2009-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2009

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(C) answer book.

4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. It is known that for any positive integer n, the coefficient of x^k in the expansion of $(1+x)^n$ is C_k^n.

(a) Prove that $C_k^n + C_{k+1}^n = C_{k+1}^{n+1}$ for all $k = 0, 1, \ldots, n-1$.

(b) Using $(1+x)^{n+1}(1+x)^n = (1+x)^{2n+1}$, prove that $\sum_{k=0}^{n} C_k^{n+1} C_k^n = C_n^{2n+1}$.

(c) Using (a) and (b), or otherwise, evaluate $\sum_{k=0}^{2008} (C_k^{2009} + C_{k+1}^{2009}) C_{k+1}^{2009}$.

(7 marks)

2. (a) Resolve $\frac{1}{(2x-1)(2x+1)(2x+3)}$ into partial fractions.

(b) Evaluate $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)(2k+3)}$.

(c) Find the greatest positive integer m such that $\sum_{k=m}^{\infty} \frac{1}{(2k-1)(2k+1)(2k+3)} > \frac{1}{4000}$.

(7 marks)

3. It is given that $f(x)$ and $g(x)$ are polynomials with real coefficients satisfying the following conditions:

(1) $f(x) = g(x) + x^3 + kx^2 + 8x + 8$, where k is a real constant;

(2) $f(x)$ and $g(x)$ are both divisible by $x+2$.

(a) Find k.

(b) Suppose that when $g(x)$ is divided by x^2-1, the remainder is $4x-1$. Find

(i) $f(1)$,

(ii) the remainder when $f(x)$ is divided by x^2-1.

(6 marks)

4. It is known that the matrix $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ represents the reflection in the straight line $y = (\tan\alpha)x$.

Let A be the matrix representing the reflection in the straight line $y = \left(\tan\frac{\pi}{8}\right)x$.

(a) Write down the matrix A.

(b) Let $B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$.

(i) Describe the geometric meaning of the transformation represented by B.

(ii) Does the matrix AB represent a rotation? Explain your answer.

(7 marks)

5. For every positive integer n, define $a_n = \frac{3^n}{(n+2)!}$.

(a) Prove that $a_n < \left(\frac{3}{4}\right)^n$.

(b) Evaluate $\lim_{n\to\infty} a_n$.

(c) Let $A_n = \sum_{k=1}^{n} a_k$. Does $\lim_{n\to\infty} A_n$ exist? Explain your answer.

(7 marks)

6. Let $S = \sum_{k=1}^{n} a_k$, where $a_1, a_2, \ldots, a_n$ are positive real numbers.

(a) Let $G = \left(\prod_{k=1}^{n} (S + a_k)\right)^{\frac{1}{n}}$. Using A.M. ≥ G.M., prove that

(i) $\sum_{k=1}^{n} \frac{1}{S + a_k} \geq \frac{n}{G}$,

(ii) $\frac{S}{G} \geq \frac{n}{n+1}$.

(b) Using (a), or otherwise, prove that $\sum_{k=1}^{n} \frac{S}{S + a_k} \geq \frac{n^2}{n+1}$.

(6 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. (a) Consider the system of linear equations in x, y, z

$$(E): \begin{cases} x + \lambda y + 2z = 1 \\ 5x - \lambda y + z = 5 \\ \lambda x - y + z = a \end{cases}, \text{ where } \lambda, a \in \mathbf{R} .$$

(i) Find the range of values of λ for which (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) Suppose that $\lambda = -1$. Find the value(s) of a for which (E) is consistent, and solve (E) for such value(s) of a.

(8 marks)

(b) Is the system of linear equations

$$\begin{cases} x - 2y + 2z = 1 \\ 5x + 2y + z = 5 \\ 2x + y - z = -3 \\ 4x + 3y - 3z = 2 \end{cases}$$

consistent? Explain your answer.

(3 marks)

(c) Find the solution(s) of the system of linear equations

$$\begin{cases} x - y + 2z = 1 \\ 5x + y + z = 5 \\ x + y - z = 1 \end{cases}$$

satisfying $4x^2 + 2y - z = 28$.

(4 marks)

8. Let $M = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$.

(a) Suppose that $M = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $r > 0$ and $0 < \theta < 2\pi$. Find r and θ.

(2 marks)

(b) A real matrix of the form $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ is called a 2×2 diagonal matrix. Find all positive integers n such that M^n is a 2×2 diagonal matrix, and evaluate M^n for such values of n.

(3 marks)

(c) Let A be a 2×2 real matrix such that $AM = MA$. Prove that $(A-M)(A^{2009} + A^{2008}M + A^{2007}M^2 + \cdots + M^{2009}) = A^{2010} - M^{2010}$.

(2 marks)

(d) Denote the 2×2 identity matrix by I. Is $2I - M$ a singular matrix? Explain your answer.

(2 marks)

(e) Evaluate $2^{2008}M + 2^{2007}M^2 + 2^{2006}M^3 + \cdots + M^{2009}$.

(6 marks)

9. (a) Consider the equation

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0 \quad \ldots\ldots\ldots\ldots\ldots (*),$$

where A, B, C and D are real constants.

Prove that $(*)$ can be written as

$$(x^2 + px)^2 + q(x^2 + px) + r = 0 \quad \ldots\ldots\ldots\ldots\ldots (**)$$

if and only if $C = \frac{A}{2}\left(B - \frac{A^2}{4}\right)$.

(5 marks)

(b) Consider the equation

$$x^4 - 20x^3 + \lambda x^2 + 230x + 120 = 0 \quad \ldots\ldots\ldots\ldots\ldots (***),$$

where λ is a real constant.

If $(***)$ can be written as $(**)$, find the value of λ and solve $(***)$.

(7 marks)

(c) Find all values of μ such that the equation $x^4 + \mu x^3 + 77x^2 + 230x + 60 = 0$ can be written as $(**)$.

(3 marks)

10. (a) Prove that $t\ln t - (1+t)\ln(1+t) < 0$ for all $t > 0$.

(2 marks)

(b) Let $a > 0$. Define $f(x) = \frac{\ln(1+a^x)}{x}$ for all $x > 0$. Prove that $f'(x) < 0$ for all $x > 0$.

(4 marks)

(c) It is given that $p \ge q > 0$.

(i) Using (b), prove that $(1+a^p)^q \le (1+a^q)^p$ for all $a > 0$.

(ii) Using (c)(i), prove that $(\lambda^p + \mu^p)^q \le (\lambda^q + \mu^q)^p$ for any 2 positive real numbers λ and μ.

(iii) Using mathematical induction, prove that $\left(\sum_{k=1}^{n} a_k^{\,p}\right)^q \le \left(\sum_{k=1}^{n} a_k^{\,q}\right)^p$ for any n positive real numbers $a_1, a_2, \ldots, a_n$.

(iv) Using (c)(iii), prove that $\left(\sum_{k=1}^{n} a_k^{\frac{1}{q}}\right)^q \le \left(\sum_{k=1}^{n} a_k^{\frac{1}{p}}\right)^p$ for any n positive real numbers $a_1, a_2, \ldots, a_n$.

(9 marks)

11. (a) (i) By solving the equation $z^{10}=1$, prove that

$$z^{10}-1=(z-1)(z+1)\prod_{k=1}^{4}\left(z^2-2z\cos\frac{k\pi}{5}+1\right).$$

(ii) Prove that if $z\neq 0$, then

$$z^5-\frac{1}{z^5}=\left(z-\frac{1}{z}\right)\prod_{k=1}^{4}\left(z+\frac{1}{z}-2\cos\frac{k\pi}{5}\right).$$

(5 marks)

(b) Let $z=\cos\theta+i\sin\theta$, where $\theta\in\mathbf{R}$. For any positive integer n,

(i) express $z^n+\frac{1}{z^n}$ in terms of $\cos n\theta$;

(ii) express $z^n-\frac{1}{z^n}$ in terms of $\sin n\theta$.

(3 marks)

(c) (i) Evaluate $\lim_{\theta\to 0}\frac{\sin 5\theta}{\sin\theta}$.

(ii) Prove that $\sin 5\theta=16\sin\theta\prod_{k=1}^{4}\left(\cos\theta-\cos\frac{k\pi}{5}\right)$ for all $\theta\in\mathbf{R}$.

Hence evaluate $\prod_{k=1}^{4}\sin\frac{k\pi}{10}$.

(7 marks)

END OF PAPER

2009-AL
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PAPER 2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
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PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
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2. Answer ALL questions in Section A, using the AL(E) answer book.
3. Answer any FOUR questions in Section B, using the AL(C) answer book.
4. Unless otherwise specified, all working must be clearly shown.

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$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Evaluate

(a) $\lim_{x \to 0} \dfrac{(e^x - 1)\sin x}{1 - \cos x}$,

(b) $\lim_{x \to 0} \dfrac{4\cot x - \sin \frac{1}{x}}{\cot x + 4 \sin \frac{1}{x}}$.

(7 marks)

2. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \begin{cases} x+3 & \text{when } x \le 2 , \\ 5 & \text{when } x > 2 . \end{cases}$

(a) Sketch the graph of $y = f(x)$.

(b) Is f a surjective function? Explain your answer.

(c) Let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = f(x-1) + f(x+1) - 6$.

(i) Sketch the graph of $y = g(x)$.

(ii) Write down the value(s) of x at which $g(x)$ is not differentiable.

(6 marks)

3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = (25 + x^2)^{\frac{-3}{2}}$.

(a) Prove that $(25 + x^2) f'(x) = -3x f(x)$.

Hence prove that $(25 + x^2) f^{(n+1)}(x) + (2n+3)x f^{(n)}(x) + n(n+2) f^{(n-1)}(x) = 0$ for all positive integers n , where $f^{(0)} = f$.

(b) Using (a), or otherwise, evaluate

(i) $f^{(5)}(0)$,

(ii) $f^{(6)}(0)$.

(6 marks)

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4. Let $\mathrm{f}: \mathbf{R} \to \mathbf{R}$ be a continuous function such that $\mathrm{f}(x) = 3 - 2\int_0^x \mathrm{f}(t)\,\mathrm{d}t$ for all $x \in \mathbf{R}$.

(a) Find $\mathrm{f}(0)$.

(b) Express $\mathrm{f}'(x)$ in terms of $\mathrm{f}(x)$.

(c) By considering the derivative of $e^{2x}\mathrm{f}(x)$, find $\mathrm{f}(x)$.

(7 marks)

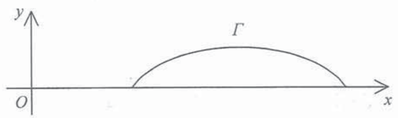
5. (a) Find

(i) $\int \cos^2 t\,\mathrm{d}t$,

(ii) $\int \cos^3 t\,\mathrm{d}t$.

(b) The figure below shows the curve

$$\Gamma: \begin{cases} x = 3\sin t + 5 \\ y = 2\cos t - 1 \end{cases}, \text{ where } \frac{-\pi}{3} \le t \le \frac{\pi}{3} .$$



The region bounded by Γ and the x-axis is revolved about the x-axis. Find the volume of the solid of revolution generated.

(7 marks)

6. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two distinct points on the hyperbola $H: y^2 - 4x^2 = 4$. The tangents to H at A and B intersect at the point $C(c, c)$.

(a) Prove that

(i) the equation of the tangent to H at A is $4x_1x - y_1y + 4 = 0$,

(ii) $c \neq 0$,

(iii) the equation of the straight line passing through A and B is $4cx - cy + 4 = 0$.

(b) Find the coordinates of the mid-point of AB in terms of c .

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. Let $\mathrm{f}(x) = \dfrac{|x|(x+16)}{x-2}$ for $x \neq 2$.

(a) (i) Is $\mathrm{f}(x)$ differentiable at $x = 0$? Explain your answer.

(ii) Find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$ for $x \neq 0$.

(4 marks)

(b) Solve each of the following inequalities:

(i) $\mathrm{f}'(x) > 0$,

(ii) $\mathrm{f}''(x) > 0$.

(2 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y = \mathrm{f}(x)$.

(4 marks)

(d) Find the asymptote(s) of the graph of $y = \mathrm{f}(x)$.

(3 marks)

(e) Sketch the graph of $y = \mathrm{f}(x)$.

(2 marks)

8. (a) Let $0 < a < \dfrac{\pi}{2}$.

(i) Prove that $\int_0^a \ln\cos(a - x)\,\mathrm{d}x = \int_0^a \ln\cos x\,\mathrm{d}x$.

(ii) Using (a)(i), prove that $\int_0^a \ln(\cos a + \sin a \tan x)\,\mathrm{d}x = 0$.

(5 marks)

(b) (i) Prove that $\int_0^1 \dfrac{\tan^{-1} x}{1+x}\,\mathrm{d}x = \dfrac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1 + \tan\theta)\,\mathrm{d}\theta$.

(ii) Evaluate $\int_0^1 \dfrac{\tan^{-1} x}{1+x}\,\mathrm{d}x$.

(6 marks)

(c) Evaluate $\lim\limits_{n\to\infty} \sum\limits_{k=2n+1}^{12n} \dfrac{1}{8n+k}\tan^{-1}\left(\dfrac{k-2n}{10n}\right)$.

(4 marks)

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9. (a) For any positive integer n, define $f_n(x) = x^n e^{-x}$ and let $I_n = \int_0^1 f_n(x)\,dx$.

(i) Prove that $f_n(x)$ is strictly increasing on $(0,1)$.

(ii) Prove that $0 < I_n < \frac{1}{e}$.

(iii) Evaluate I_1.

(iv) Express I_{n+1} in terms of I_n.

(v) Prove that $I_n = n!\left(1 - \frac{1}{e}\sum_{k=0}^{n}\frac{1}{k!}\right)$.

(12 marks)

(b) Using (a), prove that e is an irrational number.

(3 marks)

10. (a) Denote the set of positive real numbers by $\mathbf{R}^+$. Let $f:\mathbf{R}^+ \to \mathbf{R}$ be a twice differentiable function. Suppose that $f''(x) \geq 0$ for all $x \in \mathbf{R}^+$.

(i) Using Mean Value Theorem, or otherwise, prove that $f'(k) \leq f(k+1) - f(k) \leq f'(k+1)$ for all integers $k \geq 1$.

(ii) Using (a)(i), prove that $\sum_{k=1}^{n-1} f'(k) \leq f(n) - f(1) \leq \sum_{k=2}^{n} f'(k)$ for all integers $n \geq 2$.

(5 marks)

(b) Suppose that $r > 1$.

(i) Using (a), or otherwise, prove that $\frac{1}{n^r} - \sum_{k=1}^{n}\frac{1}{k^r} \leq \frac{1}{(r-1)n^{r-1}} - \frac{1}{r-1} \leq 1 - \sum_{k=1}^{n}\frac{1}{k^r}$ for all positive integers n.

(ii) Is the series $\sum_{k=1}^{\infty}\frac{1}{k^r}$ convergent? Explain your answer.

(5 marks)

(c) (i) Using (a), or otherwise, prove that $\sum_{k=1}^{n}\frac{1}{k} \geq \ln(n+1)$ for all positive integers n.

(ii) Is the series $\sum_{k=1}^{\infty}\frac{1}{k}$ convergent? Explain your answer.

(5 marks)

11. Let L_1 be the normal to the parabola $P: x^2 = 4y$ at the point $S(2s, s^2)$, where $s \neq 0$.

(a) Find the equation of L_1.

(3 marks)

(b) The straight line passing through S and the point $(0,1)$ cuts P at another point $T(2t, t^2)$. Let L_2 be the normal to P at T. The normals L_1 and L_2 intersect at the point R.

(i) Prove that $st = -1$.

(ii) Describe the geometric relationship between L_1 and L_2. Explain your answer.

(iii) Prove that $RS = \frac{1}{s^2}(1+s^2)^{\frac{3}{2}}$.

(iv) Prove that the area of ΔRST is $\frac{1}{2}\left|s + \frac{1}{s}\right|^3$.

Hence find the least area of ΔRST.

(12 marks)

END OF PAPER