# 2009 AL Pure Mathematics 純粹數學

**評卷參考** * * 此部分只設英文版本

**Marking Schemes**

These documents were prepared for markers' reference. They should not be regarded as sets of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret their contents with care.

## General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

   | | |
   |---|---|
   | 'M' marks | awarded for correct methods being used; |
   | 'A' marks | awarded for the accuracy of the answers; |
   | Marks without 'M' or 'A' | awarded for correctly completing a proof or arriving at an answer given in a question. |

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. Marks may be deducted for poor presentation (*pp*). The symbol (*pp-1*) should be used to denote 1 mark deducted for *pp*. At most deduct 1 mark from Section A and 1 mark from Section B for *pp*. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

7. In the marking scheme, 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

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**Paper 1**

| Solution | Marks |
|---|---|
| 1. (a) $C_k^n + C_{k+1}^n$ | |
| $= \dfrac{n!}{(n-k)!k!} + \dfrac{n!}{(n-k-1)!(k+1)!}$ | 1M |
| $= \dfrac{n!(k+1)+n!(n-k)}{(n-k)!(k+1)!}$ | |
| $= \dfrac{n!(n+1)}{(n-k)!(k+1)!}$ | |
| $= \dfrac{(n+1)!}{((n+1)-(k+1))!(k+1)!}$ | |
| $= C_{k+1}^{n+1}$ | 1 |
| $(1+x)^n(1+x) = (1+x)^{n+1}$ | 1M |
| $\left(\sum_{k=0}^{n} C_k^n x^k\right)(1+x) = \sum_{k=0}^{n+1} C_k^{n+1} x^k$ | |
| $C_0^n + \sum_{k=0}^{n-1}(C_k^n + C_{k+1}^n)x^{k+1} + C_n^n x^{n+1} = C_0^{n+1} + \sum_{k=0}^{n-1} C_{k+1}^{n+1} x^{k+1} + C_{n+1}^{n+1}x^{n+1}$ | |
| By comparing the coefficients of $x^{k+1}$ in both sides, we have | |
| $C_k^n + C_{k+1}^n = C_{k+1}^{n+1}$ . | 1 |
| (b) $(1+x)^{n+1}(1+x)^n = (1+x)^{2n+1}$ | |
| $\left(\sum_{k=0}^{n+1} C_k^{n+1} x^k\right)\left(\sum_{k=0}^{n} C_k^n x^k\right) = \sum_{k=0}^{2n+1} C_k^{2n+1} x^k$ | |
| $\left(\sum_{k=0}^{n+1} C_k^{n+1} x^k\right)\left(\sum_{k=0}^{n} C_{n-k}^n x^k\right) = \sum_{k=0}^{2n+1} C_k^{2n+1} x^k$ | 1M |
| $\left(\sum_{k=0}^{n+1} C_k^{n+1} x^k\right)\left(\sum_{k=0}^{n} C_k^n x^{n-k}\right) = \sum_{k=0}^{2n+1} C_k^{2n+1} x^k$ | |
| By comparing the coefficients of $x^n$ in both sides, we have | |
| $\sum_{k=0}^{n} C_k^{n+1} C_k^n = C_n^{2n+1}$ . | 1 |
| (c) $\sum_{k=0}^{2008}(C_k^{2009} + C_{k+1}^{2009})C_{k+1}^{2009}$ | |
| $= \sum_{k=0}^{2008} C_{k+1}^{2010} C_{k+1}^{2009}$ ( by (a) ) | 1M for using (a) |
| $= \sum_{k=1}^{2009} C_k^{2010} C_k^{2009}$ | |
| $= \sum_{k=0}^{2009} C_k^{2010} C_k^{2009} - C_0^{2010} C_0^{2009}$ | |
| $= C_{2009}^{4019} - C_0^{2010} C_0^{2009}$ ( by (b) ) | 1M for using (b) |
| $= C_{2009}^{4019} - 1$ | 1A |
| | ----------(7) |

| Solution | Marks |
|---|---|
| 2. (a) Let $\dfrac{1}{(2x-1)(2x+1)(2x+3)}=\dfrac{A}{2x-1}+\dfrac{B}{2x+1}+\dfrac{C}{2x+3}$ . | 1A |
| $1\equiv A(2x+1)(2x+3)+B(2x-1)(2x+3)+C(2x-1)(2x+1)$ | |
| Putting $x=\dfrac{1}{2},\dfrac{-1}{2},\dfrac{-3}{2}$ , we have $A=\dfrac{1}{8}$ , $B=\dfrac{-1}{4}$ , $C=\dfrac{1}{8}$ . | 1A for all correct |
| Thus, we have $\dfrac{1}{(2x-1)(2x+1)(2x+3)}=\dfrac{1}{8(2x-1)}-\dfrac{1}{4(2x+1)}+\dfrac{1}{8(2x+3)}$ . | |
| (b) $\displaystyle\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)(2k+3)}$ | |
| $\displaystyle=\frac{1}{8}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{2}{2k+1}+\frac{1}{2k+3}\right)$ ( by (a) ) | |
| $\displaystyle=\frac{1}{8}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)+\frac{1}{8}\sum_{k=1}^{n}\left(\frac{1}{2k+3}-\frac{1}{2k+1}\right)$ | 1M |
| $\displaystyle=\frac{1}{8}\left(1-\frac{1}{2n+1}\right)+\frac{1}{8}\left(\frac{1}{2n+3}-\frac{1}{3}\right)$ | 1A or equivalent |
| $\displaystyle=\frac{1}{12}-\frac{1}{8(2n+1)}+\frac{1}{8(2n+3)}$ | |
| $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}$ | |
| $\displaystyle=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)(2k+3)}$ | |
| $\displaystyle=\lim_{n\to\infty}\left(\frac{1}{12}-\frac{1}{8(2n+1)}+\frac{1}{8(2n+3)}\right)$ | |
| $\displaystyle=\frac{1}{12}$ | 1A |
| (c) $\displaystyle\sum_{k=m}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}$ | |
| $\displaystyle=\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}-\sum_{k=1}^{m-1}\frac{1}{(2k-1)(2k+1)(2k+3)}$ | |
| $\displaystyle=\frac{1}{12}-\left(\frac{1}{12}-\frac{1}{8(2m-1)}+\frac{1}{8(2m+1)}\right)$ ( by (b) ) | |
| $\displaystyle=\frac{1}{8}\left(\frac{1}{2m-1}-\frac{1}{2m+1}\right)$ ( which is also true for $m=1$ ) | 1A or equivalent |
| For $\displaystyle\sum_{k=m}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}>\frac{1}{4000}$ , | |
| we have $\displaystyle\frac{1}{8}\left(\frac{1}{2m-1}-\frac{1}{2m+1}\right)>\frac{1}{4000}$ . | |
| Therefore, we have $4(2m+1)(2m-1)<4000$ . | |
| So, we have $m^2<\dfrac{1001}{4}$ . | |
| As $m$ is a positive integer, we have $1\le m\le 15$ . | |
| Thus, the required greatest positive integer $m$ is 15 . | 1A |
| | ----------(7) |

| Solution | Marks |
|---|---|
| 3. (a) By (2), we have $f(-2) = g(-2) = 0$ . | 1M for either one |
| By (1), we have $f(-2) = g(-2) + (-2)^3 + k(-2)^2 + 8(-2) + 8$ . | |
| So, we have $(-2)^3 + k(-2)^2 + 8(-2) + 8 = 0$ . | |
| Thus, we have $k = 4$ . | 1A |
| (b) (i) Let $g(x) = (x^2 - 1)q_1(x) + 4x - 1$ for some polynomial $q_1(x)$ . | 1M (either one) |
| So, we have $g(1) = 4 - 1 = 3$ . | |
| By (1), we have $f(1) = g(1) + k + 17$ . | |
| Thus, with the help of (a), we have $f(1) = 24$ . | 1A |
| (ii) Let $f(x) = (x^2 - 1)q_2(x) + k_1 x + k_2$ for some polynomial $q_2(x)$ . | (either one) |
| By (b)(i), we have $k_1 + k_2 = 24$ . | |
| Note that $g(-1) = -4 - 1 = -5$ . | |
| By (1), we have $f(-1) = g(-1) + k - 1$ . | |
| So, with the help of (a), we have $f(-1) = -2$ . | 1M accept using long division |
| Therefore, we have $-k_1 + k_2 = -2$ . | |
| Solving, we have $k_1 = 13$ and $k_2 = 11$ . | |
| Thus, the required remainder is $13x + 11$ . | 1A |
| | ----------(6) |

| Solution | Marks |
|---|---|
| 4. (a) $A$ | |
| $=\begin{pmatrix}\cos\frac{\pi}{4} & \sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & -\cos\frac{\pi}{4}\end{pmatrix}$ | 1A |
| $=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$ | |
| (b) (i) Note that $B=\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}$, where $\alpha=\frac{-\pi}{8}$ . | 1M can be absorbed |
| Thus, $B$ represents the reflection in the straight line $y=-\left(\tan\frac{\pi}{8}\right)x$ . | 1A for reflection + 1A for the details |
| (ii) $AB$ | |
| $=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$ | |
| $=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ | 1A |
| $=\begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}$ | 1M |
| Thus, the matrix $AB$ represents a rotation. | 1A f.t. |
| $AB$ | |
| $=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$ | |
| $=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ | 1A |
| Note that $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ represents the rotation in the Cartesian plane anticlockwise about the origin by $\frac{\pi}{2}$ . | 1M |
| Thus, the matrix $AB$ represents a rotation. | 1A f.t. |
| | -----------(7) |

| Solution | Marks |
|---|---|
| 5. (a) Since $a_1 = \frac{1}{2} < \frac{3}{4}$ , the statement is true for $n = 1$ . | |
| Assume that $a_k < \left(\frac{3}{4}\right)^k$ for some positive integer $k$ . | |
| $a_{k+1}$ | |
| $= \frac{3^{k+1}}{(k+3)!}$ | |
| $= \left(\frac{3}{k+3}\right)\frac{3^k}{(k+2)!}$ | |
| $= \left(\frac{3}{k+3}\right)a_k$ | |
| $< \left(\frac{3}{k+3}\right)\left(\frac{3}{4}\right)^k$ | 1M for using induction assumption |
| $\le \left(\frac{3}{4}\right)^{k+1}$ | 1 |
| By mathematical induction, we have $a_n < \left(\frac{3}{4}\right)^n$ . | |
| Note that $a_1 = \frac{1}{2} < \frac{3}{4}$ . | |
| For any $n \ge 2$ , | |
| $a_n$ | |
| $= \frac{(3)(3)(3)(3)\cdots(3)}{(1)(2)(3)(4)\cdots(n+2)}$ | |
| $= \left(\frac{3}{3!}\right)\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)\left(\frac{3}{6}\right)\cdots\left(\frac{3}{n+2}\right)$ | |
| $\le \frac{3}{6}\left(\frac{3}{4}\right)^{n-1}$ | 1M |
| $< \left(\frac{3}{4}\right)^n$ | 1 |
| (b) Note that $0 < a_n < \left(\frac{3}{4}\right)^n$ for all $n \ge 1$ . | 1M accept $a_n \ge 0$ |
| Since $\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$ , we have $\lim_{n\to\infty} a_n = 0$ . | 1A |
| (c) Since $a_n > 0$ for all $n$ , the sequence $\{ A_n \}$ is strictly increasing. | 1M |
| Note that | |
| $A_n$ | |
| $< \sum_{k=1}^{n}\left(\frac{3}{4}\right)^k$ | |
| $= 3\left[1-\left(\frac{3}{4}\right)^n\right]$ | |
| $< 3$ | |
| Therefore, the sequence $\{ A_n \}$ is bounded above by $3$ . | 1M |
| Thus, $\lim_{n\to\infty} A_n$ exists. | 1A f.t. |
| | ----------(7) |

| Solution | Marks |
|---|---|
| 6. (a) (i) By A.M. ≥ G.M., we have $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{S+a_k} \geq \left(\prod_{k=1}^{n}\frac{1}{S+a_k}\right)^{\frac{1}{n}}$ . | 1M |
| Since $G=\left(\prod_{k=1}^{n}(S+a_k)\right)^{\frac{1}{n}}$ , we have $\frac{1}{G}=\left(\prod_{k=1}^{n}\frac{1}{S+a_k}\right)^{\frac{1}{n}}$ . | |
| Thus, we have $\sum_{k=1}^{n}\frac{1}{S+a_k} \geq \frac{n}{G}$ . | 1 |
| (ii) By A.M. ≥ G.M., we have $\frac{1}{n}\sum_{k=1}^{n}(S+a_k) \geq \left(\prod_{k=1}^{n}(S+a_k)\right)^{\frac{1}{n}}$ . | 1M |
| So, we have $S+\frac{1}{n}\sum_{k=1}^{n}a_k \geq G$ . | |
| Since $S=\sum_{k=1}^{n}a_k$ , we have $S+\frac{1}{n}S \geq G$ . | |
| Therefore, we have $\left(\frac{n+1}{n}\right)S \geq G$ . | |
| Thus, we have $\frac{S}{G} \geq \frac{n}{n+1}$ . | 1 |
| (b) Note that $\sum_{k=1}^{n}\frac{S}{S+a_k} \geq \frac{nS}{G}$ ( by (a)(i) ). | 1M ---- either one |
| Also note that $\frac{nS}{G} \geq \frac{n^2}{n+1}$ ( by (a)(ii) ). | ---- |
| Thus, we have $\sum_{k=1}^{n}\frac{S}{S+a_k} \geq \frac{n^2}{n+1}$ . | 1 |
| By Cauchy-Schwarz's inequality, we have | |
| $\sqrt{\sum_{k=1}^{n}\frac{S+a_k}{S}}\sqrt{\sum_{k=1}^{n}\frac{S}{S+a_k}} \geq \sum_{k=1}^{n}\sqrt{\frac{S+a_k}{S}}\sqrt{\frac{S}{S+a_k}}$ | 1M |
| $\sqrt{\sum_{k=1}^{n}\frac{S+a_k}{S}}\sqrt{\sum_{k=1}^{n}\frac{S}{S+a_k}} \geq n$ | |
| $\left(\sum_{k=1}^{n}\frac{S+a_k}{S}\right)\left(\sum_{k=1}^{n}\frac{S}{S+a_k}\right) \geq n^2$ | |
| $(n+1)\left(\sum_{k=1}^{n}\frac{S}{S+a_k}\right) \geq n^2$ | |
| $\sum_{k=1}^{n}\frac{S}{S+a_k} \geq \frac{n^2}{n+1}$ | 1 |
| | ----------(6) |

| Solution | Marks |
|---|---|
| 7. (a) (i) $(E)$ has a unique solution | |
| $\Leftrightarrow \quad \Delta \neq 0$ | |
| $\Leftrightarrow \quad \Delta = \begin{vmatrix} 1 & \lambda & 2 \\ 5 & -\lambda & 1 \\ \lambda & -1 & 1 \end{vmatrix} \neq 0$ | 1M |
| $\Leftrightarrow \quad 3\lambda^2 - 6\lambda - 9 \neq 0$ | 1A |
| $\Leftrightarrow \quad 3(\lambda+1)(\lambda-3) \neq 0$ | |
| $\Leftrightarrow \quad \lambda \neq -1$ and $\lambda \neq 3$ | 1A |
| $\Leftrightarrow \quad \lambda < -1$ , $-1 < \lambda < 3$ or $\lambda > 3$ | |
| The augmented matrix of $(E)$ is | |
| $\left(\begin{array}{ccc\|c} 1 & \lambda & 2 & 1 \\ 5 & -\lambda & 1 & 5 \\ \lambda & -1 & 1 & a \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & \lambda & 2 & 1 \\ \lambda & -1 & 1 & a \\ 2 & 0 & 1 & 2 \end{array}\right)$ | |
| $\sim \left(\begin{array}{ccc\|c} -3 & \lambda & 0 & -3 \\ \lambda-2 & -1 & 0 & a-2 \\ 2 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc\|c} (\lambda+1)(\lambda-3) & 0 & 0 & a\lambda-2\lambda-3 \\ \lambda-2 & -1 & 0 & a-2 \\ 2 & 0 & 1 & 2 \end{array}\right)$ | 1A |
| $(E)$ has a unique solution | |
| $\Leftrightarrow \quad (\lambda+1)(\lambda-3) \neq 0$ | 1M |
| $\Leftrightarrow \quad \lambda \neq -1$ and $\lambda \neq 3$ | 1A |
| $\Leftrightarrow \quad \lambda < -1$ , $-1 < \lambda < 3$ or $\lambda > 3$ | |
| When $(E)$ has a unique solution, | |
| $x = \dfrac{\begin{vmatrix} 1 & \lambda & 2 \\ 5 & -\lambda & 1 \\ a & -1 & 1 \end{vmatrix}}{\Delta} = \dfrac{3(a\lambda-2\lambda-3)}{\Delta}$ | 1M for Cramer's rule |
| $= \dfrac{a\lambda-2\lambda-3}{(\lambda+1)(\lambda-3)}$ | |
| $y = \dfrac{\begin{vmatrix} 1 & 1 & 2 \\ 5 & 5 & 1 \\ \lambda & a & 1 \end{vmatrix}}{\Delta} = \dfrac{-9(\lambda-a)}{\Delta}$ | |
| $= \dfrac{3(a-\lambda)}{(\lambda+1)(\lambda-3)}$ | |
| $z = \dfrac{\begin{vmatrix} 1 & \lambda & 1 \\ 5 & -\lambda & 5 \\ \lambda & -1 & a \end{vmatrix}}{\Delta} = \dfrac{6\lambda(\lambda-a)}{\Delta}$ | |
| $= \dfrac{2\lambda(\lambda-a)}{(\lambda+1)(\lambda-3)}$ | 1A + 1A (1A for any one, 1A for all) |

| Solution | Marks |
|---|---|
| When $(E)$ has a unique solution, the augmented matrix of $(E)$ becomes | |
| $\left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{a\lambda-2\lambda-3}{(\lambda+1)(\lambda-3)} \\ \lambda-2 & -1 & 0 & a-2 \\ 2 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{a\lambda-2\lambda-3}{(\lambda+1)(\lambda-3)} \\ 0 & 1 & 0 & \frac{3(a-\lambda)}{(\lambda+1)(\lambda-3)} \\ 2 & 0 & 1 & 2 \end{array}\right)$ | 1M |
| $\sim \left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{a\lambda-2\lambda-3}{(\lambda+1)(\lambda-3)} \\ 0 & 1 & 0 & \frac{3(a-\lambda)}{(\lambda+1)(\lambda-3)} \\ 0 & 0 & 1 & \frac{2\lambda(\lambda-a)}{(\lambda+1)(\lambda-3)} \end{array}\right)$ | |
| $\therefore \quad x=\frac{a\lambda-2\lambda-3}{(\lambda+1)(\lambda-3)}$ , $y=\frac{3(a-\lambda)}{(\lambda+1)(\lambda-3)}$ , $z=\frac{2\lambda(\lambda-a)}{(\lambda+1)(\lambda-3)}$ . | 1A + 1A (1A for any one, 1A for all) |
| (ii) When $\lambda=-1$ , the augmented matrix of $(E)$ becomes | |
| $\left(\begin{array}{ccc\|c} 1 & -1 & 2 & 1 \\ 5 & 1 & 1 & 5 \\ -1 & -1 & 1 & a \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -1 & 2 & 1 \\ 0 & 6 & -9 & 0 \\ 0 & 0 & 0 & a+1 \end{array}\right)$ | |
| $(E)$ is consistent when $a=-1$ . | 1A |
| Solving, the solution set is $\{(1-t,3t,2t):t\in\mathbf{R}\}$ . | 1A or equivalent |
| | ---------(8) |
| (b) Putting $\lambda=-2$ and $a=3$ in (a), we have, by (a)(i), the solution of the first three linear equations of the system is $x=-1$ , $y=3$ and $z=4$ . | 1A for correct values of $\lambda$ and $a$<br>1M |
| Note that $4(-1)+3(3)-3(4)=-7\neq 2$ . | |
| Thus, the system of linear equations is inconsistent. | 1A f.t. |
| | ---------(3) |
| (c) Putting $\lambda=-1$ and $a=-1$ in (a), we have, by (a)(ii), the solution of the system of linear equations is $x=1-t$ , $y=3t$ and $z=2t$ , where $t\in\mathbf{R}$ . | 1A for correct values of $\lambda$ and $a$ |
| Putting $x=1-t$ , $y=3t$ and $z=2t$ in $4x^2+2y-z=28$ , | 1M |
| we have $t^2-t-6=0$ . | 1A or equivalent |
| Solving, we have $t=-2$ or $t=3$ . | |
| Thus, the required solutions are $(3,-6,-4)$ and $(-2,9,6)$ . | 1A for all correct |
| | --------(4) |

| Solution | Marks |
|---|---|
| 8. (a) Since $M = r\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$, we have $r\cos\theta = -1$ and $r\sin\theta = \sqrt{3}$ . | 1M for equating entries |
| Note that $r > 0$ and $0 < \theta < 2\pi$ . | |
| Thus, we have $r = 2$ and $\theta = \frac{2\pi}{3}$ . | 1A for both correct |
| $M$ $= \begin{pmatrix}-1 & -\sqrt{3} \\ \sqrt{3} & -1\end{pmatrix}$ $= 2\begin{pmatrix}\frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2}\end{pmatrix}$ | 1M |
| $= 2\begin{pmatrix}\cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3}\end{pmatrix}$ | |
| Thus, we have $r = 2$ and $\theta = \frac{2\pi}{3}$ . | 1A for both correct |
| | ---------(2) |
| (b) $M^n$ $= 2^n\begin{pmatrix}\cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3}\end{pmatrix}^n$ | |
| $= 2^n\begin{pmatrix}\cos\frac{2n\pi}{3} & -\sin\frac{2n\pi}{3} \\ \sin\frac{2n\pi}{3} & \cos\frac{2n\pi}{3}\end{pmatrix}$ | |
| $M^n$ is a $2\times 2$ diagonal matrix if and only if $2^n\sin\frac{2n\pi}{3} = 0$ . | 1M |
| As $n$ is a positive integer, we have $n = 3, 6, 9, \ldots$ . | 1A |
| When $n = 3, 6, 9, \ldots$ , we have $M^n = 2^n I$ . | 1A |
| | ---------(3) |
| (c) $(A-M)(A^{2009} + A^{2008}M + A^{2007}M^2 + \cdots + M^{2009})$ | |
| $= A^{2010} + A^{2009}M + \cdots + AM^{2009} - MA^{2009} - MA^{2008}M - \cdots - M^{2010}$ | 1M |
| $= A^{2010} + A^{2009}M + \cdots + AM^{2009} - A^{2009}M - A^{2008}M^2 - \cdots - M^{2010}$ | |
| $= A^{2010} - M^{2010}$ | 1 |
| | ---------(2) |

| Solution | Marks |
| --- | --- |
| (d) Since $2I-M=\begin{pmatrix}3 & \sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}$ , we have $\det(2I-M)=12$ . | 1M |
| So, we have $\det(2I-M)\neq 0$ . | |
| Thus, $2I-M$ is not a singular matrix. | 1A f.t. |
| Note that $2I-M=\begin{pmatrix}3 & \sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}$ . | |
| Also note that $\begin{pmatrix}\frac{1}{4} & \frac{-\sqrt{3}}{12}\\ \frac{\sqrt{3}}{12} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}3 & \sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}=I=\begin{pmatrix}3 & \sqrt{3}\\ -\sqrt{3} & 3\end{pmatrix}\begin{pmatrix}\frac{1}{4} & \frac{-\sqrt{3}}{12}\\ \frac{\sqrt{3}}{12} & \frac{1}{4}\end{pmatrix}$ . | 1M |
| Thus, $2I-M$ is not a singular matrix. | 1A f.t. |
| | ---------(2) |
| (e) Note that $(2I)M=M(2I)$ . | 1M withhold 1M if checking is omitted |
| Putting $A=2I$ in (c), we have | |
| $(2I-M)(2^{2009}I+2^{2008}M+2^{2007}M^2+\cdots+M^{2009})=2^{2010}I-M^{2010}$ . | 1M |
| Note that $2010=(3)(670)$ . | |
| By (b), we have $M^{2010}=2^{2010}I$ . | 1M for using (b) |
| Therefore, we have $(2I-M)(2^{2009}I+2^{2008}M+2^{2007}M^2+\cdots+M^{2009})=0$ . | 1A |
| By (d), $2I-M$ is a non-singular matrix. | |
| Hence, the inverse of $(2I-M)$ exists. | |
| So, we have $2^{2009}I+2^{2008}M+2^{2007}M^2+\cdots+M^{2009}=(2I-M)^{-1}0$ . | 1M |
| Therefore, we have $2^{2009}I+2^{2008}M+2^{2007}M^2+\cdots+M^{2009}=0$ . | |
| Thus, we have $2^{2008}M+2^{2007}M^2+2^{2006}M^3+\cdots+M^{2009}=-2^{2009}I$ . | 1A |
| $(2I-M)(2^{2008}M+2^{2007}M^2+2^{2006}M^3+\cdots+M^{2009})$ | 1M |
| $=2^{2009}M+2^{2008}M^2+\cdots+2M^{2009}-2^{2008}M^2-\cdots-2M^{2009}-M^{2010}$ | 1M |
| $=2^{2009}M-M^{2010}$ | |
| $=2^{2009}M-2^{2010}I$ ( by (b) ) | 1M for using (b) |
| $=-2^{2009}(2I-M)$ | 1A |
| By (d), $2I-M$ is a non-singular matrix. | |
| Hence, the inverse of $(2I-M)$ exists. | |
| $2^{2008}M+2^{2007}M^2+2^{2006}M^3+\cdots+M^{2009}$ | |
| $=-2^{2009}(2I-M)^{-1}(2I-M)$ | 1M |
| $=-2^{2009}I$ | 1A |
| | ---------(6) |

| Solution | Marks |
|---|---|
| 9. (a) "$\Rightarrow$" | |
| If (*) can be written as (**), then $A=2p$ , $B=p^2+q$ and $C=pq$ . | 1M |
| So, we have $p=\frac{A}{2}$ , $q=\left(B-\frac{A^2}{4}\right)$ and $C=pq$ . | |
| Thus, we have $C=\frac{A}{2}\left(B-\frac{A^2}{4}\right)$ . | 1 |
| "$\Leftarrow$" | |
| Assume that $C=\frac{A}{2}\left(B-\frac{A^2}{4}\right)$ . | |
| $x^4+Ax^3+Bx^2+Cx+D$ | |
| $=\left((x^2)^2+2\left(\frac{A}{2}\right)x^3+\left(\frac{A}{2}\right)^2x^2\right)-\frac{A^2}{4}x^2+Bx^2+Cx+D$ | 1M |
| $=\left(x^2+\frac{A}{2}x\right)^2+\left(B-\frac{A^2}{4}\right)x^2+Cx+D$ | 1 |
| $=\left(x^2+\frac{A}{2}x\right)^2+\left(B-\frac{A^2}{4}\right)x^2+\frac{A}{2}\left(B-\frac{A^2}{4}\right)x+D$ | |
| $=\left(x^2+\frac{A}{2}x\right)^2+\left(B-\frac{A^2}{4}\right)\left(x^2+\frac{A}{2}x\right)+D$ | 1 |
| By letting $p=\frac{A}{2}$ , $q=B-\frac{A^2}{4}$ and $r=D$ , (*) can be written as (**). | |
| | ----------(5) |
| (b) By (a), we have $230=\left(\frac{-20}{2}\right)\left(\lambda-\frac{(-20)^2}{4}\right)$ . | 1M for using (a) |
| So, we have $-23=\lambda-100$ .<br>Solving, we have $\lambda=77$ . | 1A |
| $x^4-20x^3+77x^2+230x+120=0$ | |
| $\Leftrightarrow\ (x^2-10x)^2-23(x^2-10x)+120=0$ | 1M for using (a) + 1A |
| $\Leftrightarrow\ ((x^2-10x)-15)((x^2-10x)-8)=0$ | 1M + 1A |
| $\Leftrightarrow\ (x^2-10x-15)(x^2-10x-8)=0$ | |
| $\Leftrightarrow\ x^2-10x-15=0$ or $x^2-10x-8=0$ | |
| $\Leftrightarrow\ x=5-2\sqrt{10}$ , $x=5+2\sqrt{10}$ , $x=5-\sqrt{33}$ or $x=5+\sqrt{33}$ | 1A for all being correct<br>----------(7) |
| (c) By (a), we have $230=\left(\frac{\mu}{2}\right)\left(77-\frac{\mu^2}{4}\right)$ . | 1M for obtaining a cubic equation |
| With the help of (b), $\mu=-20$ satisfies the equation $230=\left(\frac{\mu}{2}\right)\left(77-\frac{\mu^2}{4}\right)$ . | 1M for using (b) |
| Simplifying, we have $\mu^3-308\mu+1840=0$ . | |
| Therefore, we have $(\mu+20)(\mu^2-20\mu+92)=0$ . | |
| Thus, we have $\mu=-20$ , $\mu=10-2\sqrt{2}$ or $\mu=10+2\sqrt{2}$ . | 1A for all being correct<br>----------(3) |

| Solution | Marks |
|---|---|
| 10. (a) $t\ln t-(1+t)\ln(1+t)$ | |
| $=t\ln t-t\ln(1+t)-\ln(1+t)$ | |
| $=t\ln\frac{t}{1+t}-\ln(1+t)$ | |
| Since $0<\frac{t}{1+t}<1$ , we have $\ln\frac{t}{1+t}<0$ . | 1M |
| Note that $\ln(1+t)>0$ . | |
| Thus, we have $t\ln t-(1+t)\ln(1+t)<0$ for all $t>0$ . | 1 |
| Note that $0<t^t<(1+t)^t<(1+t)^{1+t}$ . | |
| So, we have $\ln t^t<\ln(1+t)^{1+t}$ . | 1M |
| Therefore, we have $t\ln t<(1+t)\ln(1+t)$ . | |
| Thus, we have $t\ln t-(1+t)\ln(1+t)<0$ for all $t>0$ . | 1 |
| Note that $t\ln t-(1+t)\ln(1+t)=\ln\left[\left(\frac{t}{1+t}\right)^t\left(\frac{1}{1+t}\right)\right]$ . | |
| As $0<\frac{t}{1+t}<1$ and $0<\frac{1}{1+t}<1$ , we have $0<\left(\frac{t}{1+t}\right)^t\left(\frac{1}{1+t}\right)<1$ . | |
| Therefore, we have $\ln\left[\left(\frac{t}{1+t}\right)^t\left(\frac{1}{1+t}\right)\right]<0$ . | 1M |
| Thus, we have $t\ln t-(1+t)\ln(1+t)<0$ for all $t>0$ . | 1 |
| | ----------(2) |
| (b) Define $f(x)=\frac{\ln(1+a^x)}{x}$ for all $x>0$ . | |
| $f'(x)$ | |
| $=\frac{\frac{x}{1+a^x}(\ln a)a^x-\ln(1+a^x)}{x^2}$ | 1M |
| $=\frac{(x\ln a)a^x-(1+a^x)\ln(1+a^x)}{x^2(1+a^x)}$ | |
| $=\frac{a^x\ln a^x-(1+a^x)\ln(1+a^x)}{x^2(1+a^x)}$ | 1A |
| Since $a^x>0$ , we have $a^x\ln a^x-(1+a^x)\ln(1+a^x)<0$ ( by (a) ). | 1M for using (a) |
| Thus, we have $f'(x)<0$ for all $x>0$ . | 1 |
| | ----------(4) |
| (c) (i) By (b), $f(x)=\frac{\ln(1+a^x)}{x}$ is a strictly decreasing function. | |
| Since $p\geq q>0$ , we have $\frac{\ln(1+a^p)}{p}\leq\frac{\ln(1+a^q)}{q}$ . | 1M |
| So, we have $\ln(1+a^p)^q\leq\ln(1+a^q)^p$ . | |
| Note that $\ln x$ is a strictly increasing function. | |
| Thus, we have $(1+a^p)^q\leq(1+a^q)^p$ for all $a>0$ . | 1 |

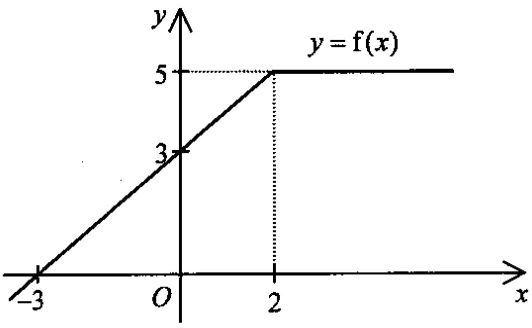
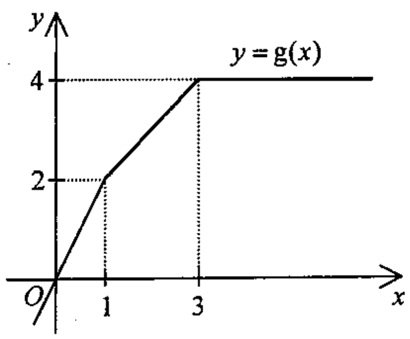
| Solution | Marks |
| --- | --- |
| (ii) Note that $\frac{\mu}{\lambda}>0$ for any 2 positive real numbers $\lambda$ and $\mu$. | |
| Putting $a=\frac{\mu}{\lambda}$ in (c)(i), we have $\left(1+\left(\frac{\mu}{\lambda}\right)^p\right)^q\le\left(1+\left(\frac{\mu}{\lambda}\right)^q\right)^p$ . | 1M accept $a=\frac{\lambda}{\mu}$ |
| Therefore, we have $\frac{(\lambda^p+\mu^p)^q}{\lambda^{pq}}\le\frac{(\lambda^q+\mu^q)^p}{\lambda^{qp}}$ . | |
| Thus, we have $(\lambda^p+\mu^p)^q\le(\lambda^q+\mu^q)^p$ . | 1 |
| (iii) Since $(a_1{}^p)^q=(a_1{}^q)^p$ , the statement is true for $n=1$ .<br>Assume that the statement is true for $n=m$ .<br>Then, for any $(m+1)$ positive real numbers $a_1,a_2,\dots,a_{m+1}$ , | |
| $\left(\sum_{k=1}^{m+1}a_k{}^p\right)^q$ | |
| $=\left(\left(\sum_{k=1}^{m}a_k{}^p\right)+a_{m+1}{}^p\right)^q$ | |
| $\le\left(\left(\sum_{k=1}^{m}a_k{}^q\right)^{\frac{p}{q}}+a_{m+1}{}^p\right)^q$ ( by induction assumption ) | 1M for using induction assumption |
| $\le\left(\left(\sum_{k=1}^{m}a_k{}^q\right)^{\frac{q}{q}}+a_{m+1}{}^q\right)^p$ ( by (c)(ii) ) | 1M for using (c)(ii) |
| $=\left(\sum_{k=1}^{m}a_k{}^q+a_{m+1}{}^q\right)^p$ | |
| $=\left(\sum_{k=1}^{m+1}a_k{}^q\right)^p$ | 1 |
| By mathematical induction, we have $\left(\sum_{k=1}^{n}a_k{}^p\right)^q\le\left(\sum_{k=1}^{n}a_k{}^q\right)^p$ for any $n$ positive real numbers $a_1,a_2,\dots,a_n$ . | |
| (iv) Note that $\frac{1}{q}\ge\frac{1}{p}>0$ . | |
| By (c)(iii), we have $\left(\sum_{k=1}^{n}a_k{}^{\frac{1}{q}}\right)^{\frac{1}{p}}\le\left(\sum_{k=1}^{n}a_k{}^{\frac{1}{p}}\right)^{\frac{1}{q}}$ . | 1M for using (c)(iii) |
| So, we have $\left[\left(\sum_{k=1}^{n}a_k{}^{\frac{1}{q}}\right)^{\frac{1}{p}}\right]^{pq}\le\left[\left(\sum_{k=1}^{n}a_k{}^{\frac{1}{p}}\right)^{\frac{1}{q}}\right]^{pq}$ . | |
| Thus, we have $\left(\sum_{k=1}^{n}a_k{}^{\frac{1}{q}}\right)^q\le\left(\sum_{k=1}^{n}a_k{}^{\frac{1}{p}}\right)^p$ . | 1 |
| | ----------(9) |

| Solution | Marks |
|---|---|
| 11. (a) (i) By solving the equation $z^{10}=1$ , | |
| the roots are $z=\cos\frac{k\pi}{5}+i\sin\frac{k\pi}{5}$ , where $k=-4,-3,-2,\ldots,5$ . | 1A |
| $z^{10}-1$ | |
| $=\prod_{k=-4}^{5}\left(z-\left(\cos\frac{k\pi}{5}+i\sin\frac{k\pi}{5}\right)\right)$ | 1M |
| $=(z-1)(z+1)\prod_{k=1}^{4}\left(z-\left(\cos\frac{k\pi}{5}+i\sin\frac{k\pi}{5}\right)\right)\left(z-\left(\cos\frac{k\pi}{5}-i\sin\frac{k\pi}{5}\right)\right)$ | 1M |
| $=(z-1)(z+1)\prod_{k=1}^{4}\left(\left(z-\cos\frac{k\pi}{5}\right)+i\sin\frac{k\pi}{5}\right)\left(\left(z-\cos\frac{k\pi}{5}\right)-i\sin\frac{k\pi}{5}\right)$ | |
| $=(z-1)(z+1)\prod_{k=1}^{4}\left(\left(z-\cos\frac{k\pi}{5}\right)^2+\sin^2\frac{k\pi}{5}\right)$ | |
| $=(z-1)(z+1)\prod_{k=1}^{4}\left(z^2-2z\cos\frac{k\pi}{5}+\cos^2\frac{k\pi}{5}+\sin^2\frac{k\pi}{5}\right)$ | |
| $=(z-1)(z+1)\prod_{k=1}^{4}\left(z^2-2z\cos\frac{k\pi}{5}+1\right)$ | 1 |
| (ii) $z^5-\frac{1}{z^5}$ | |
| $=\frac{1}{z^5}(z^{10}-1)$ | |
| $=\frac{1}{z^5}(z-1)(z+1)\prod_{k=1}^{4}\left(z^2-2z\cos\frac{k\pi}{5}+1\right)$ | |
| $=\left(\frac{z^2-1}{z}\right)\prod_{k=1}^{4}\left(\frac{1}{z}\left(z^2-2z\cos\frac{k\pi}{5}+1\right)\right)$ | |
| $=\left(z-\frac{1}{z}\right)\prod_{k=1}^{4}\left(z-2\cos\frac{k\pi}{5}+\frac{1}{z}\right)$ | |
| $=\left(z-\frac{1}{z}\right)\prod_{k=1}^{4}\left(z+\frac{1}{z}-2\cos\frac{k\pi}{5}\right)$ . | 1 |
| | ---------(5) |
| (b) $z=\cos\theta+i\sin\theta$ | |
| $z^n=\cos n\theta+i\sin n\theta$ | 1A for either correct |
| $\frac{1}{z^n}=z^{-n}=\cos(-n\theta)+i\sin(-n\theta)=\cos n\theta-i\sin n\theta$ | |
| (i) $z^n+\frac{1}{z^n}=2\cos n\theta$ | 1A |
| (ii) $z^n-\frac{1}{z^n}=2i\sin n\theta$ | 1A |
| | ---------(3) |

| Solution | Marks |
|---|---|
| (c) (i) $\lim_{\theta\to 0}\frac{\sin 5\theta}{\sin\theta}$ | |
| $=5\lim_{\theta\to 0}\left(\frac{\sin 5\theta}{5\theta}\frac{\theta}{\sin\theta}\right)$ | 1M can be absorbed |
| $=5\left(\lim_{\theta\to 0}\frac{\sin 5\theta}{5\theta}\right)\left(\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\right)$ | |
| $=5$ | 1A |
| $\lim_{\theta\to 0}\frac{\sin 5\theta}{\sin\theta}$ <br> $=\lim_{\theta\to 0}\frac{5\cos 5\theta}{\cos\theta}$ <br> $=\frac{5}{1}$ <br> $=5$ | 1M can be absorbed <br> 1A |
| (ii) Putting $z=\cos\theta+i\sin\theta$ in (a)(ii) and using (b), we have | |
| $2i\sin 5\theta=2i\sin\theta\prod_{k=1}^{4}\left(2\cos\theta-2\cos\frac{k\pi}{5}\right)$ . | |
| Hence, we have $\sin 5\theta=16\sin\theta\prod_{k=1}^{4}\left(\cos\theta-\cos\frac{k\pi}{5}\right)$ . | 1 |
| So, we have $\frac{\sin 5\theta}{\sin\theta}=16\prod_{k=1}^{4}\left(\cos\theta-\cos\frac{k\pi}{5}\right)$ for all $\sin\theta\neq 0$ . | |
| By (c)(i), we have $\lim_{\theta\to 0}\frac{\sin 5\theta}{\sin\theta}=5$ . | |
| Therefore, we have $16\lim_{\theta\to 0}\prod_{k=1}^{4}\left(\cos\theta-\cos\frac{k\pi}{5}\right)=5$ . | 1M |
| So, we have $16\prod_{k=1}^{4}\left(1-\cos\frac{k\pi}{5}\right)=5$ . | |
| Hence, we have $16\prod_{k=1}^{4}\left(2\sin^2\frac{k\pi}{10}\right)=5$ . | 1M |
| Therefore, we have $256\prod_{k=1}^{4}\sin^2\frac{k\pi}{10}=5$ . | |
| So, we have $\left(\prod_{k=1}^{4}\sin\frac{k\pi}{10}\right)^2=\frac{5}{256}$ . | |
| Note that $\sin\frac{k\pi}{10}>0$ for all $k=1,2,3,4$ . | 1M withhold 1M if reasons are omitted |
| Thus, we have $\prod_{k=1}^{4}\sin\frac{k\pi}{10}=\frac{\sqrt{5}}{16}$ . | 1A |
| | ---------(7) |

**Paper 2**

| Solution | Marks |
|---|---|
| 1. (a) $\lim_{x\to 0}\frac{(e^x-1)\sin x}{1-\cos x}$ | |
| $=\lim_{x\to 0}\frac{e^x\sin x+(e^x-1)\cos x}{\sin x}$ | 1M |
| $=\lim_{x\to 0}\frac{e^x\sin x+e^x\cos x+e^x\cos x-(e^x-1)\sin x}{\cos x}$ | 1M |
| $=\lim_{x\to 0}\frac{2e^x\cos x+\sin x}{\cos x}$ | |
| $=2$ | 1A |
| $\lim_{x\to 0}\frac{(e^x-1)\sin x}{1-\cos x}$ | |
| $=\lim_{x\to 0}\frac{2(e^x-1)\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$ | 1M |
| $=\lim_{x\to 0}\frac{e^x-1}{\tan\frac{x}{2}}$ | |
| $=2\lim_{x\to 0}\frac{e^x}{\sec^2\frac{x}{2}}$ | 1M |
| $=2$ | 1A |
| (b) Note that $\left\|\sin\frac{1}{x}\right\|\le 1$ for all $x\neq 0$ . | 1M |
| Therefore, we have $\left\|\tan x\sin\frac{1}{x}\right\|\le\|\tan x\|$ for all $x\in\left(\frac{-\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$ . | |
| So, we have $-\|\tan x\|\le\tan x\sin\frac{1}{x}\le\|\tan x\|$ for all $x\in\left(\frac{-\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$ . | |
| Since $\lim_{x\to 0}(-\|\tan x\|)=0=\lim_{x\to 0}\|\tan x\|$ , we have | 1M |
| $\lim_{x\to 0}\tan x\sin\frac{1}{x}=0$ ( by Sandwich Theorem ). | 1A |
| Note that $\left\|\sin\frac{1}{x}\right\|\le 1$ for all $x\neq 0$ . | 1M |
| Also note that $\lim_{x\to 0}\tan x=0$ . | 1M |
| Thus, we have $\lim_{x\to 0}\tan x\sin\frac{1}{x}=0$ . | 1A |
| $\lim_{x\to 0}\frac{4\cot x-\sin\frac{1}{x}}{\cot x+4\sin\frac{1}{x}}$ | |
| $=\lim_{x\to 0}\frac{4-\tan x\sin\frac{1}{x}}{1+4\tan x\sin\frac{1}{x}}$ | |
| $=4$ | 1A<br>----------(7) |

| Solution | Marks |
|---|---|
| 2. (a)  | 1A for all being correct |
| (b) Note that $f(x) \neq 6 \in \mathbf{R}$ for all $x \in \mathbf{R}$ . | 1M |
| Thus, f is not a surjective function. | 1A f.t. |
| (c) (i) Since $g(x) = f(x-1) + f(x+1) - 6$ , we have $$g(x) = \begin{cases} 2x & \text{when } x \le 1 , \\ x+1 & \text{when } 1 < x \le 3 , \\ 4 & \text{when } x > 3 . \end{cases}$$  | 1M for the shape of the graph<br>1A for all being correct |
| (ii) $g(x)$ is not differentiable at $x = 1$ and $x = 3$ . | 1A for $x = 1$ and $x = 3$<br>----------(6) |

| Solution | Marks |
|---|---|
| 3. (a) $f(x)=(25+x^2)^{\frac{-3}{2}}$ | |
| $f'(x)=\left(\frac{-3}{2}\right)(2x)(25+x^2)^{\frac{-5}{2}}$ | |
| $(25+x^2)f'(x)=(-3x)(25+x^2)^{\frac{-3}{2}}$ | |
| $(25+x^2)f'(x)=-3xf(x)$ | 1 |
| Differentiating both sides $n$ times with respect to $x$, we have | |
| $(25+x^2)f^{(n+1)}(x)+n(2x)f^{(n)}(x)+C_2^n(2)f^{(n-1)}(x)=-3xf^{(n)}(x)+n(-3)f^{(n-1)}(x)$ | 1M |
| $(25+x^2)f^{(n+1)}(x)+2nxf^{(n)}(x)+n(n-1)f^{(n-1)}(x)=-3xf^{(n)}(x)-3nf^{(n-1)}(x)$ | |
| $(25+x^2)f^{(n+1)}(x)+(2n+3)xf^{(n)}(x)+n(n+2)f^{(n-1)}(x)=0$ | 1 |
| (b) Putting $x=0$ in (a), we have $25f^{(n+1)}(0)+n(n+2)f^{(n-1)}(0)=0$ . | |
| So, we have $f^{(n+1)}(0)=\frac{-n(n+2)}{25}f^{(n-1)}(0)$ for all positive integers $n$ . | 1M can be absorbed |
| (i) Since $f'(0)=0$ , we have $f^{(5)}(0)=0$ . | 1A |
| (ii) $f^{(6)}(0)$ | |
| $=\frac{-7}{5}f^{(4)}(0)$ | |
| $=\left(\frac{-7}{5}\right)\left(\frac{-3}{5}\right)f''(0)$ | |
| $=\left(\frac{-7}{5}\right)\left(\frac{-3}{5}\right)\left(\frac{-3}{25}\right)f(0)$ | |
| $=\left(\frac{-7}{5}\right)\left(\frac{-3}{5}\right)\left(\frac{-3}{25}\right)\left(\frac{1}{125}\right)$ | |
| $=\frac{-63}{78125}$ | 1A −0.0008064 |
| | ----------(6) |

| Solution | Marks |
|---|---|
| 4. (a) $f(0)$ | |
| $= 3 - 2\int_0^0 f(t)\,dt$ | |
| $= 3 - 0$ | |
| $= 3$ | 1A |
| (b) Since $f(x) = 3 - 2\int_0^x f(t)\,dt$ for all $x \in \mathbf{R}$ , we have $f'(x) = -2f(x)$ . | 1A |
| (c) Let $g(x) = e^{2x} f(x)$ for all $x \in \mathbf{R}$ . | |
| Then, we have $g'(x) = 2e^{2x} f(x) + e^{2x} f'(x)$ . | 1M |
| Therefore, we have $g'(x) = e^{2x}(f'(x) + 2f(x)) = 0$ ( by (b) ). | 1A |
| So, we have $g(x) = a$ for all $x \in \mathbf{R}$ , where $a$ is a constant. | 1A |
| Note that $g(0) = f(0) = 3$ ( by (a) ). | 1M for using (a) |
| Hence, we have $a = 3$ . | |
| Thus, we have $f(x) = 3e^{-2x}$ . | 1A |
| | ----------(7) |

| Solution | Marks |
|---|---|
| 5. (a) (i) $\int \cos^2 t \, dt$ | |
| $= \frac{1}{2}\int (1+\cos 2t)\, dt$ | 1M |
| $= \frac{t}{2} + \frac{\sin 2t}{4} + \text{constant}$ | 1A pp–1 for omitting constant |
| (ii) $\int \cos^3 t \, dt$ | |
| $= \int \cos^2 t \, d\sin t$ | 1M |
| $= \int (1-\sin^2 t)\, d\sin t$ | |
| $= \sin t - \frac{\sin^3 t}{3} + \text{constant}$ | 1A pp–1 for omitting constant |
| (b) The required volume | |
| $= \pi \int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} (2\cos t - 1)^2 (3\cos t)\, dt$ | 1M |
| $= 3\pi \int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} (4\cos^3 t - 4\cos^2 t + \cos t)\, dt$ | |
| $= 3\pi \left[ 4\sin t - \frac{4\sin^3 t}{3} - 2t - \sin 2t + \sin t \right]_{\frac{-\pi}{3}}^{\frac{\pi}{3}}$ ( by (a)(i) and (a)(ii) ) | 1M for using (a)(i) and (a)(ii) |
| $= 9\sqrt{3}\pi - 4\pi^2$ | 1A |
| The required volume | |
| $= 2\pi \int_0^{\frac{\pi}{3}} (2\cos t - 1)(6\sin t) \lvert -2\sin t \rvert \, dt$ | 1M |
| $= 24\pi \int_0^{\frac{\pi}{3}} (2\cos t - 1)(1-\cos^2 t)\, dt$ | |
| $= 24\pi \int_0^{\frac{\pi}{3}} (-1 + 2\cos t + \cos^2 t - 2\cos^3 t)\, dt$ | |
| $= 24\pi \left[ -t + 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4} - 2\sin t + \frac{2\sin^3 t}{3} \right]_0^{\frac{\pi}{3}}$ ( by (a)(i) and (a)(ii) ) | 1M for using (a)(i) and (a)(ii) |
| $= 9\sqrt{3}\pi - 4\pi^2$ | 1A |
| | ----------(7) |

| Solution | Marks |
|---|---|
| 6. (a) (i) Since $y^2 - 4x^2 = 4$ , we have $2y\frac{dy}{dx} - 8x = 0$ . | |
| So, we have $\frac{dy}{dx} = \frac{4x}{y}$ . | 1M |
| The equation of the tangent to $H$ at $A$ is | |
| $y - y_1 = \frac{4x_1}{y_1}(x - x_1)$ | |
| $y_1 y - {y_1}^2 = 4x_1 x - 4{x_1}^2$ | |
| $4x_1 x - y_1 y + {y_1}^2 - 4{x_1}^2 = 0$ | |
| Note that ${y_1}^2 - 4{x_1}^2 = 4$ . | |
| Thus, the equation of the tangent to $H$ at $A$ is $4x_1 x - y_1 y + 4 = 0$ . | 1 |
| (ii) Since $C$ lies on the tangent to $H$ at $A$, we have $4x_1 c - y_1 c + 4 = 0$ . | |
| If $c = 0$ , then we have $4 = 0$ which is impossible. | |
| Thus, we have $c \neq 0$ . | 1 |
| (iii) The equation of the tangent to $H$ at $B$ is $4x_2 x - y_2 y + 4 = 0$ . | |
| Note that $C$ lies on the two tangents. | |
| So, we have $4x_1 c - y_1 c + 4 = 0$ and $4x_2 c - y_2 c + 4 = 0$ . | |
| Therefore, both $A$ and $B$ lie on the straight line $4cx - cy + 4 = 0$ . | |
| Thus, the required equation is $4cx - cy + 4 = 0$ . | 1 |
| (b) Let $(h, k)$ be the coordinates of the mid-point of $AB$ . | |
| By (a)(iii), the equation of $AB$ is $4cx - cy + 4 = 0$ . | |
| By (a)(ii), we have $c \neq 0$ . | |
| Putting $y = 4x + \frac{4}{c}$ in $y^2 - 4x^2 = 4$ , we have | |
| $\left(4x + \frac{4}{c}\right)^2 - 4x^2 = 4$ | 1M |
| $\left(2x + \frac{2}{c}\right)^2 - x^2 = 1$ | |
| $4x^2 + \frac{8x}{c} + \frac{4}{c^2} - x^2 = 1$ | |
| $3x^2 + \frac{8x}{c} + \frac{4}{c^2} - 1 = 0$ | |
| Hence, $x_1$ and $x_2$ are the roots of the equation $3x^2 + \frac{8x}{c} + \frac{4}{c^2} - 1 = 0$ . | |
| So, we have $x_1 + x_2 = \frac{-8}{3c}$ . | |
| Therefore, we have $h = \frac{x_1 + x_2}{2} = \frac{-4}{3c}$ . | 1A |
| Since the mid-point $(h, k)$ lies on $AB$, we have $k = 4h + \frac{4}{c}$ . | |
| So, we have $k = 4\left(\frac{-4}{3c}\right) + \frac{4}{c} = \frac{-4}{3c}$ . | 1A |
| Thus, the coordinates of the mid-point of $AB$ are $\left(\frac{-4}{3c}, \frac{-4}{3c}\right)$ . | |
| | -----------(7) |

| Solution | Marks |
|---|---|
| 7. (a) (i) $\because \quad \lim_{h\to 0^-}\frac{f(h)-f(0)}{h}$ $=\lim_{h\to 0^-}\frac{\lvert h\rvert(h+16)}{h(h-2)}$ $=\lim_{h\to 0^-}\frac{(-h)(h+16)}{h(h-2)}$ $=\lim_{h\to 0^-}\frac{-h-16}{h-2}$ $=8$ | |
| $\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$ $=\lim_{h\to 0^+}\frac{\lvert h\rvert(h+16)}{h(h-2)}$ $=\lim_{h\to 0^+}\frac{h(h+16)}{h(h-2)}$ $=\lim_{h\to 0^+}\frac{h+16}{h-2}$ $=-8$ | |
| $\therefore \quad \lim_{h\to 0^-}\frac{f(h)-f(0)}{h}\neq\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$ | 1M |
| Thus, $f(x)$ is not differentiable at $x=0$ . | 1A f.t. |
| (ii) $f(x)=\begin{cases}\dfrac{x(x+16)}{x-2} & \text{if } x>0 \text{ and } x\neq 2\\ \dfrac{-x(x+16)}{x-2} & \text{if } x<0\end{cases}$ | |
| $f'(x)=\begin{cases}\dfrac{x^2-4x-32}{(x-2)^2} & \text{if } x>0 \text{ and } x\neq 2\\ \dfrac{-(x^2-4x-32)}{(x-2)^2} & \text{if } x<0\end{cases}$ | 1A ( accept if $x\neq 2$ is omitted ) |
| $=\begin{cases}\dfrac{(x+4)(x-8)}{(x-2)^2} & \text{if } x>0 \text{ and } x\neq 2\\ \dfrac{-(x+4)(x-8)}{(x-2)^2} & \text{if } x<0\end{cases}$ | |
| $f''(x)=\begin{cases}\dfrac{72}{(x-2)^3} & \text{if } x>0 \text{ and } x\neq 2\\ \dfrac{-72}{(x-2)^3} & \text{if } x<0\end{cases}$ | 1A ( accept if $x\neq 2$ is omitted ) |
| | ----------(4) |

| Solution | Marks |
|---|---|

(b) Note that $f'(x)=0 \Leftrightarrow x=-4$ or $x=8$ .

| $x$ | $(-\infty,-4)$ | $-4$ | $(-4,0)$ | $0$ | $(0,2)$ | $(2,8)$ | $8$ | $(8,\infty)$ |
|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ | $\nexists$ | $-$ | $-$ | $0$ | $+$ |
| $f''(x)$ | $+$ | $+$ | $+$ | $\nexists$ | $-$ | $+$ | $+$ | $+$ |
| $f(x)$ | ↘ | $-8$ | ↗ | $0$ | ↘ | ↘ | $32$ | ↗ |

(i) $f'(x)>0 \Leftrightarrow -4<x<0$ or $x>8$ — 1A

(ii) $f''(x)>0 \Leftrightarrow x<0$ or $x>2$ — 1A

----------(2)

(c) The relative minimum points are $(-4,-8)$ and $(8,32)$ . — 1A + 1A

The relative maximum point is $(0,0)$ . — 1A

The point of inflexion is $(0,0)$ . — 1A

----------(4)

(d) $\because \lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\frac{x(x+16)}{x-2}=-\infty$ and $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\frac{x(x+16)}{x-2}=\infty$

$\therefore$ the vertical asymptote is $x=2$ . — 1A

$$\because \lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x+16}{x-2}=1$$

$$\lim_{x\to+\infty}(f(x)-x)=\lim_{x\to+\infty}\left(\frac{x(x+16)}{x-2}-x\right)=\lim_{x\to+\infty}\frac{18x}{x-2}=18$$ — 1M (either one)

$$\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{-x-16}{x-2}=-1$$

$$\lim_{x\to-\infty}(f(x)+x)=\lim_{x\to-\infty}\left(\frac{-x(x+16)}{x-2}+x\right)=\lim_{x\to-\infty}\frac{-18x}{x-2}=-18$$

$\therefore$ the oblique asymptotes are $y=x+18$ and $y=-x-18$ . — 1A for both correct

$\because \lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\frac{x(x+16)}{x-2}=-\infty$ and $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\frac{x(x+16)}{x-2}=\infty$

$\therefore$ the vertical asymptote is $x=2$ . — 1A

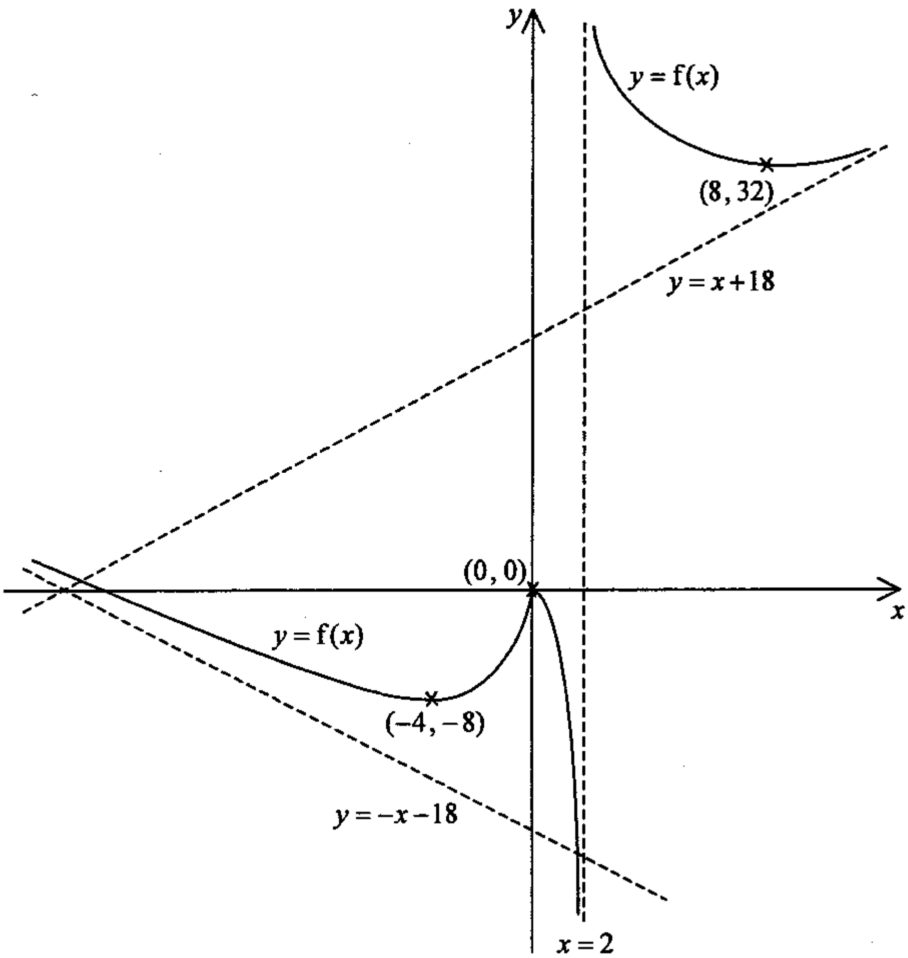
$$\because f(x)=\begin{cases} x+18+\dfrac{36}{x-2} & \text{if } x>0 \text{ and } x\neq 2 \\ -x-18-\dfrac{36}{x-2} & \text{if } x<0 \end{cases}$$ — 1M (either one)

$\therefore$ the oblique asymptotes are $y=x+18$ and $y=-x-18$ . — 1A for both correct

----------(3)

| Solution | Marks |
|---|---|

(e)




1M for the shape of the curve
1A for all being correct

----------(2)

| Solution | Marks |
|---|---|
| **8.** (a) (i) Let $u = a - x$ . | 1M |
| Then, we have | |
| $\int_0^a \ln\cos(a-x)\,dx$ | |
| $= -\int_a^0 \ln\cos u\,du$ | |
| $= \int_0^a \ln\cos u\,du$ | |
| $= \int_0^a \ln\cos x\,dx$ | 1 |
| (ii) $\int_0^a \ln\cos(a-x)\,dx - \int_0^a \ln\cos x\,dx = 0$ ( by (a)(i) ) | 1M |
| $\int_0^a (\ln\cos(a-x) - \ln\cos x)\,dx = 0$ | |
| $\int_0^a \ln\left(\frac{\cos(a-x)}{\cos x}\right)dx = 0$ | 1M |
| $\int_0^a \ln\left(\frac{\cos a\cos x + \sin a\sin x}{\cos x}\right)dx = 0$ | |
| $\int_0^a \ln(\cos a + \sin a\tan x)\,dx = 0$ | 1 |
| | ----------(5) |
| (b) (i) Let $x = \tan\theta$ . Then, we have | 1M |
| $\int_0^1 \frac{\tan^{-1}x}{1+x}\,dx$ | |
| $= \int_0^{\frac{\pi}{4}} \frac{\theta\sec^2\theta}{1+\tan\theta}\,d\theta$ | |
| $= [\theta\ln(1+\tan\theta)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$ | 1M for using integration by parts |
| $= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$ | 1 |
| $\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$ | |
| $= [\theta\ln(1+\tan\theta)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\theta\sec^2\theta}{1+\tan\theta}\,d\theta$ | 1M for using integration by parts |
| $= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \frac{\theta\sec^2\theta}{1+\tan\theta}\,d\theta$ | |
| $= \frac{\pi}{4}\ln 2 - \int_0^1 \frac{\tan^{-1}x}{1+x}\,dx$ ( by letting $x = \tan\theta$ ) | 1M |
| Thus, we have $\int_0^1 \frac{\tan^{-1}x}{1+x}\,dx = \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$ . | 1 |

| Solution | Marks |
|---|---|
| (ii) Note that $0<\frac{\pi}{4}<\frac{\pi}{2}$ . Putting $a=\frac{\pi}{4}$ in (a)(ii), we have | 1M |
| $\int_0^{\frac{\pi}{4}} \ln\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\tan x\right)dx=0$ | |
| $\int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan x\right)dx=0$ | |
| $\int_0^{\frac{\pi}{4}} \left(\ln(1+\tan x)-\ln\sqrt{2}\right)dx=0$ | 1M |
| $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$ | |
| $=\int_0^{\frac{\pi}{4}} \ln\sqrt{2}\,dx$ | |
| $=\frac{\pi}{8}\ln 2$ | |
| $\int_0^1 \frac{\tan^{-1}x}{1+x}dx$ | |
| $=\frac{\pi}{4}\ln 2-\frac{\pi}{8}\ln 2$ ( by (b)(i) ) | |
| $=\frac{\pi}{8}\ln 2$ | 1A |
| Let $u=\frac{\pi}{4}-\theta$ . Then, we have | 1M |
| $\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$ | |
| $=-\int_{\frac{\pi}{4}}^0 \ln\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)du$ | |
| $=\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan u}{1+\tan u}\right)du$ | |
| $=\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right)du$ | |
| $=\int_0^{\frac{\pi}{4}} (\ln 2-\ln(1+\tan u))\,du$ | |
| $=\frac{\pi}{4}\ln 2-\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$ | 1M |
| So, we have $\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta=\frac{\pi}{8}\ln 2$ . | |
| $\int_0^1 \frac{\tan^{-1}x}{1+x}dx$ | |
| $=\frac{\pi}{4}\ln 2-\frac{\pi}{8}\ln 2$ ( by (b)(i) ) | |
| $=\frac{\pi}{8}\ln 2$ | 1A |
| | ----------(6) |

| Solution | Marks |
| --- | --- |
| (c) $\lim_{n\to\infty}\sum_{k=2n+1}^{12n}\frac{1}{8n+k}\tan^{-1}\left(\frac{k-2n}{10n}\right)$ | |
| $=\lim_{n\to\infty}\sum_{l=1}^{10n}\frac{1}{10n+l}\tan^{-1}\left(\frac{l}{10n}\right)$ | 1M |
| $=\lim_{n\to\infty}\frac{1}{10n}\sum_{l=1}^{10n}\frac{1}{1+\frac{l}{10n}}\tan^{-1}\left(\frac{l}{10n}\right)$ | 1M |
| $=\int_0^1\frac{\tan^{-1}x}{1+x}\mathrm{d}x$ | 1A |
| $=\frac{\pi}{8}\ln 2$ | 1M |
| $\lim_{n\to\infty}\sum_{k=2n+1}^{12n}\frac{1}{8n+k}\tan^{-1}\left(\frac{k-2n}{10n}\right)$ | |
| $=\lim_{n\to\infty}\sum_{k=2n+1}^{12n}\frac{1}{10n+k-2n}\tan^{-1}\left(\frac{k-2n}{10n}\right)$ | 1M |
| $=\lim_{n\to\infty}\frac{1}{10n}\sum_{k=2n+1}^{12n}\frac{1}{1+\frac{k-2n}{10n}}\tan^{-1}\left(\frac{k-2n}{10n}\right)$ | 1M |
| $=\int_0^1\frac{\tan^{-1}x}{1+x}\mathrm{d}x$ | 1A |
| $=\frac{\pi}{8}\ln 2$ | 1M |
| $\lim_{n\to\infty}\sum_{k=2n+1}^{12n}\frac{1}{8n+k}\tan^{-1}\left(\frac{k-2n}{10n}\right)$ | |
| $=\lim_{n\to\infty}\frac{1}{n}\sum_{k=2n+1}^{12n}\frac{1}{8+\frac{k}{n}}\tan^{-1}\left(\frac{1}{10}\left(\frac{k}{n}-2\right)\right)$ | 1M |
| $=\int_2^{12}\frac{\tan^{-1}\left(\frac{x-2}{10}\right)}{8+x}\mathrm{d}x$ | |
| Let $y=\frac{x-2}{10}$ . | 1M |
| Then, we have $x+8=10(y+1)$ and $\frac{\mathrm{d}x}{\mathrm{d}y}=10$ . | |
| So, we have $\int_2^{12}\frac{\tan^{-1}\left(\frac{x-2}{10}\right)}{8+x}\mathrm{d}x=\int_0^1\frac{\tan^{-1}y}{1+y}\mathrm{d}y$ . | 1A |
| Thus, we have $\lim_{n\to\infty}\sum_{k=2n+1}^{12n}\frac{1}{8n+k}\tan^{-1}\left(\frac{k-2n}{10n}\right)=\frac{\pi}{8}\ln 2$ . | 1M |
| | ----------(4) |

| Solution | Marks |
| --- | --- |
| 9. (a) (i) $\frac{d}{dx}f_n(x)$ | |
| $= nx^{n-1}e^{-x} - x^n e^{-x}$ | 1M |
| $= (n-x)x^{n-1}e^{-x}$ | |
| $> 0$ for all $x \in (0,1)$ | 1 |
| Thus, $f_n(x)$ is strictly increasing on $(0,1)$ . | |
| (ii) Note that $f_n(x)$ is continuous on $[0,1]$ . | |
| By (a)(i), we have $f_n(0) < f_n(x) < f_n(1)$ for all $x \in (0,1)$ . | |
| So, we have $0 < f_n(x) < \frac{1}{e}$ for all $x \in (0,1)$ . | 1M |
| Therefore, we have $\int_0^1 0\,dx < \int_0^1 f_n(x)\,dx < \int_0^1 \frac{1}{e}\,dx$ . | 1M |
| Thus, we have $0 < I_n < \frac{1}{e}$ . | 1 + 1 |
| (iii) $I_1$ | |
| $= \int_0^1 x e^{-x} dx$ | |
| $= \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x} dx$ | 1M for using integration by parts (either one) |
| $= 1 - \frac{2}{e}$ | 1A |
| (iv) $I_{n+1}$ | |
| $= \int_0^1 x^{n+1} e^{-x} dx$ | |
| $= \left[-x^{n+1}e^{-x}\right]_0^1 + (n+1)\int_0^1 x^n e^{-x} dx$ | |
| $= \frac{-1}{e} + (n+1)I_n$ | 1A |
| (v) $I_n$ | |
| $= \frac{-1}{e} + nI_{n-1}$ ( by (a)(iv) ) | |
| $= \frac{-1}{e} + n\left(\frac{-1}{e} + (n-1)I_{n-2}\right)$ ( by (a)(iv) ) | 1M for using (a)(iv) repeatedly |
| $= \frac{-1}{e} + n\left(\frac{-1}{e}\right) + \cdots + n(n-1)\cdots(3)\left(\frac{-1}{e}\right) + n(n-1)\cdots(2)I_1$ | |
| $= \frac{-1}{e}\sum_{k=2}^{n}\frac{n!}{k!} + n(n-1)\cdots(2)\left(1 - \frac{2}{e}\right)$ ( by (a)(iii) ) | 1M for using (a)(iii) |
| $= \frac{-n!}{e}\sum_{k=0}^{n}\frac{1}{k!} + n!$ | |
| $= n!\left(1 - \frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\right)$ ( which is also true for $n = 1$ ) | 1 |

| Solution | Marks |
| --- | --- |
| Note that $I_1 = 1-\frac{2}{e} = 1!\left(1-\frac{1}{e}\sum_{k=0}^{1}\frac{1}{k!}\right)$ . | |
| So, the statement is true for $n=1$ . | |
| Assume that $I_m = m!\left(1-\frac{1}{e}\sum_{k=0}^{m}\frac{1}{k!}\right)$ for some positive integer $m$ . | 1M |
| $I_{m+1}$ | |
| $=\frac{-1}{e}+(m+1)I_m$ ( by (a)(iv) ) | |
| $=\frac{-1}{e}+(m+1)m!\left(1-\frac{1}{e}\sum_{k=0}^{m}\frac{1}{k!}\right)$ ( by induction assumption ) | 1M |
| $=\frac{-1}{e}+(m+1)m!\left(1-\frac{1}{e}\sum_{k=0}^{m}\frac{1}{k!}\right)$ | |
| $=\frac{-1}{e}+(m+1)!\left(1-\frac{1}{e}\sum_{k=0}^{m}\frac{1}{k!}\right)$ | |
| $=(m+1)!\left(1-\frac{1}{e}\sum_{k=0}^{m+1}\frac{1}{k!}\right)$ | 1 |
| Therefore, the statement is true for $n=m+1$ . | |
| By mathematical induction, we have $I_n = n!\left(1-\frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\right)$ . | |
| | ---------(12) |
| (b) Assume that $e$ is a rational number. | |
| Then, we have $e=\frac{p}{q}$ for some positive integers $p$ and $q$ . | 1M |
| By (a)(v) and (a)(ii), we have $0 < n!\left(1-\frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\right) < \frac{1}{e}$ for all $n$ . | 1M for using (a)(v) and (a)(ii) |
| Hence, we have $0 < n!\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right) < 1$ for all positive integers $n$ . | |
| Therefore, we have $0 < n!\left(\frac{p}{q}\right)-\sum_{k=0}^{n}\frac{n!}{k!} < 1$ for all positive integers $n$ . | |
| Note that $\sum_{k=0}^{q}\frac{q!}{k!}$ is an integer. | |
| Further note that $q!(\frac{p}{q})$ is also an integer. | |
| So, $q!\left(\frac{p}{q}\right)-\sum_{k=0}^{q}\frac{q!}{k!}$ is an integer. | |
| However, there is no integer in the open interval $(0,1)$ . | |
| Thus, $e$ is an irrational number. | 1 |
| | ---------(3) |

| Solution | Marks |
|---|---|
| 10. (a) (i) Let $k$ be an integer greater than or equal to $1$. | |
| By Mean Value Theorem, we have | |
| $\frac{f(k+1)-f(k)}{(k+1)-k} = f'(c)$ for some $c \in (k, k+1)$ . | 1M |
| So, we have $f(k+1)-f(k)=f'(c)$ for some $c \in (k, k+1)$ . | |
| Since $f''(x) \geq 0$ for all $x \in \mathbf{R}^+$ , $f'(x)$ is increasing. | 1M |
| Therefore, we have $f'(k) \leq f'(c) \leq f'(k+1)$ . | |
| Thus, we have $f'(k) \leq f(k+1)-f(k) \leq f'(k+1)$ for all integers $k \geq 1$ . | 1 |
| Let $k$ be an integer greater than or equal to $1$. | |
| Since $f''(x) \geq 0$ for all $x \in \mathbf{R}^+$ , $f'(x)$ is increasing. | 1M |
| So, we have $f'(k) \leq f'(t) \leq f'(k+1)$ for all $t \in [k, k+1]$ . | |
| Therefore, we have $\int_k^{k+1} f'(k)\,dt \leq \int_k^{k+1} f'(t)\,dt \leq \int_k^{k+1} f'(k+1)\,dt$ . | 1M |
| Thus, we have $f'(k) \leq f(k+1)-f(k) \leq f'(k+1)$ for all integers $k \geq 1$ . | 1 |
| (ii) Let $n$ be an integer greater than or equal to $2$. | |
| By (a)(i), we have $\sum_{k=1}^{n-1} f'(k) \leq \sum_{k=1}^{n-1}\left(f(k+1)-f(k)\right) \leq \sum_{k=1}^{n-1} f'(k+1)$ . | 1M |
| Thus, we have $\sum_{k=1}^{n-1} f'(k) \leq f(n)-f(1) \leq \sum_{k=2}^{n} f'(k)$ for all integers $n \geq 2$ . | 1 |
| | ----------(5) |
| (b) (i) Let $f(x)=\frac{1}{(r-1)x^{r-1}}$ for all $x \in \mathbf{R}^+$ . | 1M |
| So, we have $f''(x)=\frac{r}{x^{r+1}} \geq 0$ for all $x \in \mathbf{R}^+$ . | 1M withhold 1M if checking is omitted |
| By (a)(ii), we have | |
| $-1-\frac{1}{2^r}-\cdots-\frac{1}{(n-1)^r} \leq \frac{1}{(r-1)n^{r-1}}-\frac{1}{r-1} \leq -\frac{1}{2^r}-\frac{1}{3^r}-\cdots-\frac{1}{n^r}$ . | |
| Thus, we have $\frac{1}{n^r}-\sum_{k=1}^{n}\frac{1}{k^r} \leq \frac{1}{(r-1)n^{r-1}}-\frac{1}{r-1} \leq 1-\sum_{k=1}^{n}\frac{1}{k^r}$ . | 1 |
| (ii) Let $S_n=\sum_{k=1}^{n}\frac{1}{k^r}$ . | |
| Since $\frac{1}{k^r}>0$ , the sequence $\{S_n\}$ is strictly increasing. | |
| By (b)(i), we have $S_n \leq 1+\frac{1}{r-1}-\frac{1}{(r-1)n^{r-1}}$ . | |
| So, the sequence $\{S_n\}$ is bounded above by $1+\frac{1}{r-1}$ . | 1M |
| Therefore, $\lim_{n\to\infty} S_n$ exists. | |
| Thus, the series $\sum_{k=1}^{\infty}\frac{1}{k^r}$ is convergent. | 1A f.t. |
| | ----------(5) |

| Solution | Marks |
| --- | --- |
| (c) (i) Let $f(x) = -\ln x$ for all $x \in \mathbf{R}^+$ . | 1M |
| So, we have $f''(x) = \frac{1}{x^2} \geq 0$ for all $x \in \mathbf{R}^+$ . | 1M withhold 1M if checking is omitted |
| By (a)(ii), we have $-1 - \frac{1}{2} - \cdots - \frac{1}{n} \leq -\ln(n+1) + \ln 1$ . | |
| Thus, we have $\sum_{k=1}^{n} \frac{1}{k} \geq \ln(n+1)$ for all positive integers $n$ . | 1 |
| (ii) Note that $\ln(n+1) \to \infty$ as $n \to \infty$ . | 1M |
| By (c)(i), we have $\sum_{k=1}^{n} \frac{1}{k} \to \infty$ as $n \to \infty$ . | |
| Thus, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is not convergent. | 1A f.t. |
| | ----------(5) |

| Solution | Marks |
|---|---|
| 11. (a) Differentiating both sides of $x^2 = 4y$ with respect to $x$, we have | |
| $\frac{dy}{dx} = \frac{x}{2}$ | |
| $\left.\frac{dy}{dx}\right\|_{(2s,s^2)} = s$ | 1M can be absorbed |
| The equation of $L_1$ is | |
| $y - s^2 = \frac{-1}{s}(x - 2s)$ | 1M |
| $x + sy - 2s - s^3 = 0$ | 1A |
| | ----------(3) |
| (b) (i) $\frac{t^2 - s^2}{2t - 2s} = \frac{t^2 - 1}{2t - 0}$ | 1M |
| $\frac{t+s}{2} = \frac{t^2 - 1}{2t}$ | |
| $(t+s)t = t^2 - 1$ | |
| $t^2 + st = t^2 - 1$ | |
| $st = -1$ | 1 |
| (ii) Note that the slope of $L_1$ is $\frac{-1}{s}$ and the slope of $L_2$ is $\frac{-1}{t}$. | 1M |
| Hence, the product of the slopes of $L_1$ and $L_2$ is $-1$. | |
| Thus, $L_1$ and $L_2$ are perpendicular to each other. | 1A |
| (iii) Note that the equation of $L_2$ is $x + ty - 2t - t^3 = 0$. | |
| Solving $x + sy - 2s - s^3 = 0$ and $x + ty - 2t - t^3 = 0$, | 1M for solving |
| we have $(s - t)y = 2(s - t) + (s^3 - t^3)$. | |
| So, we have $y = 2 + (s^2 + st + t^2)$. | |
| With the help of $st = -1$, we have $y = s^2 + t^2 + 1$. | |
| Since $x + s(s^2 + t^2 + 1) - 2s - s^3 = 0$, we have $x = s - st^2$. | |
| With the help of $st = -1$, we have $x = s + t$. | |
| Hence, the coordinates of $R$ are $(s + t, s^2 + t^2 + 1)$. | 1M |
| $RS$ | |
| $= \sqrt{(s-t)^2 + (-t^2 - 1)^2}$ | 1M |
| $= \sqrt{\left(s + \frac{1}{s}\right)^2 + \left(\frac{1}{s^2} + 1\right)^2}$ | |
| $= \sqrt{\left(\frac{1+s^2}{s}\right)^2 + \left(\frac{1+s^2}{s^2}\right)^2}$ | |
| $= \sqrt{\frac{(1+s^2)^3}{s^4}}$ | |
| $= \frac{1}{s^2}(1+s^2)^{\frac{3}{2}}$ | 1 |

| Solution | Marks |
| --- | --- |
| (iv) By (b)(iii), we have $RS=\frac{1}{s^2}(1+s^2)^{\frac{3}{2}}$ . | |
| So, we have $RT=\frac{1}{t^2}(1+t^2)^{\frac{3}{2}}$ . | |
| Therefore, we have $RT=s^2\left(1+\frac{1}{s^2}\right)^{\frac{3}{2}}$ . | |
| The area of $\Delta RST$ | |
| $=\frac{1}{2}(RS)(RT)$ | |
| $=\frac{1}{2}(1+s^2)^{\frac{3}{2}}\left(1+\frac{1}{s^2}\right)^{\frac{3}{2}}$ | 1M |
| $=\frac{1}{2}\left(s^2+2+\frac{1}{s^2}\right)^{\frac{3}{2}}$ | |
| $=\frac{1}{2}\left(\left(s+\frac{1}{s}\right)^2\right)^{\frac{3}{2}}$ | |
| $=\frac{1}{2}\left\|s+\frac{1}{s}\right\|^3$ | 1 |
| The area of $\Delta RST$ | |
| $=\left\|\frac{1}{2}\begin{vmatrix} 2t & t^2 & 1 \\ 2s & s^2 & 1 \\ s+t & s^2+t^2+1 & 1 \end{vmatrix}\right\|$ | 1M |
| $=\left\|\frac{1}{2}\begin{vmatrix} 2t & s^2 & 1 \\ 2s-2t & s^2-t^2 & 0 \\ s-t & s^2+1 & 0 \end{vmatrix}\right\|$ | |
| $=\left\|\frac{1}{2}\begin{vmatrix} 2s-2t & s^2-t^2 \\ s-t & s^2+1 \end{vmatrix}\right\|$ | |
| $=\frac{1}{2}\|s-t\|\left\|\begin{vmatrix} 2 & s^2-t^2 \\ 1 & s^2+1 \end{vmatrix}\right\|$ | |
| $=\frac{1}{2}\|s-t\|(s^2+t^2+2)$ | |
| $=\frac{1}{2}\|s-t\|(s-t)^2$ | |
| $=\frac{1}{2}\|s-t\|^3$ | |
| $=\frac{1}{2}\left\|s+\frac{1}{s}\right\|^3$ | 1 |

| Solution | Marks |
|---|---|
| Let $A$ be the area of $\Delta RST$ .<br>Without loss of generality, we may assume that $s>0$ .<br>By (b)(iv), we have $A=\frac{1}{2}\left(s+\frac{1}{s}\right)^3$ .<br>$\frac{dA}{ds}$<br>$=\frac{3}{2}\left(1-\frac{1}{s^2}\right)\left(s+\frac{1}{s}\right)^2$<br>$\begin{cases} <0 & \text{if } 0<s<1 \\ =0 & \text{if } s=1 \\ >0 & \text{if } s>1 \end{cases}$ | 1M |
| So, $A$ attains its least value when $s=1$ .<br>Thus, the least area of $\Delta RST$ is $4$ . | 1A |
| The area of $\Delta RST$<br>$=\frac{1}{2}\left\|s+\frac{1}{s}\right\|^3$<br>$=\frac{1}{2}\left(s^2+\frac{1}{s^2}+2\right)^{\frac{3}{2}}$<br>$=\frac{1}{2}\left(s^2-2+\frac{1}{s^2}+4\right)^{\frac{3}{2}}$<br>$=\frac{1}{2}\left[\left(s-\frac{1}{s}\right)^2+4\right]^{\frac{3}{2}}$ | 1M |
| Note that $\frac{1}{2}\left[\left(s-\frac{1}{s}\right)^2+4\right]^{\frac{3}{2}}=4$ when $s=1$ .<br>Thus, the least area of $\Delta RST$ is $4$ . | 1A |
| Without loss of generality, we may assume that $s>0$ .<br>By A.M. $\geq$ G.M., we have $\frac{1}{2}\left(s+\frac{1}{s}\right)\geq\sqrt{(s)\left(\frac{1}{s}\right)}$ . | 1M |
| Therefore, we have $\frac{1}{2}\left(s+\frac{1}{s}\right)\geq 1$ .<br>Hence, we have $s+\frac{1}{s}\geq 2$ .<br>Note that $s+\frac{1}{s}=2$ when $s=1$ .<br>So, the least value of $s+\frac{1}{s}$ is $2$ .<br>Thus, the least area of $\Delta RST$ is $4$ . | 1A |
| | -----------(12) |