+2 \end{array}\right)$	1A
(E) has a unique solution	
$\Leftrightarrow a^2-4 \neq 0$	1M
$\Leftrightarrow a^2 \neq 4$	
$\Leftrightarrow a \neq -2$ and $a \neq 2$	1A
$\Leftrightarrow a<-2$, $-2<a<2$ or $a>2$	
When (E) has a unique solution,	
$x = \dfrac{\begin{vmatrix} 2 & 1 & 1 \\ 2 & 0 & -4 \\ b & 4 & a+4 \end{vmatrix}}{4-a^2} = \dfrac{2(16-2b-a)}{4-a^2}$	1M for Cramer's Rule
$y = \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ a & 2 & -4 \\ 3 & b & a+4 \end{vmatrix}}{4-a^2} = \dfrac{-22-6a+4b+ab-2a^2}{4-a^2}$	
$z = \dfrac{\begin{vmatrix} 1 & 1 & 2 \\ a & 0 & 2 \\ 3 & 4 & b \end{vmatrix}}{4-a^2} = \dfrac{-2+8a-ab}{4-a^2}$	1A+1A (1A for any one, 1A for all)

Solution	Marks
In this case, the augmented matrix of (E)	
$\sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & a+1 & b-6 \\ 0 & 0 & 1 & \dfrac{-2+8a-ab}{4-a^2} \end{array}\right)$	1M
$\sim \left(\begin{array}{ccc\|c} 1 & 0 & 0 & \dfrac{2(16-2b-a)}{4-a^2} \\ 0 & 1 & 0 & \dfrac{-22-6a+4b+ab-2a^2}{4-a^2} \\ 0 & 0 & 1 & \dfrac{-2+8a-ab}{4-a^2} \end{array}\right)$	
$\therefore\ x=\dfrac{2(16-2b-a)}{4-a^2}$, $y=\dfrac{-22-6a+4b+ab-2a^2}{4-a^2}$, $z=\dfrac{-2+8a-ab}{4-a^2}$.	1A+1A (1A for any one, 1A for all)
(ii) When $a=2$, the augmented matrix of (E) becomes	
$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 2 & 0 & -4 & 2 \\ 3 & 4 & 6 & b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -6 & -2 \\ 0 & 1 & 3 & b-6 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & b-7 \end{array}\right)$.	
So, (E) is consistent when $b=7$.	1A
Thus, the solution set is $\{(1+2s,\ 1-3s,\ s): s\in\mathbf{R}\}$.	1A or equivalent
	----------(8)
(b) Note that the augmented matrix of the first three equations of (F) is	
$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & -1 \\ 3 & 4 & 2 & \lambda \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & \lambda-9 \end{array}\right)$.	
So, the first three equations of (F) are solvable when $\lambda=9$.	1A
Under this case, the solution set for the first three equations is $\{(-1-2t,\ t+3,\ t): t\in\mathbf{R}\}$.	1A or equivalent
Hence, (F) is consistent when $\lambda=9$ and $\mu=7(-1-2t)+17(3+t)-3t$.	1M
Thus, we have $\lambda=9$ and $\mu=44$.	1A
	---------(4)
(c) Putting $a=\dfrac{2}{3}$ and $b=5$ in (a), the first three equations of (G) have the	1M
unique solution $x=3$, $y=-1$ and $z=0$.	1A
Note that $5(3)-2(-1)-18(0)=17\neq 16$.	
Thus, (G) is inconsistent.	1A f.t.
	---------(3)

Solution	Marks
8. (a) '$\Rightarrow$'	
Since λ is a repeated root of the equation $p(x)=0$,	
$p(x)=(x-\lambda)^2 q_1(x)$, where $q_1(x)$ is a polynomial.	1M
$p'(x)=2(x-\lambda)q_1(x)+(x-\lambda)^2 q_1'(x)$	
Thus, we have $p(\lambda)=p'(\lambda)=0$.	1
'$\Leftarrow$'	
As $p(\lambda)=0$, we have $p(x)=(x-\lambda)q_2(x)$, where $q_2(x)$ is a polynomial.	1M
$p'(x)=q_2(x)+(x-\lambda)q_2'(x)$	
As $p'(\lambda)=0$, we have $q_2(\lambda)+(\lambda-\lambda)q_2'(\lambda)=0$.	
Hence, we have $q_2(\lambda)=0$.	
So, we have $q_2(x)=(x-\lambda)q_3(x)$, where $q_3(x)$ is a polynomial.	
Therefore, we have $p(x)=(x-\lambda)^2 q_3(x)$.	1
Thus, λ is a repeated root of the equation $p(x)=0$.	
	----------(4)
(b) (i) Since $f(\mu)=0$ and $f(0)=1$, we have $f(\mu)\neq f(0)$.	
Thus, we have $\mu\neq 0$.	1
(ii) Since $f(\mu)=0$, we have $\mu^6+a\mu^5+b\mu^4+c\mu^3+b\mu^2+a\mu+1=0$.	
So, we have $1+a\left(\frac{1}{\mu}\right)+b\left(\frac{1}{\mu}\right)^2+c\left(\frac{1}{\mu}\right)^3+b\left(\frac{1}{\mu}\right)^4+a\left(\frac{1}{\mu}\right)^5+\left(\frac{1}{\mu}\right)^6=0$.	1M (either one)
Hence, we have $f\left(\frac{1}{\mu}\right)=0$.	1
Note that μ is a repeated real root of $f(x)=0$.	
By (a), we have $\mu^6+a\mu^5+b\mu^4+c\mu^3+b\mu^2+a\mu+1=0$ and	
$6\mu^5+5a\mu^4+4b\mu^3+3c\mu^2+2b\mu+a=0$.	1M
Therefore, we have $a\mu^5+2b\mu^4+3c\mu^3+4b\mu^2+5a\mu+6=0$.	1M
So, we have $a+2b\left(\frac{1}{\mu}\right)+3c\left(\frac{1}{\mu}\right)^2+4b\left(\frac{1}{\mu}\right)^3+5a\left(\frac{1}{\mu}\right)^4+6\left(\frac{1}{\mu}\right)^5=0$.	(either one)
Hence, we have $f'\left(\frac{1}{\mu}\right)=0$.	1
By (a), $\frac{1}{\mu}$ is a repeated root of $f(x)=0$.	
	----------(6)
(c) (i) Note that $g(2)=0$.	1M
Also note that $g'(x)=24x^5-80x^4+68x^3-21x^2+34x-16$.	
As $g'(2)=0$, 2 is a repeated root of the equation $g(x)=0$.	1A f.t.
(ii) By (c)(i) and (b)(ii), $\frac{1}{2}$ is also a repeated root of the equation $g(x)=0$.	1M
Therefore, we have $g(x)=(x-2)^2(2x-1)^2(x^2+x+1)$.	1M
Note that $1^2-4(1)(1)=-3<0$.	
Thus, $g(x)$ cannot be factorized as a product of linear polynomials with real coefficients.	1A f.t.
	----------(5)

Solution	Marks
9. (a) (i) $x_n - y_n$	
$= \frac{5}{6}x_{n-1} + \frac{1}{6}y_{n-1} - \left(\frac{2}{9}x_{n-1} + \frac{7}{9}y_{n-1}\right)$	
$= \frac{11}{18}(x_{n-1} - y_{n-1})$	1M
$= \left(\frac{11}{18}\right)^{n-1}(x_1 - y_1)$	
> 0	1
Thus, we have $x_n > y_n$.	
(ii) $x_{n+1} - x_n$	
$= \frac{5}{6}x_n + \frac{1}{6}y_n - x_n$	
$= \frac{1}{6}(y_n - x_n)$	
< 0	1
Thus, $\{ x_n \}$ is a strictly decreasing sequence.	
$y_{n+1} - y_n$	
$= \frac{2}{9}x_n + \frac{7}{9}y_n - y_n$	
$= \frac{2}{9}(x_n - y_n)$	
> 0	1
Thus, $\{ y_n \}$ is a strictly increasing sequence.	
(iii) By (a)(i) and (a)(ii), we have $y_1 \le y_n < x_n \le x_1$.	
$\{ x_n \}$ is a strictly decreasing sequence and is bounded below by y_1 .	1A
$\{ y_n \}$ is a strictly increasing sequence and is bounded above by x_1 .	1A
Thus, $\{ x_n \}$ and $\{ y_n \}$ are convergent sequences.	
(iv) Let $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$.	
Then, we have $A = \frac{5}{6}A + \frac{1}{6}B$.	1M
So, we have $A = B$.	1
Thus, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.	
(v) $4x_{n+1} + 3y_{n+1}$	
$= \frac{10}{3}x_n + \frac{2}{3}y_n + \frac{2}{3}x_n + \frac{7}{3}y_n$	
$= 4x_n + 3y_n$	1
(vi) By (a)(v), we have $4x_n + 3y_n = 4x_{n-1} + 3y_{n-1} = \cdots = 4x_1 + 3y_1$.	
So, we have $\lim_{n\to\infty}(4x_n + 3y_n) = 4x_1 + 3y_1$.	1M
By (a)(iii), we have $4\lim_{n\to\infty} x_n + 3\lim_{n\to\infty} y_n = 4x_1 + 3y_1$.	
By (a)(iv), we have $7\lim_{n\to\infty} x_n = 4x_1 + 3y_1$.	1M
Thus, we have $\lim_{n\to\infty} x_n = \frac{4x_1 + 3y_1}{7}$.	1A
	---------(12)

Solution	Marks
(b) If $x_1 < y_1$, then $-x_1 > -y_1$.	
Define $a_n = -x_n$ and $b_n = -y_n$ for $n = 1, 2, 3, \dots$.	1M
Then, we have $a_1 > b_1$, $a_{n+1} = \frac{5}{6} a_n + \frac{1}{6} b_n$ and $b_{n+1} = \frac{2}{9} a_n + \frac{7}{9} b_n$.	
By (a)(iii), $\{ a_n \}$ and $\{ b_n \}$ are convergent sequences.	1A f.t.
Thus, $\{ x_n \}$ and $\{ y_n \}$ are convergent sequences.	1A f.t.
If $x_1 < y_1$, then $x_n - y_n$ $= \left(\frac{11}{18} \right)^{n-1} (x_1 - y_1)$ < 0 Therefore, we have $x_n < y_n$ for all $n = 1, 2, 3, \dots$. $x_{n+1} - x_n$ $= \frac{1}{6} (y_n - x_n)$ > 0 Therefore, we have $x_{n+1} > x_n$ for all $n = 1, 2, 3, \dots$. $y_{n+1} - y_n$ $= \frac{2}{9} (x_n - y_n)$ < 0 Therefore, we have $y_{n+1} < y_n$ for all $n = 1, 2, 3, \dots$.	
Note that $x_1 \le x_n < y_n \le y_1$ for all $n = 1, 2, 3, \dots$.	1M
$\{ x_n \}$ is a strictly increasing sequence and is bounded above by y_1 . $\{ y_n \}$ is a strictly decreasing sequence and is bounded below by x_1 .	1A f.t.
Thus, $\{ x_n \}$ and $\{ y_n \}$ are convergent sequences.	1A f.t.
	---------(3)

Solution	Marks
10. (a) '$\Rightarrow$'	
Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are real numbers.	
Then, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}=\begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.	
Hence, we have $a^2+b^2=c^2+d^2=1$ and $ac+bd=0$.	1M
Therefore, the points (a,b) and (c,d) lie on the circle $x^2+y^2=1$.	
So, there exist $\theta,\phi\in\mathbf{R}$ such that $\begin{cases} a=\cos\theta \\ b=\sin\theta \end{cases}$ and $\begin{cases} c=\sin\phi \\ d=\cos\phi \end{cases}$.	1M
Then, we have $M=\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\phi & \cos\phi \end{pmatrix}$.	
Since $ac+bd=0$, we have $\cos\theta\sin\phi+\sin\theta\cos\phi=0$.	
So, we have $\sin(\theta+\phi)=0$.	1M
Therefore, we have $\theta+\phi=n\pi$, where $n\in\mathbf{Z}$.	1M
Hence, we have $\phi=n\pi-\theta$, where $n\in\mathbf{Z}$.	
Note that when n is an even number, we have $\begin{cases} \sin(n\pi-\theta)=-\sin\theta \\ \cos(n\pi-\theta)=\cos\theta \end{cases}$.	
Also note that when n is an odd number, we have $\begin{cases} \sin(n\pi-\theta)=\sin\theta \\ \cos(n\pi-\theta)=-\cos\theta \end{cases}$.	
So, we have $\begin{cases} \sin\phi=-\sin\theta \\ \cos\phi=\cos\theta \end{cases}$ or $\begin{cases} \sin\phi=\sin\theta \\ \cos\phi=-\cos\theta \end{cases}$.	
Thus, we have $M=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ or $M=\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.	1
'$\Leftarrow$'	
If $M=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, then $MM^T=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.	
So, we have $MM^T=\begin{pmatrix} \cos^2\theta+\sin^2\theta & 0 \\ 0 & \cos^2\theta+\sin^2\theta \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.	
If $M=\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, then $MM^T=\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.	
So, we have $MM^T=\begin{pmatrix} \cos^2\theta+\sin^2\theta & 0 \\ 0 & \cos^2\theta+\sin^2\theta \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.	
Thus, M is orthogonal.	1
	---------(6)

Solution	Marks
(b) (i) Since $M=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, the statement is true for $n=1$.	
Assume that $M^k=\begin{pmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{pmatrix}$, where k is a positive integer.	
M^{k+1}	
$=M^kM$	
$=\begin{pmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ (by induction assumption)	
$=\begin{pmatrix}\cos k\theta\cos\theta-\sin k\theta\sin\theta & \cos k\theta\sin\theta+\sin k\theta\cos\theta\\ -\sin k\theta\cos\theta-\cos k\theta\sin\theta & -\sin k\theta\sin\theta+\cos k\theta\cos\theta\end{pmatrix}$	
$=\begin{pmatrix}\cos(k+1)\theta & \sin(k+1)\theta\\ -\sin(k+1)\theta & \cos(k+1)\theta\end{pmatrix}$	1
By mathematical induction, we have $M^n=\begin{pmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{pmatrix}$.	
(ii) Note that $M^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.	1A
Thus, we have $M^n=\begin{cases}M & \text{if } n=1,3,5,\dots\\ I & \text{if } n=2,4,6,\dots\end{cases}$.	1A
	---------(3)
(c) By (a) and (b), we have $X=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$.	1A
So, we have $\begin{pmatrix}\cos 400\theta & \sin 400\theta\\ -\sin 400\theta & \cos 400\theta\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$.	
Hence, we have $\cos 400\theta=-1$ and $\sin 400\theta=0$.	1M
Then, we have $400\theta=(2n-1)\pi$, where $n\in\mathbf{Z}$.	1A
Therefore, we have $X=\begin{pmatrix}\cos\frac{(2n-1)\pi}{400} & \sin\frac{(2n-1)\pi}{400}\\ -\sin\frac{(2n-1)\pi}{400} & \cos\frac{(2n-1)\pi}{400}\end{pmatrix}$, where $n\in\mathbf{Z}$.	1A
Thus, we have $X=\begin{pmatrix}\cos\frac{(2n-1)\pi}{400} & \sin\frac{(2n-1)\pi}{400}\\ -\sin\frac{(2n-1)\pi}{400} & \cos\frac{(2n-1)\pi}{400}\end{pmatrix}$, where $n=1,2,\dots,400$.	
	---------(4)

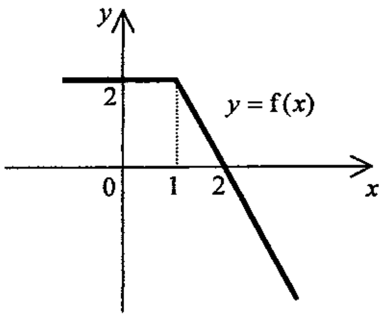
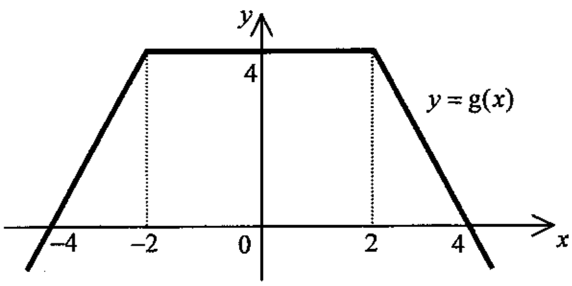
Solution	Marks
(d) By (a), there exists $\theta \in \mathbf{R}$ such that $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ or $M = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.	
By (b), we have $M^{401} = \begin{pmatrix} \cos 401\theta & \sin 401\theta \\ -\sin 401\theta & \cos 401\theta \end{pmatrix}$ or $M^{401} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.	1M for either one
Thus, M^{401} is orthogonal (by (a)).	1A f.t.
$(M^{401})(M^{401})^T$ $= (M^{401})(M^T)^{401}$ $= M^{400}(MM^T)(M^T)^{400}$ $= M^{400}(M^T)^{400}$ $= \cdots$ $= MM^T$ $= I$ Thus, M^{401} is orthogonal.	 1M 1A f.t.
	---------(2)

Solution	Marks
11. (a) (i) Let $f(x)=\ln x-x+1$ for all $x>0$.	
Note that $f'(x)=\frac{1-x}{x}\begin{cases}>0 \text{ if } 0<x<1,\\ =0 \text{ if } x=1,\\ <0 \text{ if } x>1.\end{cases}$	1M for testing
Therefore, $f(x)$ attains its absolute maximum at $x=1$.	
So, we have $f(x)\le f(1)$ for all $x>0$.	
Since $f(1)=0$, we have $\ln x-x+1\le 0$ for all $x>0$.	
Thus, we have $\ln x\le x-1$ for all $x>0$.	1
(ii) By (a)(i), we have $\ln b_k\le b_k-1$ for $k=1,2,\ldots,n$.	
Hence, we have $a_k\ln b_k\le a_kb_k-a_k$ for $k=1,2,\ldots,n$.	
Then, we have $\ln b_k^{a_k}\le a_kb_k-a_k$ for $k=1,2,\ldots,n$.	
So, we have $\sum_{k=1}^{n}\ln b_k^{a_k}\le\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}a_k$.	1M
Therefore, we have $\ln\left(b_1^{a_1}b_2^{a_2}\cdots b_n^{a_n}\right)\le 0$.	
Thus, we have $b_1^{a_1}b_2^{a_2}\cdots b_n^{a_n}\le 1$.	1
	---------(4)
(b) (i) Note that	
$\ln x^x-\ln x$	
$=x\ln x-\ln x$	
$=(x-1)\ln x$	
$\begin{cases}>0 \text{ when } 0<x<1,\\ =0 \text{ when } x=1,\\ >0 \text{ when } x>1.\end{cases}$	1M for dividing cases
Therefore, we have $\ln x^x\ge\ln x$ for all $x>0$.	
Since $\ln x$ is strictly increasing, we have $x^x\ge x$ for all $x>0$.	1
Let $g(x)=x^x-x$ for all $x>0$.	
So, we have $g'(x)=x^x(\ln x+1)-1$.	
Hence, we have $g'(x)\begin{cases}<0 \text{ if } 0<x<1,\\ =0 \text{ if } x=1,\\ >0 \text{ if } x>1.\end{cases}$	1M for testing
Therefore, $g(x)$ attains its absolute minimum at $x=1$.	
So, we have $g(x)\ge g(1)$ for all $x>0$.	
Since $g(1)=0$, we have $x^x-x\ge 0$ for all $x>0$.	
Thus, we have $x^x\ge x$ for all $x>0$.	1
(ii) By (b)(i), we have $c_k^{c_k}\ge c_k>0$ for $k=1,2,\ldots,n$.	
Multiplying, we have $c_1^{c_1}c_2^{c_2}\cdots c_n^{c_n}\ge c_1c_2\cdots c_n$.	
Thus, we have $c_1^{c_1}c_2^{c_2}\cdots c_n^{c_n}\ge 1$.	1
	---------(3)

Solution	Marks
(c) (i) Let $S_1=\sum_{r=1}^{n}x_r$ and $S_2=\sum_{r=1}^{n}x_r^{\ 2}$.	
For each $k=1,2,\dots,n$, define $a_k=\dfrac{x_k}{S_1}$ and $b_k=\dfrac{x_k S_1}{S_2}$.	1M for either one
Then, we have $\sum_{k=1}^{n}a_k=\dfrac{S_1}{S_1}=1$ and $\sum_{k=1}^{n}a_k b_k=\dfrac{S_2}{S_2}=1$.	1M withhold 1M if checking is omitted
By (a)(ii), we have $b_1^{\ a_1}b_2^{\ a_2}\cdots b_n^{\ a_n}\le 1$.	
So, we have $\left(\dfrac{x_1 S_1}{S_2}\right)^{\frac{x_1}{S_1}}\left(\dfrac{x_2 S_1}{S_2}\right)^{\frac{x_2}{S_1}}\cdots\left(\dfrac{x_n S_1}{S_2}\right)^{\frac{x_n}{S_1}}\le 1$.	1M for using (a)(ii)
Hence, we have $\left(\dfrac{x_1 S_1}{S_2}\right)^{x_1}\left(\dfrac{x_2 S_1}{S_2}\right)^{x_2}\cdots\left(\dfrac{x_n S_1}{S_2}\right)^{x_n}\le 1$.	
Therefore, we have $x_1^{\ x_1}x_2^{\ x_2}\cdots x_n^{\ x_n}\le\left(\dfrac{S_2}{S_1}\right)^{x_1+x_2+\cdots+x_n}$	
Thus, we have $x_1^{\ x_1}x_2^{\ x_2}\cdots x_n^{\ x_n}\le\left(\dfrac{x_1^{\ 2}+x_2^{\ 2}+\cdots+x_n^{\ 2}}{x_1+x_2+\cdots+x_n}\right)^{x_1+x_2+\cdots+x_n}$	1
(ii) Let $G=\sqrt[n]{x_1x_2\cdots x_n}$.	
For each $k=1,2,\dots,n$, define $c_k=\dfrac{x_k}{G}$.	1M
Then, we have $c_1c_2\cdots c_n=\dfrac{x_1x_2\cdots x_n}{G^n}=1$.	1M withhold 1M if checking is omitted
By (b)(ii), we have $c_1^{\ c_1}c_2^{\ c_2}\cdots c_n^{\ c_n}\ge 1$.	
So, we have $\left(\dfrac{x_1}{G}\right)^{\frac{x_1}{G}}\left(\dfrac{x_2}{G}\right)^{\frac{x_2}{G}}\cdots\left(\dfrac{x_n}{G}\right)^{\frac{x_n}{G}}\ge 1$.	1M for using (b)(ii)
Therefore, we have $\left(\dfrac{x_1}{G}\right)^{x_1}\left(\dfrac{x_2}{G}\right)^{x_2}\cdots\left(\dfrac{x_n}{G}\right)^{x_n}\ge 1$.	
Hence, we have $x_1^{\ x_1}x_2^{\ x_2}\cdots x_n^{\ x_n}\ge G^{x_1+x_2+\cdots+x_n}$.	
Thus, we have $x_1^{\ x_1}x_2^{\ x_2}\cdots x_n^{\ x_n}\ge\left(\sqrt[n]{x_1x_2\cdots x_n}\right)^{x_1+x_2+\cdots+x_n}$.	1
	---------(8)

Solution	Marks
1. (a) $\lim_{x\to 0}\dfrac{x-\sin x}{x\sin x}$	
$=\lim_{x\to 0}\dfrac{1-\cos x}{\sin x+x\cos x}$	1M
$=\lim_{x\to 0}\dfrac{\sin x}{\cos x+\cos x-x\sin x}$	
$=\lim_{x\to 0}\dfrac{\sin x}{2\cos x-x\sin x}$	
$=0$	1A
(b) Since $\mathrm{f}(x)$ is differentiable at $x=0$, $\mathrm{f}(x)$ is continuous at $x=0$.	1M
So, we have $\lim_{x\to 0^-}\mathrm{f}(x)=\mathrm{f}(0)$.	
Note that $\lim_{x\to 0^-}\mathrm{f}(x)=\lim_{x\to 0^-}\left(\sin x+\dfrac{ax}{\sin x}\right)=\lim_{x\to 0^-}\sin x+a\lim_{x\to 0^-}\dfrac{x}{\sin x}=a$.	
Since $\mathrm{f}(0)=3$, we have $a=3$.	1A
$\lim_{x\to 0^+}\dfrac{\mathrm{f}(x)-\mathrm{f}(0)}{x-0}$	1M (either one)
$=\lim_{x\to 0^+}\dfrac{3+bx+x^2-3}{x-0}$	
$=\lim_{x\to 0^+}(b+x)$	
$=b$	1A (either one)
$\lim_{x\to 0^-}\dfrac{\mathrm{f}(x)-\mathrm{f}(0)}{x-0}$	
$=\lim_{x\to 0^-}\dfrac{\sin x+\dfrac{3x}{\sin x}-3}{x-0}$	
$=\lim_{x\to 0^-}\dfrac{\sin x}{x}+3\lim_{x\to 0^-}\dfrac{x-\sin x}{x\sin x}$	
$=1+3(0)$ (by (a))	
$=1$	
Since $\mathrm{f}(x)$ is differentiable at $x=0$, we have	
$\lim_{x\to 0^+}\dfrac{\mathrm{f}(x)-\mathrm{f}(0)}{x-0}=\lim_{x\to 0^-}\dfrac{\mathrm{f}(x)-\mathrm{f}(0)}{x-0}$.	
Thus, we have $b=1$.	1A
	----------(7)

Solution	Marks
2. (a) $f(0)$ $= 3 + \int_0^0 g(t)\,dt$ $= 3$ $g(0)$ $= \int_0^0 f(t)\,dt - 1$ $= -1$	1A (for both correct)
(b) $f'(x)$ $= g(x)$ $g'(x)$ $= f(x)$	1A (for both correct)
(c) Let $h(x) = (f(x))^2 - (g(x))^2$ for all $x \in \mathbf{R}$.	
Then, we have $h'(x) = 2f(x)f'(x) - 2g(x)g'(x)$.	1M
Therefore, we have $h'(x) = 2f(x)g(x) - 2g(x)f(x) = 0$ (by (b)) .	1A
So, we have $h(x) = a$ for all $x \in \mathbf{R}$, where a is a constant.	1A
Note that $f(0) = 3$ and $g(0) = -1$ (by (a)) .	
Hence, we have $a = 8$.	
Thus, we have $(f(x))^2 - (g(x))^2 = 8$ for all $x \in \mathbf{R}$.	1
	----------(6)

Solution	Marks
3. (a) (i) Since $f(x) = 3 - x - \lvert x - 1 \rvert$, we have	
$f(x) = \begin{cases} 2 & \text{when } x \le 1, \\ 4 - 2x & \text{when } x > 1. \end{cases}$	
	1M for the shape 1A for all correct
(ii) $\because$ $f(0) = 2 = f(1)$ but $0 \ne 1$	
$\therefore$ f is not an injective function.	1A f.t.
(b) (i) $g(x)$	
$= f(x - 1) + f(-x - 1)$	
$g(-x)$	
$= f(-x - 1) + f(-(-x) - 1)$	
$= f(x - 1) + f(-x - 1)$	
Hence, we have $g(x) = g(-x)$ for all $x \in \mathbf{R}$.	1
Thus, g is an even function.	
(ii) Since $g(x) = f(x - 1) + f(-x - 1)$, we have the following sketch:	
	1M for the shape 1A for all correct
$\int_{-4}^{4} g(x)\,dx$	
= the area of the region bounded by the graph of $y = g(x)$ and the x-axis	
$= \dfrac{(4+8)(4)}{2}$	
$= 24$	1A
Note that g is an even function and $g(x) = \begin{cases} 4 & \text{when } 0 \le x \le 2, \\ 8 - 2x & \text{when } x > 2. \end{cases}$	
$\int_{-4}^{4} g(x)\,dx$	
$= 2\left(\int_{0}^{2} 4\,dx + \int_{2}^{4} (8 - 2x)\,dx \right)$	
$= 2\left(\left[4x \right]_0^2 + \left[8x - x^2 \right]_2^4 \right)$	
$= 24$	1A
	----------(7)

Solution	Marks
4. (a) $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right)$	
$=\dfrac{\left(\frac{1}{x}\right)(x)-\ln x}{x^2}$	1M for quotient rule or product rule
$=\dfrac{1-\ln x}{x^2}$	1A
(b) $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{2n}\dfrac{1-\ln\left(1+\frac{k}{n}\right)}{\left(1+\frac{k}{n}\right)^2}$	
$=\displaystyle\int_1^3\frac{1-\ln x}{x^2}dx$	1A
$=\left[\dfrac{\ln x}{x}\right]_1^3$ (by (a))	1M for using the result of (a)
$=\dfrac{\ln 3}{3}$	1A
$\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{2n}\dfrac{1-\ln\left(1+\frac{k}{n}\right)}{\left(1+\frac{k}{n}\right)^2}$	
$=\displaystyle\int_0^2\frac{1-\ln(1+x)}{(1+x)^2}dx$	1A
$=\displaystyle\int_1^3\frac{1-\ln t}{t^2}dt$ (by letting $t=1+x$)	
$=\left[\dfrac{\ln t}{t}\right]_1^3$ (by (a))	1M for using the result of (a)
$=\dfrac{\ln 3}{3}$	1A
$\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{k=1}^{2n}\dfrac{1-\ln\left(1+\frac{k}{n}\right)}{\left(1+\frac{k}{n}\right)^2}$	
$=2\lim\limits_{n\to\infty}\dfrac{1}{2n}\sum\limits_{k=1}^{2n}\dfrac{1-\ln\left(1+2\left(\frac{k}{2n}\right)\right)}{\left(1+2\left(\frac{k}{2n}\right)\right)^2}$	
$=2\displaystyle\int_0^1\frac{1-\ln(1+2x)}{(1+2x)^2}dx$	1A
$=\displaystyle\int_1^3\frac{1-\ln t}{t^2}dt$ (by letting $t=1+2x$)	
$=\left[\dfrac{\ln t}{t}\right]_1^3$ (by (a))	1M for using the result of (a)
$=\dfrac{\ln 3}{3}$	1A
	----------(5)

Solution	Marks
5. (a) Letting $t=x-1$, we have $\frac{dt}{dx}=1$.	1M
$\int (5-x)\sqrt{x-1}\,dx$	
$=\int (4-t)\sqrt{t}\,dt$	1A
$=\int (4t^{\frac{1}{2}}-t^{\frac{3}{2}})\,dt$	
$=\frac{8}{3}t^{\frac{3}{2}}-\frac{2}{5}t^{\frac{5}{2}}+\text{constant}$	
$=\frac{8}{3}(x-1)^{\frac{3}{2}}-\frac{2}{5}(x-1)^{\frac{5}{2}}+\text{constant}$	1A pp−1 for omitting constant
Letting $t^2=x-1$, we have $2t\frac{dt}{dx}=1$.	1M
$\int (5-x)\sqrt{x-1}\,dx$	
$=\int (4-t^2)(2t^2)\,dt$	1A
$=\int (8t^2-2t^4)\,dt$	
$=\frac{8}{3}t^3-\frac{2}{5}t^5+\text{constant}$	
$=\frac{8}{3}(x-1)^{\frac{3}{2}}-\frac{2}{5}(x-1)^{\frac{5}{2}}+\text{constant}$	1A pp−1 for omitting constant
(b) Putting $y=0$ in $y=(5-x)^{\frac{1}{2}}(x-1)^{\frac{1}{4}}$, we have $x=1$ or $x=5$.	
The required volume	
$=\int_1^5 \pi y^2\,dx$	1M+1A
$=\pi\int_1^5 (5-x)\sqrt{x-1}\,dx$	
$=\pi\left[\frac{8}{3}(x-1)^{\frac{3}{2}}-\frac{2}{5}(x-1)^{\frac{5}{2}}\right]_1^5$	1M for using (a)
$=\frac{128\pi}{15}$	1A
	----------(7)

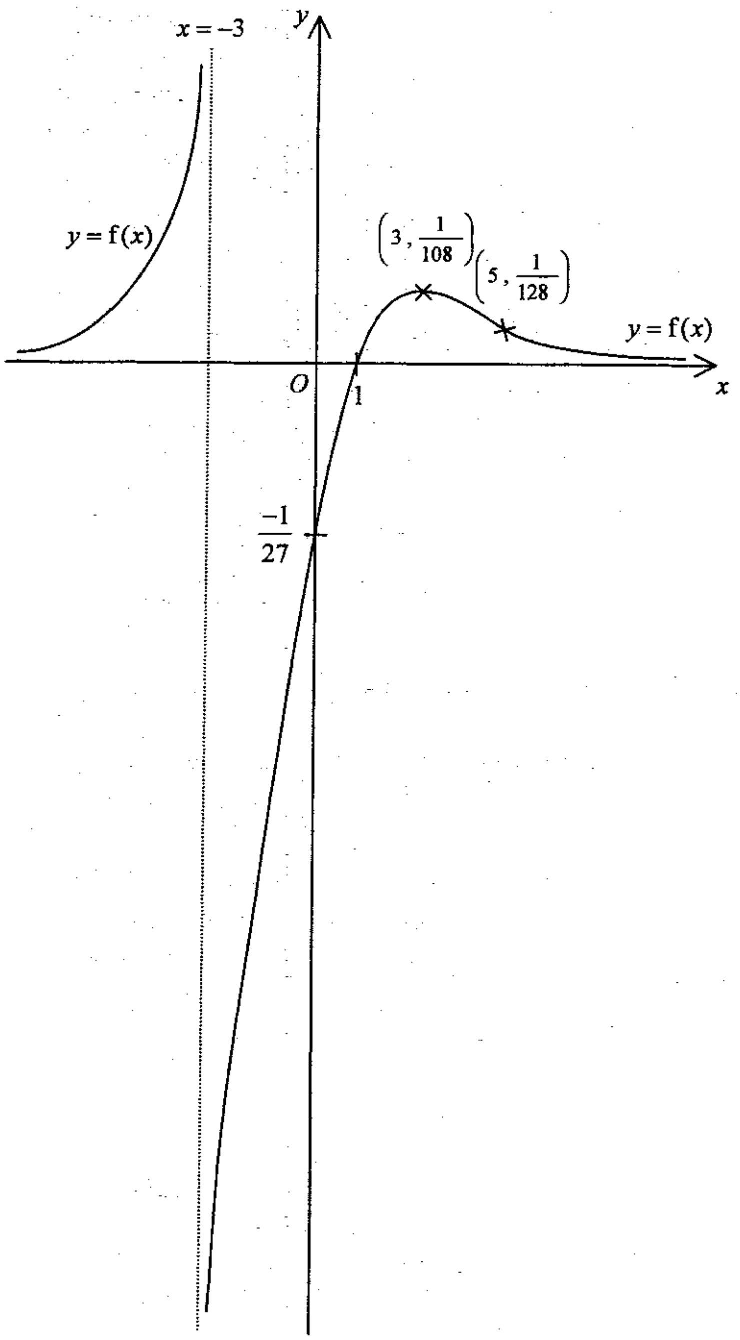
Solution	Marks
6. (a) Note that $1^3-13(1)+12=0$.	1M
So, $x-1$ is a factor of $x^3-13x+12$.	
$x^3-13x+12$	
$=(x-1)(x^2+x-12)$	
$=(x-1)(x-3)(x+4)$	1A
(b) (i) Case 1: $p\neq 0$	
Since $x=4t$ and $y=2t^2+1$, we have $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{4t}{4}=t$.	1M
The equation of the normal to Γ at P is	
$y-2p^2-1=\frac{-1}{p}(x-4p)$	1M
$-py+2p^3+p=x-4p$	
$x+py-2p^3-5p=0$	1A or equivalent
Case 2: $p=0$	
Under this case, P is the point $(0,1)$.	
The equation of the normal to Γ at P is $x=0$.	1A
Thus, by combining the above two cases, the equation of the normal is $x+py-2p^3-5p=0$.	
(ii) Putting $x=-24$ and $y=31$ in $x+py-2p^3-5p=0$,	
we have $-24+31p-2p^3-5p=0$.	1M
Simplifying, we have $p^3-13p+12=0$.	
By (a), we have $(p-1)(p-3)(p+4)=0$.	
Thus, we have $p=1$, $p=3$ or $p=-4$.	1A for all correct
	----------(8)

Solution	Marks
7. (a) $f'(x)=\dfrac{-2(x-3)}{(x+3)^4}$	1A or equivalent
$f''(x)=\dfrac{6(x-5)}{(x+3)^5}$	1A or equivalent
	----------(2)

(b) Note that $f(x)=0 \Leftrightarrow x=1$, $f'(x)=0 \Leftrightarrow x=3$ and $f''(x)=0 \Leftrightarrow x=5$.

x	$(-\infty,-3)$	$(-3, 1)$	1	$(1,3)$	3	$(3,5)$	5	$(5,\infty)$
$f(x)$	+	−	0	+	+	+	+	+
$f'(x)$	+	+	+	+	0	−	−	−
$f''(x)$	+	−	−	−	−	−	0	+

(i) $f(x)>0 \Leftrightarrow x<-3$ or $x>1$	1A
(ii) $f'(x)>0 \Leftrightarrow x<-3$ or $-3<x<3$	1A accept $x<3$
(iii) $f''(x)>0 \Leftrightarrow x<-3$ or $x>5$	1A
	------------(3)
(c) From the table in (b), the maximum point is $\left(3,\dfrac{1}{108}\right)$ and the point of	1A
inflexion is $\left(5,\dfrac{1}{128}\right)$.	1A
	------------(2)
(d) $\because$ $\lim_{x\to-3^-} f(x)=\lim_{x\to-3^-}\dfrac{x-1}{(x+3)^3}=+\infty$ and	
$\lim_{x\to-3^+} f(x)=\lim_{x\to-3^+}\dfrac{x-1}{(x+3)^3}=-\infty$	
$\therefore$ the vertical asymptote is $x=-3$.	1A
$\because$ $\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\dfrac{x-1}{(x+3)^3}=0$	
$\therefore$ the horizontal asymptote is $y=0$.	1A
	------------(2)

Solution	Marks
(e) 	1M for the maximum point and the point of inflexion 1A for the asymptotes 1A for all correct ----------(3)
(f) With the help of the graph of $y = f(x)$, we have $n(k) = \begin{cases} 1 \text{ when } k \le 0 \text{ or } k > \frac{1}{108}, \\ 2 \text{ when } k = \frac{1}{108}, \\ 3 \text{ when } 0 < k < \frac{1}{108}. \end{cases}$	1M for considering different cases 1A for $n(k) = 1$, 2 or 3 1A for all correct ----------(3)

Solution	Marks
8. (a) $y=\dfrac{1}{1+x^2}$	
$(1+x^2)y=1$	
Differentiating both sides $(n+2)$ times, we have	1M
$(1+x^2)y^{(n+2)}+C_1^{n+2}(2x)y^{(n+1)}+C_2^{n+2}(2)y^{(n)}=0$	
$(1+x^2)y^{(n+2)}+2(n+2)x\,y^{(n+1)}+(n+2)(n+1)y^{(n)}=0$	1
	----------(2)
(b) (i) $(1+x^2)y^{(n+2)}+2(n+2)x\,y^{(n+1)}+(n+2)(n+1)y^{(n)}=0$	
$(1+x^2)^{n+2}\left((1+x^2)y^{(n+2)}+2(n+2)x\,y^{(n+1)}+(n+2)(n+1)y^{(n)}\right)=0$	
$f_{n+2}(x)+2(n+2)x\,f_{n+1}(x)+(n+1)(n+2)(1+x^2)f_n(x)=0$	1
(ii) $f_n(x)=(1+x^2)^{n+1}y^{(n)}$	
$\dfrac{df_n(x)}{dx}=(1+x^2)^{n+1}y^{(n+1)}+(n+1)(1+x^2)^n(2x)y^{(n)}$	1A
$\dfrac{d^2f_n(x)}{dx^2}=(1+x^2)^{n+1}y^{(n+2)}+2(n+1)(1+x^2)^n(2x)y^{(n+1)}$	
$+\left(2x(n+1)n(1+x^2)^{n-1}(2x)+(n+1)(1+x^2)^n(2)\right)y^{(n)}$	1A
$(1+x^2)\dfrac{d^2f_n(x)}{dx^2}=(1+x^2)^{n+2}y^{(n+2)}+4(n+1)(1+x^2)^{n+1}x\,y^{(n+1)}$	
$+\left(4nx^2+2(1+x^2)\right)(n+1)(1+x^2)^n y^{(n)}$	
$-2nx\dfrac{df_n(x)}{dx}=-2nx\left((1+x^2)^{n+1}y^{(n+1)}+(n+1)(1+x^2)^n(2x)y^{(n)}\right)$	
$=-2n(1+x^2)^{n+1}x\,y^{(n+1)}-4n(n+1)(1+x^2)^n x^2y^{(n)}$	
$(1+x^2)\dfrac{d^2f_n(x)}{dx^2}-2nx\dfrac{df_n(x)}{dx}+n(n+1)f_n(x)$	
$=(1+x^2)^{n+2}y^{(n+2)}+\left(4(n+1)-2n\right)(1+x^2)^{n+1}x\,y^{(n+1)}$	
$+\left(4nx^2+2(1+x^2)-4nx^2+n(1+x^2)\right)(n+1)(1+x^2)^n y^{(n)}$	
$=(1+x^2)^{n+1}\left((1+x^2)y^{(n+2)}+2(n+2)x\,y^{(n+1)}+(n+1)(n+2)y^{(n)}\right)$	
$=0$	1
	----------(4)

Solution	Marks
(c) Note that $f_1(x)=(1+x^2)^2y'=(1+x^2)^2\left(\dfrac{-2x}{(1+x^2)^2}\right)=-2x$.	
So, $f_1(x)$ is a polynomial of degree 1 and the coefficient of x is $(-1)^1 2!$. Therefore, the statement is true for $n=1$.	1
Also note that $y''=\dfrac{-2}{(1+x^2)^2}+\dfrac{(-2x)(-2)(2x)}{(1+x^2)^3}=\dfrac{6x^2-2}{(1+x^2)^3}$.	
Hence, $f_2(x)=6x^2-2$ which is a polynomial of degree 2 and the coefficient of x^2 is $6=(-1)^2 3!$. So, the statement is also true for $n=2$.	1
Assuming that the statement is true for $n=k$ and $n=k+1$, $f_k(x)$ and $f_{k+1}(x)$ are polynomials with leading terms $(-1)^k(k+1)!x^k$ and $(-1)^{k+1}(k+2)!x^{k+1}$ respectively.	1M
Note that $f_{k+2}(x)=-2(k+2)xf_{k+1}(x)-(k+1)(k+2)(1+x^2)f_k(x)$. So, $f_{k+2}(x)$ is a polynomial of degree at most $k+2$.	1
The coefficient of x^{k+2} in $f_{k+2}(x)$ $=-2(k+2)(-1)^{k+1}(k+2)!-(k+1)(k+2)(-1)^k(k+1)!$	1M
$=(-1)^{k+2}(k+2)!(2(k+2)-(k+1))$ $=(-1)^{k+2}(k+3)!$	1
Hence, $f_{k+2}(x)$ is a polynomial of degree $k+2$ and the coefficient of x^{k+2} is $(-1)^{k+2}(k+3)!$ So, the statement is also true for $n=k+2$. Thus, $f_n(x)$ is a polynomial of degree n and the coefficient of x^n is $(-1)^n(n+1)!$.	
	----------(6)
(d) Let a be the coefficient of x^{n-1} in $f_n(x)$.	
By comparing the coefficients of x^{n-1} in both sides of $(1+x^2)\dfrac{d^2f_n(x)}{dx^2}-2nx\dfrac{df_n(x)}{dx}+n(n+1)f_n(x)=0$, we have $(n-1)(n-2)a-2n(n-1)a+n(n+1)a=0$. So, we have $2a=0$.	1M
Hence, we have $a=0$.	1A
With the help of (c), we have $\sum_{k=1}^{n}\alpha_k=\dfrac{(-1)^{n+1}a}{(n+1)!}$.	
Thus, we have $\sum_{k=1}^{n}\alpha_k=0$.	1M
	----------(3)

Solution	Marks
9. (a) Since $\frac{2^1}{1!}=2<4=\frac{4}{1}$, the result holds for $n=1$.	
Since $\frac{2^2}{2!}=\frac{4}{2}$, the result also holds for $n=2$.	
When $n\geq 3$,	
$\frac{2^n}{n!}$	
$=\frac{(2)(2)(2)\cdots(2)}{(1)(2)(3)\cdots(n)}$	
$=2\left(\frac{2}{1}\right)\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{2}{n-1}\right)\left(\frac{1}{n}\right)$	
$=4\left(\frac{2}{2}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{2}{n-1}\right)\left(\frac{1}{n}\right)$	1M
$\leq 4(1)^{n-2}\left(\frac{1}{n}\right)$	
$=\frac{4}{n}$	
Thus, we have $\frac{2^n}{n!}\leq\frac{4}{n}$ for all positive integers n .	1
Since $\frac{2^1}{1!}=2<4=\frac{4}{1}$, the statement is true for $n=1$.	
Since $\frac{2^2}{2!}=\frac{4}{2}$, the statement is also true for $n=2$.	
Assume that $\frac{2^k}{k!}\leq\frac{4}{k}$ for some integer $k\geq 2$.	
$\frac{2^{k+1}}{(k+1)!}$	
$=\frac{2^k}{k!}\left(\frac{2}{k+1}\right)$	
$\leq\left(\frac{4}{k}\right)\left(\frac{2}{k+1}\right)$	1M for using induction assumption
$=\left(\frac{4}{k+1}\right)\left(\frac{2}{k}\right)$	
$\leq\frac{4}{k+1}$	
By mathematical induction, we have $\frac{2^n}{n!}\leq\frac{4}{n}$.	1
Note that $0<\frac{2^n}{n!}\leq\frac{4}{n}$ for all integers n .	
Since $\lim\limits_{n\to\infty}\frac{4}{n}=0$, we have $\lim\limits_{n\to\infty}\frac{2^n}{n!}=0$.	1
	----------(3)

Solution	Marks
(b) (i) I_1	
$= \int_1^e x^{-3} \ln x \, dx$	
$= \left[\frac{-x^{-2} \ln x}{2} \right]_1^e + \frac{1}{2} \int_1^e x^{-3} \, dx$	1M for integration by parts (either one)
$= \frac{1}{4} - \frac{3}{4e^2}$	1A
(ii) I_{n+1}	
$= \int_1^e x^{-3} (\ln x)^{n+1} dx$	
$= \left[\frac{-x^{-2} (\ln x)^{n+1}}{2} \right]_1^e + \frac{n+1}{2} \int_1^e x^{-3} (\ln x)^n dx$	(1M for integration by parts, either one)
$= \frac{n+1}{2} I_n - \frac{1}{2e^2}$	1A
(iii) I_n	
$= \frac{n}{2} I_{n-1} - \frac{1}{2e^2}$	
$= \frac{n}{2}\left(\frac{n-1}{2} I_{n-2} - \frac{1}{2e^2} \right) - \frac{1}{2e^2}$	1M
$= \cdots$	
$= \frac{n(n-1)\cdots 2}{2^{n-1}} I_1 - \frac{1}{2e^2}\left(1 + \frac{n}{2} + \frac{n(n-1)}{2^2} + \cdots + \frac{n(n-1)\cdots 3}{2^{n-2}} \right)$	
$= \frac{n(n-1)\cdots 2}{2^{n-1}}\left(\frac{1}{4} - \frac{3}{4e^2} \right) - \frac{1}{2e^2}\left(1 + \frac{n}{2} + \frac{n(n-1)}{2^2} + \cdots + \frac{n(n-1)\cdots 3}{2^{n-2}} \right)$	
$= \frac{n(n-1)\cdots 2}{2^{n-1}}\left(\frac{1}{4} - \frac{1}{2e^2} - \frac{1}{4e^2} \right) - \frac{1}{2e^2}\left(1 + \frac{n}{2} + \frac{n(n-1)}{2^2} + \cdots + \frac{n(n-1)\cdots 3}{2^{n-2}} \right)$	
$= \frac{n!}{2^{n+1}} - \frac{1}{2e^2}\left(1 + \frac{n}{2} + \cdots + \frac{n(n-1)\cdots 3}{2^{n-2}} + \frac{n(n-1)\cdots 2}{2^{n-1}} + \frac{n(n-1)\cdots 2}{2^n} \right)$	
$= \frac{n!}{2^{n+1}} - \frac{n!}{2e^2}\left(\frac{1}{n!} + \frac{1}{(n-1)!\,2} + \cdots + \frac{1}{2!\,2^{n-2}} + \frac{1}{2^{n-1}} + \frac{1}{2^n} \right)$	
$= n!\left(\frac{1}{2^{n+1}} - \frac{1}{2e^2} \sum_{k=0}^{n} \frac{1}{(n-k)!\,2^k} \right)$	
$= n!\left(\frac{1}{2^{n+1}} - \frac{1}{e^2} \sum_{k=0}^{n} \frac{1}{(n-k)!\,2^{k+1}} \right)$	1

Solution	Marks
Note that $1!\left(\frac{1}{2^{1+1}}-\frac{1}{e^2}\sum_{k=0}^{1}\frac{1}{(1-k)!2^{k+1}}\right)=\frac{1}{4}-\frac{3}{4e^2}=I_1$ (by (b)(i)).	
So, the statement is true for $n=1$.	
Assume that $I_s=s!\left(\frac{1}{2^{s+1}}-\frac{1}{e^2}\sum_{k=0}^{s}\frac{1}{(s-k)!2^{k+1}}\right)$, where $s\geq 1$.	
I_{s+1}	
$=\frac{s+1}{2}I_s-\frac{1}{2e^2}$	
$=\frac{(s+1)!}{2}\left(\frac{1}{2^{s+1}}-\frac{1}{e^2}\sum_{k=0}^{s}\frac{1}{(s-k)!2^{k+1}}\right)-\frac{1}{2e^2}$	1M
$=(s+1)!\left(\frac{1}{2^{s+2}}-\frac{1}{e^2}\sum_{k=0}^{s}\frac{1}{(s-k)!2^{k+2}}\right)-\frac{1}{2e^2}$	
$=(s+1)!\left(\frac{1}{2^{s+2}}-\frac{1}{e^2}\sum_{k=1}^{s+1}\frac{1}{(s+1-k)!2^{k+1}}\right)-\frac{1}{2e^2}$	
$=(s+1)!\left(\frac{1}{2^{s+2}}-\frac{1}{e^2}\sum_{k=0}^{s+1}\frac{1}{(s+1-k)!2^{k+1}}\right)$	
So, the statement is true for $n=s+1$.	
By induction, we have $I_n=n!\left(\frac{1}{2^{n+1}}-\frac{1}{e^2}\sum_{k=0}^{n}\frac{1}{(n-k)!2^{k+1}}\right)$.	1
(iv) For all $x\in[1,e]$, we have $e^{-2}\leq x^{-2}\leq 1$ and $x^{-1}(\ln x)^n\geq 0$.	
So, we have $e^{-2}x^{-1}(\ln x)^n\leq x^{-3}(\ln x)^n\leq x^{-1}(\ln x)^n$ for all $x\in[1,e]$.	1
Then, we have $e^{-2}\int_1^e x^{-1}(\ln x)^n\,dx\leq\int_1^e x^{-3}(\ln x)^n\,dx\leq\int_1^e x^{-1}(\ln x)^n\,dx$	1M
Therefore, we have $e^{-2}\left[\frac{(\ln x)^{n+1}}{(n+1)}\right]_1^e\leq I_n\leq\left[\frac{(\ln x)^{n+1}}{n+1}\right]_1^e$.	
Thus, we have $\frac{1}{e^2(n+1)}\leq I_n\leq\frac{1}{n+1}$.	1
	----------(8)

Solution	Marks
(c) $\frac{2^{n+1}}{n!}I_n$	
$=1-\frac{1}{e^2}\sum_{k=0}^{n}\frac{2^{n+1}}{(n-k)!2^{k+1}}$ (by (b)(iii))	1M (for either)
$=1-\frac{1}{e^2}\sum_{k=0}^{n}\frac{2^{n-k}}{(n-k)!}$	
$=1-\frac{1}{e^2}\sum_{k=0}^{n}\frac{2^{k}}{k!}$	1M (for either)
By (b)(iv), we have $\frac{2^{n+1}}{e^2(n+1)!}\le\frac{2^{n+1}}{n!}I_n\le\frac{2^{n+1}}{(n+1)!}$ for any positive integer n .	
Since $\lim_{n\to\infty}\frac{2^n}{n!}=0$ (by (a)), we have $\lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}=0=\lim_{n\to\infty}\frac{2^{n+1}}{e^2(n+1)!}$.	1M for using (a)
So, we have $\lim_{n\to\infty}\frac{2^{n+1}}{n!}I_n=0$.	
Therefore, we have $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{2^k}{k!}=e^2$.	
Thus, we have $\sum_{k=0}^{\infty}\frac{2^k}{k!}=e^2$.	1A
By (b)(iii) and (b)(iv), we have $\frac{1}{e^2(n+1)}\le n!\left(\frac{1}{2^{n+1}}-\frac{1}{e^2}\sum_{k=0}^{n}\frac{1}{(n-k)!2^{k+1}}\right)\le\frac{1}{n+1}$.	1M for using (b)(iii) or (b)(iv)
Hence, we have $e^2\left(1-\frac{2^{n+1}}{(n+1)!}\right)\le\sum_{k=0}^{n}\frac{2^{n-k}}{(n-k)!}\le e^2-\frac{2^{n+1}}{(n+1)!}$.	
So, we have $e^2\left(1-\frac{2^{n+1}}{(n+1)!}\right)\le\sum_{k=0}^{n}\frac{2^{k}}{k!}\le e^2-\frac{2^{n+1}}{(n+1)!}$.	1M
Since $\lim_{n\to\infty}\frac{2^n}{n!}=0$ (by (a)), we have $\lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}=0$.	1M for using (a)
Then, we have $\lim_{n\to\infty}e^2\left(1-\frac{2^{n+1}}{(n+1)!}\right)=e^2=\lim_{n\to\infty}\left(e^2-\frac{2^{n+1}}{(n+1)!}\right)$.	
Therefore, we have $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{2^k}{k!}=e^2$.	
Thus, we have $\sum_{k=0}^{\infty}\frac{2^k}{k!}=e^2$.	1A
	----------(4)

Solution	Marks
10. (a) (i) (1) $\mathrm{H}'(x)$	
$= 2\mathrm{f}(x)\mathrm{g}(x)\left(\int_0^x \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t\right) - (\mathrm{f}(x))^2 \int_0^x (\mathrm{g}(t))^2\,\mathrm{d}t - (\mathrm{g}(x))^2 \int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t$	1M
$= -\int_0^x \left((\mathrm{f}(x))^2(\mathrm{g}(t))^2 - 2\mathrm{f}(t)\mathrm{g}(t)\mathrm{f}(x)\mathrm{g}(x) + (\mathrm{f}(t))^2(\mathrm{g}(x))^2\right)\mathrm{d}t$	1A
$= -\int_0^x (\mathrm{f}(x)\mathrm{g}(t) - \mathrm{f}(t)\mathrm{g}(x))^2\,\mathrm{d}t$	
≤ 0	1
Thus, H is decreasing on $\mathbf{I}$.	
(2) By (a)(i)(1), we have $\mathrm{H}(x) \le \mathrm{H}(0)$ for all $x \in \mathbf{I}$.	1M for using (a)(i)(1)
So, we have $\left(\int_0^x \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t\right)^2 - \int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t \int_0^x (\mathrm{g}(t))^2\,\mathrm{d}t \le 0$.	
Thus, we have $\left(\int_0^x \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t\right)^2 \le \left(\int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t\right)\left(\int_0^x (\mathrm{g}(t))^2\,\mathrm{d}t\right)$.	1
Note that the result is true for $x = 0$.	
Let $x \in (0, 1]$.	
Case 1: $\int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t = 0$	
Under this case, we have $\mathrm{f}(t) = 0$ for all $t \in [0, x]$. Hence, the result is true.	
Case 2: $\int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t > 0$	
Since $\int_0^x (\lambda\,\mathrm{f}(t) - \mathrm{g}(t))^2\,\mathrm{d}t \ge 0$ for all $\lambda \in \mathbf{R}$, we have	1M
$\left(\int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t\right)\lambda^2 - 2\left(\int_0^x \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t\right)\lambda + \int_0^x (\mathrm{g}(t))^2\,\mathrm{d}t \ge 0$.	
So, we have $\left(-2\int_0^x \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t\right)^2 - 4\int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t \int_0^x (\mathrm{g}(t))^2\,\mathrm{d}t \le 0$.	
Thus, we have $\left(\int_0^x \mathrm{f}(t)\mathrm{g}(t)\,\mathrm{d}t\right)^2 \le \left(\int_0^x (\mathrm{f}(t))^2\,\mathrm{d}t\right)\left(\int_0^x (\mathrm{g}(t))^2\,\mathrm{d}t\right)$.	1
(ii) (1) Note that $\mathrm{h}(x) = \mathrm{h}(x) - \mathrm{h}(0) = \int_0^x \mathrm{h}'(t)\,\mathrm{d}t$ for all $x \in \mathbf{I}$.	1A
$(\mathrm{h}(x))^2$	
$= \left(\int_0^x \mathrm{h}'(t)\,\mathrm{d}t\right)^2$	
$\le \left(\int_0^x \mathrm{d}t\right)\left(\int_0^x (\mathrm{h}'(t))^2\,\mathrm{d}t\right)$ (by (a)(i)(2))	1M for using (a)(i)(2)
$= x\int_0^x (\mathrm{h}'(t))^2\,\mathrm{d}t$	1
Thus, we have $(\mathrm{h}(x))^2 \le x\int_0^x (\mathrm{h}'(t))^2\,\mathrm{d}t$ for all $x \in \mathbf{I}$.	

Solution	Marks
(2) By (a)(ii)(1), we have $(\mathrm{h}(x))^2 \le x\int_0^x (\mathrm{h}'(t))^2\,\mathrm{d}t$ for all $x \in \mathbf{I}$.	
So, we have $(\mathrm{h}(x))^2 \le x\int_0^1 (\mathrm{h}'(t))^2\,\mathrm{d}t$ for all $x \in \mathbf{I}$.	1M
Therefore, we have $\int_0^1 (\mathrm{h}(x))^2\,\mathrm{d}x \le \left(\int_0^1 x\,\mathrm{d}x\right)\left(\int_0^1 (\mathrm{h}'(t))^2\,\mathrm{d}t\right)$.	1M
Note that $\int_0^1 x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$.	
Hence, we have $\int_0^1 (\mathrm{h}(x))^2\,\mathrm{d}x \le \frac{1}{2}\int_0^1 (\mathrm{h}'(t))^2\,\mathrm{d}t$.	
Thus, we have $\int_0^1 (\mathrm{h}(x))^2\,\mathrm{d}x \le \frac{1}{2}\int_0^1 (\mathrm{h}'(x))^2\,\mathrm{d}x$.	1
Let $\mathrm{G}(x) = \int_0^x (\mathrm{h}'(t))^2\,\mathrm{d}t$ for all $x \in \mathbf{I}$.	
$\int_0^1 (\mathrm{h}(x))^2\,\mathrm{d}x$	
$\le \int_0^1 x\mathrm{G}(x)\,\mathrm{d}x$ (by (a)(ii)(1))	1M
$= \left[\frac{x^2}{2}\mathrm{G}(x)\right]_0^1 - \frac{1}{2}\int_0^1 x^2\,\mathrm{G}'(x)\,\mathrm{d}x$	
$= \frac{1}{2}\mathrm{G}(1) - \frac{1}{2}\int_0^1 x^2\,(\mathrm{h}'(x))^2\,\mathrm{d}x$	1M
$= \frac{1}{2}\int_0^1 (\mathrm{h}'(t))^2\,\mathrm{d}t - \frac{1}{2}\int_0^1 x^2\,(\mathrm{h}'(x))^2\,\mathrm{d}x$	
$\le \frac{1}{2}\int_0^1 (\mathrm{h}'(t))^2\,\mathrm{d}t$	
$= \frac{1}{2}\int_0^1 (\mathrm{h}'(x))^2\,\mathrm{d}x$	1
	----------(11)
(b) Let $\mathrm{h}(x) = \ln(\sec x + \tan x)$.	
Then, we have $\mathrm{h}'(x) = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) = \sec x$.	1A
So, we have $\mathrm{h}(0) = 0$ and h' is ontinuous on $\mathbf{I}$.	1M withhold 1M if checking is omitted
$\int_0^1 (\ln(\sec x + \tan x))^2\,\mathrm{d}x$	
$\le \frac{1}{2}\int_0^1 \sec^2 x\,\mathrm{d}x$ (by (a)(ii)(2))	1M for using (a)(ii)(2)
$= \frac{1}{2}[\tan x]_0^1$	
$= \frac{1}{2}\tan 1$	1

Solution	Marks
Let $\mathrm{h}(x)=\ln(\sec x+\tan x)$.	
Then, we have $\mathrm{h}'(x)=\frac{1}{\sec x+\tan x}(\sec x\tan x+\sec^2 x)=\sec x$.	1A
So, we have $\mathrm{h}(0)=0$ and h' is continuous on $\mathbf{I}$.	1M withhold 1M if checking is omitted
By (a)(ii)(1), we have $(\ln(\sec x+\tan x))^2\le x\int_0^x \sec^2 t\,dt$ for all $x\in\mathbf{I}$.	1M for using (a)(ii)(1)
Therefore, we have $(\ln(\sec x+\tan x))^2\le x\tan x$ for all $x\in\mathbf{I}$.	
Hence, we have $(\ln(\sec x+\tan x))^2\le(\tan 1)\,x$ for all $x\in\mathbf{I}$.	
So, we have $\int_0^1(\ln(\sec x+\tan x))^2\,\mathrm{d}x\le\tan 1\int_0^1 x\,\mathrm{d}x$.	
Thus, we have $\int_0^1(\ln(\sec x+\tan x))^2\,\mathrm{d}x\le\frac{1}{2}\tan 1$.	1
	----------(4)

Solution	Marks
11 (a) Assume that $\sin\theta = 0$ or $\sin\phi = 0$.	
Then, P or Q lies on the y-axis.	
Note that PQ passes through A .	
Hence, both P and Q lie on the y-axis.	1M
So, the tangents to E at P and Q are both parallel to the x-axis.	
Therefore, the two tangents do not intersect.	1M
It is impossible.	
Thus, we have $\sin\theta \neq 0$ and $\sin\phi \neq 0$.	1
	------------(3)
(b) (i) The equation of the tangent to E at P is	
$\frac{(12\sin\theta)\,x}{144} + \frac{(20\cos\theta)\,y}{400} = 1$.	1M
Thus, the required equation is $(5\sin\theta)\,x + (3\cos\theta)\,y = 60$.	1A
Case 1: $\cos\theta \neq 0$	
Differentiating both sides of $\frac{x^2}{144} + \frac{y^2}{400} = 1$ with respect to x ,	
we have $\left.\frac{dy}{dx}\right\|_P = \frac{-5\sin\theta}{3\cos\theta}$.	
The equation of the tangent to E at P is	
$y - 20\cos\theta = \frac{-5\sin\theta}{3\cos\theta}(x - 12\sin\theta)$	1M
$(5\sin\theta)\,x + (3\cos\theta)\,y = 60$	1A
Case 2: $\cos\theta = 0$ and $\sin\theta = 1$	
The equation of the tangent to E at P is $x = 12$.	
Case 3: $\cos\theta = 0$ and $\sin\theta = -1$	
The equation of the tangent to E at P is $x = -12$.	
Thus, by combining the above three cases, the required equation is $(5\sin\theta)\,x + (3\cos\theta)\,y = 60$.	
The equation of the tangent to E at Q is $(5\sin\phi)\,x + (3\cos\phi)\,y = 60$.	1M
(ii) Note that the tangents to E at P and Q intersect at the point R .	
Hence, the system of linear equations $\begin{cases} (5\sin\theta)\,x + (3\cos\theta)\,y = 60 \\ (5\sin\phi)\,x + (3\cos\phi)\,y = 60 \end{cases}$	
has a unique solution.	
Therefore, we have $\begin{vmatrix} 5\sin\theta & 3\cos\theta \\ 5\sin\phi & 3\cos\phi \end{vmatrix} \neq 0$.	1M
So, we have $15\sin\theta\cos\phi - 15\sin\phi\cos\theta \neq 0$.	
Then, we have $\sin\theta\cos\phi - \sin\phi\cos\theta \neq 0$.	
Thus, we have $\sin(\theta - \phi) \neq 0$.	1

Solution	Marks
Assume that $\sin(\theta-\phi)=0$.	
Then, we have $\theta-\phi=n\pi$, where n is an integer.	
So, we have $\theta=n\pi+\phi$, where n is an integer.	
Therefore, we have $\tan\theta=\tan\phi$.	
Hence, we have $\dfrac{-5\tan\theta}{3}=\dfrac{-5\tan\phi}{3}$.	
By (b)(i), the slopes of the tangents to E at P and Q are the same.	
Hence, the tangents to E at P and Q are parallel to each other.	1M
Therefore, the two tangents do not intersect.	
It is impossible.	
Thus, we have $\sin(\theta-\phi)\neq 0$.	1
	------------(5)
(c) (i) As PQ passes through $A(0,16)$,	
we have $\dfrac{20\cos\theta-16}{12\sin\theta-0}=\dfrac{20\cos\phi-16}{12\sin\phi-0}$.	1M
Then, we have $\dfrac{5\cos\theta-4}{\sin\theta}=\dfrac{5\cos\phi-4}{\sin\phi}$.	
Therefore, we have $(5\cos\phi-4)\sin\theta=(5\cos\theta-4)\sin\phi$.	
Thus, we have $4(\sin\theta-\sin\phi)=5\sin(\theta-\phi)$.	1
(ii) The equation of the tangent to E at P is $(5\sin\theta)\,x+(3\cos\theta)\,y=60$.	
The equation of the tangent to E at Q is $(5\sin\phi)\,x+(3\cos\phi)\,y=60$.	
Solving, we have $y\sin(\theta-\phi)=20(\sin\theta-\sin\phi)$.	
By (c)(i), we have $y\sin(\theta-\phi)=25\sin(\theta-\phi)$.	
By (b)(ii), we have $y=25$.	1
Putting $y=25$ in $(5\sin\theta)x+(3\cos\theta)y=60$, we have $x=\dfrac{3(4-5\cos\theta)}{\sin\theta}$.	1
Thus, the coordinates of R are $\left(\dfrac{3(4-5\cos\theta)}{\sin\theta},25\right)$.	
	------------(4)
(d) Note that $PA^2=(12\sin\theta)^2+(20\cos\theta-16)^2=16(5-4\cos\theta)^2$.	1M (any one)
Also note that $RA^2=\left(\dfrac{3(4-5\cos\theta)}{\sin\theta}\right)^2+(25-16)^2=\dfrac{9(5-4\cos\theta)^2}{\sin^2\theta}$.	
So, we have $PA^2+RA^2=\dfrac{(5-4\cos\theta)^2(9+16\sin^2\theta)}{\sin^2\theta}$.	
Further note that $PR^2=\left(12\sin\theta-\dfrac{3(4-5\cos\theta)}{\sin\theta}\right)^2+(20\cos\theta-25)^2$.	
Simplifying, we have $PR^2=\dfrac{(5-4\cos\theta)^2(9+16\sin^2\theta)}{\sin^2\theta}$.	
Hence, we have $PA^2+RA^2=PR^2$.	1M
Thus, ΔPAR is a right-angled triangle.	1A f.t.

Solution	Marks
Case 1: $\cos\theta \neq \frac{4}{5}$	
The slope of RA $= \frac{25-16}{\frac{3(4-5\cos\theta)}{\sin\theta}-0}$	1M (either one)
$= \frac{3\sin\theta}{4-5\cos\theta}$	
The slope of PA $= \frac{20\cos\theta-16}{12\sin\theta}$	
$= \frac{5\cos\theta-4}{3\sin\theta}$	
So, the product of the slope of RA and the slope of PA is -1.	1M
Thus, ΔPAR is a right-angled triangle.	1A f.t.
Case 2: $\cos\theta = \frac{4}{5}$	
Under this case, RA is vertical and PA is horizontal.	
So, we have $\angle PAR = 90°$.	
Thus, ΔPAR is a right-angled triangle.	
By combining the above two cases, ΔPAR is a right-angled triangle.	
The slope of PQ $= \frac{20\cos\theta-16}{12\sin\theta-0}$	
$= \frac{5\cos\theta-4}{3\sin\theta}$	
So, the equation of PQ is $y-16=\frac{5\cos\theta-4}{3\sin\theta}(x-0)$.	
Hence, the equation of PQ is $(5\cos\theta-4)x-3\sin\theta y+48\sin\theta=0$.	
The perpendicular distance from R to PQ $= \frac{\left\lvert (5\cos\theta-4)\frac{3(4-5\cos\theta)}{\sin\theta}-3\sin\theta(25)+48\sin\theta \right\rvert}{\sqrt{(5\cos\theta-4)^2+9\sin^2\theta}}$	1M
$= \frac{3(5\cos\theta-4)^2+9\sin^2\theta}{\sin\theta\sqrt{(5\cos\theta-4)^2+9\sin^2\theta}}$	
$= \frac{3\sqrt{(5\cos\theta-4)^2+9\sin^2\theta}}{\sin\theta}$	
$= \frac{3\sqrt{(5-4\cos\theta)^2}}{\sin\theta}$	
$= \frac{3(5-4\cos\theta)}{\sin\theta}$	
Also note that $RA=\sqrt{\left(\frac{3(4-5\cos\theta)}{\sin\theta}\right)^2+(25-16)^2}=\frac{3(5-4\cos\theta)}{\sin\theta}$.	1M
Hence, the perpendicular distance from R to PQ is equal to RA.	
So, PA is perpendicular to RA.	
Thus, ΔPAR is a right-angled triangle.	1A f.t.
	------------(3)