2010-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2010

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(C) answer book.

4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

SECTION A (40 marks)

Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Let n be a positive integer. Denote the coefficient of x^k in the expansion of $(1+2x)^n$ by a_k.

Prove that

(a) $\sum_{k=0}^{n} a_k = 3^n$,

(b) $\sum_{k=1}^{n} k a_k = 2n3^{n-1}$,

(c) $\sum_{k=0}^{n} (3k+1) a_k = (2n+1)3^n$.

(7 marks)

2. (a) Resolve $\frac{x}{(x^2-1)(x^2-4)}$ into partial fractions.

(b) By differentiating $\frac{x}{(x^2-1)(x^2-4)}$, or otherwise, resolve $\frac{3x^4-5x^2-4}{(x^2-1)^2(x^2-4)^2}$ into partial fractions.

(c) Evaluate $\sum_{k=3}^{\infty} \frac{3k^4-5k^2-4}{(k^2-1)^2(k^2-4)^2}$.

(6 marks)

3. Let $f(x)=x^3+g(x)$, where $g(x)$ is a quadratic polynomial with real coefficients. When $f(x)$ is divided by $(x-1)(x-4)$ and when $f(x)$ is divided by $(x-4)^2$, the remainders are $-x+k$ and $kx-10$ respectively, where k is a real number. Find

(a) k ,

(b) $g(x)$,

(c) the remainder when $(g(x))^3$ is divided by $x+1$.

(7 marks)

Go on to the next page

4. (a) Write down the matrix which represents the anticlockwise rotation about the origin by $\frac{\pi}{2}$ in the Cartesian plane.

(b) Let O be the origin. It is given that O, $P(1,3)$ and Q are the vertices of an isosceles triangle, where $\angle POQ = \frac{\pi}{2}$ and Q lies in the second quadrant.

(i) Find the coordinates of Q.

(ii) Let T be the transformation which transforms the points $(1,0)$ and $(0,1)$ to the points $(0,-1)$ and $(-1,0)$ respectively and M be the 2×2 real matrix which represents the transformation T.

(1) Write down the matrix M.

(2) Describe the geometric meaning of the transformation T.

(3) The transformation T transforms O, P and Q to O', P' and Q' respectively. Find the area of $\Delta O'P'Q'$.

(7 marks)

5. Let $S = \{ z \in \mathbf{C} : z\bar{z} = (12+16i)z + (12-16i)\bar{z} - 375 \}$.

(a) Prove that S is represented by a circle on the Argand diagram. Also find the centre and the radius of the circle.

(b) Find $z_1 \in S$ such that $|z_1| \le |z|$ for all $z \in S$.

(7 marks)

6. Let α, β, γ and δ be positive real numbers. Prove that

(a) $(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right) \ge 16$,

(b) $\frac{3}{\beta+\gamma+\delta}+\frac{3}{\gamma+\delta+\alpha}+\frac{3}{\delta+\alpha+\beta}+\frac{3}{\alpha+\beta+\gamma} \ge \frac{16}{\alpha+\beta+\gamma+\delta}$,

(c) $\frac{\alpha}{\beta+\gamma+\delta}+\frac{\beta}{\gamma+\delta+\alpha}+\frac{\gamma}{\delta+\alpha+\beta}+\frac{\delta}{\alpha+\beta+\gamma} \ge \frac{4}{3}$.

(6 marks)

SECTION B (60 marks)

Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. (a) Consider the system of linear equations in x, y, z

$$(E):\begin{cases} x + y + z = 2 \\ ax - 4z = 2 \\ 3x + 4y + (a+4)z = b \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Find the range of values of a for which (E) has a unique solution, and solve (E) when (E) has a unique solution.

(ii) Suppose that $a = 2$. Find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b.

(8 marks)

(b) Consider the system of linear equations in x, y, z

$$(F):\begin{cases} x + y + z = 2 \\ x + 2z = -1 \\ 3x + 4y + 2z = \lambda \\ 7x + 17y - 3z = \mu \end{cases}, \text{ where } \lambda, \mu \in \mathbf{R}.$$

Find the values of λ and μ for which (F) is consistent.

(4 marks)

(c) Consider the system of linear equations in x, y, z

$$(G):\begin{cases} x + y + z = 2 \\ x - 6z = 3 \\ 9x + 12y + 14z = 15 \\ 5x - 2y - 18z = 16 \end{cases}.$$

Is (G) consistent? Explain your answer.

(3 marks)

8. (a) Let λ be a real number and $p(x)$ be a polynomial with real coefficients. Prove that λ is a repeated root of the equation $p(x) = 0$ if and only if $p(\lambda) = p'(\lambda) = 0$.

(4 marks)

(b) Let $f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1$, where a, b and c are real numbers. Suppose that a real number μ is a repeated root of the equation $f(x) = 0$. Prove that

(i) $\mu \ne 0$,

(ii) $\frac{1}{\mu}$ is a repeated root of the equation $f(x) = 0$.

(6 marks)

(c) Let $g(x) = 4x^6 - 16x^5 + 17x^4 - 7x^3 + 17x^2 - 16x + 4$.

(i) Find a repeated root of the equation $g(x) = 0$.

(ii) Can $g(x)$ be factorized as a product of linear polynomials with real coefficients? Explain your answer.

(5 marks)

Go on to the next page

9. Let x_1 and y_1 be real numbers. For any positive integer n, define

$$x_{n+1} = \frac{5}{6}x_n + \frac{1}{6}y_n \quad \text{and} \quad y_{n+1} = \frac{2}{9}x_n + \frac{7}{9}y_n .$$

(a) Suppose that $x_1 > y_1$.

(i) Prove that $x_n > y_n$.

(ii) Prove that $\{x_n\}$ is a strictly decreasing sequence and $\{y_n\}$ is a strictly increasing sequence.

(iii) Prove that $\{x_n\}$ and $\{y_n\}$ are convergent sequences.

(iv) Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

(v) Prove that $4x_{n+1} + 3y_{n+1} = 4x_n + 3y_n$.

(vi) Express $\lim_{n\to\infty} x_n$ in terms of x_1 and y_1.

(12 marks)

(b) Suppose that $x_1 < y_1$. Are $\{x_n\}$ and $\{y_n\}$ convergent sequences? Explain your answer.

(3 marks)

10. Denote the 2×2 identity matrix by I. A 2×2 real matrix M is said to be orthogonal if and only if $MM^T = I$, where M^T is the transpose of M.

(a) Prove that a 2×2 real matrix M is orthogonal if and only if there exists $\theta \in \mathbf{R}$ such that

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} .$$

(6 marks)

(b) (i) Suppose that $M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Prove that $M^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$ for all positive integers n.

(ii) Suppose that $M = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$. Evaluate M^n for all positive integers n.

(3 marks)

(c) Find all 2×2 real orthogonal matrices X such that $X^{400} = -I$.

(4 marks)

(d) Suppose that M is a 2×2 real orthogonal matrix. Is M^{401} orthogonal? Explain your answer.

(2 marks)

11. (a) (i) Prove that $\ln x \le x - 1$ for all $x > 0$.

(ii) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = a_1b_1 + a_2b_2 + \cdots + a_nb_n = 1$. Prove that $b_1^{a_1} b_2^{a_2} \cdots b_n^{a_n} \le 1$.

(4 marks)

(b) (i) Prove that $x^x \ge x$ for all $x > 0$.

(ii) Let $c_1, c_2, \ldots, c_n$ be positive real numbers satisfying $c_1c_2\cdots c_n = 1$. Prove that $c_1^{c_1} c_2^{c_2} \cdots c_n^{c_n} \ge 1$.

(3 marks)

(c) Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

(i) $$x_1^{x_1} x_2^{x_2} \cdots x_n^{x_n} \le \left(\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} \right)^{x_1 + x_2 + \cdots + x_n} ,$$

(ii) $$x_1^{x_1} x_2^{x_2} \cdots x_n^{x_n} \ge \left(\sqrt[n]{x_1 x_2 \cdots x_n} \right)^{x_1 + x_2 + \cdots + x_n} .$$

(8 marks)

END OF PAPER

2010-AL
P MATH
PAPER 2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2010

PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(E) answer book.
3. Answer any FOUR questions in Section B, using the AL(C) answer book.
4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Evaluate $\lim_{x\to 0}\dfrac{x-\sin x}{x\sin x}$.

(b) Let a and b be real constants and $f:(-\pi,\pi)\to\mathbf{R}$ be defined by

$$f(x)=\begin{cases}\sin x+\dfrac{ax}{\sin x} & \text{when } -\pi<x<0\,,\\ 3+bx+x^2 & \text{when } 0\le x<\pi\,.\end{cases}$$

If $f(x)$ is differentiable at $x=0$, find a and b .

(7 marks)

2. Let $f:\mathbf{R}\to\mathbf{R}$ and $g:\mathbf{R}\to\mathbf{R}$ be continuous functions such that

$$f(x)=3+\int_0^x g(t)\,dt \quad\text{and}\quad g(x)=\int_0^x f(t)\,dt-1 \quad\text{for all } x\in\mathbf{R}\,.$$

(a) Find $f(0)$ and $g(0)$.

(b) Express $f'(x)$ and $g'(x)$ in terms of $f(x)$ and $g(x)$.

(c) By differentiating $h(x)=(f(x))^2-(g(x))^2$, prove that $(f(x))^2-(g(x))^2=8$ for all $x\in\mathbf{R}$.

(6 marks)

3. Let $f:\mathbf{R}\to\mathbf{R}$ be defined by $f(x)=3-x-|x-1|$.

(a) (i) Sketch the graph of $y=f(x)$.

(ii) Is f an injective function? Explain your answer.

(b) Let $g:\mathbf{R}\to\mathbf{R}$ be defined by $g(x)=f(x-1)+f(-x-1)$.

(i) Prove that g is an even function.

(ii) Sketch the graph of $y=g(x)$.

Hence, or otherwise, evaluate $\int_{-4}^{4} g(x)\,dx$.

(7 marks)

Go on to the next page

4. (a) Find $\frac{d}{dx}\left(\frac{\ln x}{x}\right)$.

(b) Evaluate $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{2n}\frac{1-\ln\left(1+\frac{k}{n}\right)}{\left(1+\frac{k}{n}\right)^2}$.

(5 marks)

5. (a) Find $\int(5-x)\sqrt{x-1}\,dx$.

(b) Let D be the region bounded by the curve $y=(5-x)^{\frac{1}{2}}(x-1)^{\frac{1}{4}}$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(7 marks)

6. (a) Factorize $x^3-13x+12$ as a product of linear polynomials.

(b) Consider the curve $\Gamma:\begin{cases}x=4t\\ y=2t^2+1\end{cases}$, where $t\in\mathbf{R}$. Let P be the point $(4p, 2p^2+1)$.

(i) Find the equation of the normal to Γ at P .

(ii) Find p such that the normal to Γ at P passes through the point $(-24, 31)$.

(8 marks)

SECTION B (60 marks)

Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. Let $f:\mathbf{R}\setminus\{-3\}\to\mathbf{R}$ be defined by $f(x)=\frac{x-1}{(x+3)^3}$.

(a) Find $f'(x)$ and $f''(x)$.

(2 marks)

(b) Solve

(i) $f(x)>0$,

(ii) $f'(x)>0$,

(iii) $f''(x)>0$.

(3 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y=f(x)$.

(2 marks)

(d) Find the asymptote(s) of the graph of $y=f(x)$.

(2 marks)

(e) Sketch the graph of $y=f(x)$.

(3 marks)

(f) Let $n(k)$ be the number of points of intersection of the graph of $y=f(x)$ and the horizontal line $y=k$. Using the graph of $y=f(x)$, find $n(k)$ for any $k\in\mathbf{R}$.

(3 marks)

8. Let $y=\frac{1}{1+x^2}$. For any positive integer n , define $f_n(x)=(1+x^2)^{n+1}y^{(n)}$.

(a) Prove that $(1+x^2)y^{(n+2)}+2(n+2)xy^{(n+1)}+(n+2)(n+1)y^{(n)}=0$.

(2 marks)

(b) Prove that

(i) $f_{n+2}(x)+2(n+2)xf_{n+1}(x)+(n+1)(n+2)(1+x^2)f_n(x)=0$,

(ii) $(1+x^2)\frac{d^2f_n(x)}{dx^2}-2nx\frac{df_n(x)}{dx}+n(n+1)f_n(x)=0$.

(4 marks)

(c) Using mathematical induction, prove that $f_n(x)$ is a polynomial of degree n and the coefficient of x^n is $(-1)^n(n+1)!$.

(6 marks)

(d) Denote the roots of the equation $f_n(x)=0$ by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Evaluate $\sum_{k=1}^{n}\alpha_k$.

(3 marks)

Go on to the next page

9. (a) Prove that $\frac{2^n}{n!} \le \frac{4}{n}$ for all positive integers n.

Hence prove that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

(3 marks)

(b) For any positive integer n, define $I_n = \int_1^e x^{-3}(\ln x)^n \, dx$.

(i) Evaluate I_1.

(ii) Express I_{n+1} in terms of I_n.

(iii) Prove that $I_n = n!\left(\frac{1}{2^{n+1}} - \frac{1}{e^2}\sum_{k=0}^{n}\frac{1}{(n-k)!2^{k+1}}\right)$.

(iv) Prove that $e^{-2}x^{-1}(\ln x)^n \le x^{-3}(\ln x)^n \le x^{-1}(\ln x)^n$ for all $x \in [1, e]$.

Hence prove that $\frac{1}{e^2(n+1)} \le I_n \le \frac{1}{n+1}$.

(8 marks)

(c) Using (a) and (b), evaluate $\sum_{k=0}^{\infty}\frac{2^k}{k!}$.

(4 marks)

10. (a) Denote the interval $[0, 1]$ by $\mathbf{I}$.

(i) Let $f : \mathbf{I} \to \mathbf{R}$ and $g : \mathbf{I} \to \mathbf{R}$ be continuous functions.

(1) Define $H(x) = \left(\int_0^x f(t)g(t)\,dt\right)^2 - \left(\int_0^x (f(t))^2\,dt\right)\left(\int_0^x (g(t))^2\,dt\right)$ for all $x \in \mathbf{I}$.

Prove that H is decreasing on $\mathbf{I}$.

(2) Prove that $\left(\int_0^x f(t)g(t)\,dt\right)^2 \le \left(\int_0^x (f(t))^2\,dt\right)\left(\int_0^x (g(t))^2\,dt\right)$ for all $x \in \mathbf{I}$.

(ii) Let h be a real-valued function such that h' is continuous on $\mathbf{I}$ and $h(0) = 0$.

Prove that

(1) $(h(x))^2 \le x\int_0^x (h'(t))^2\,dt$ for all $x \in \mathbf{I}$,

(2) $\int_0^1 (h(x))^2\,dx \le \frac{1}{2}\int_0^1 (h'(x))^2\,dx$.

(11 marks)

(b) Using (a)(ii), or otherwise, prove that $\int_0^1 (\ln(\sec x + \tan x))^2\,dx \le \frac{1}{2}\tan 1$.

(4 marks)

11. A straight line passing through the point $A(0, 16)$ cuts the ellipse $E : \frac{x^2}{144} + \frac{y^2}{400} = 1$ at two distinct points $P(12\sin\theta, 20\cos\theta)$ and $Q(12\sin\phi, 20\cos\phi)$. The tangents to E at P and Q intersect at the point R.

(a) Prove that $\sin\theta \ne 0$ and $\sin\phi \ne 0$.

(3 marks)

(b) (i) Find the equation of the tangent to E at P. Also write down the equation of the tangent to E at Q.

(ii) Prove that $\sin(\theta - \phi) \ne 0$.

(5 marks)

(c) Prove that

(i) $4(\sin\theta - \sin\phi) = 5\sin(\theta - \phi)$,

(ii) the coordinates of R are $\left(\frac{3(4 - 5\cos\theta)}{\sin\theta}, 25\right)$.

(4 marks)

(d) Is ΔPAR a right-angled triangle? Explain your answer.

(3 marks)

END OF PAPER