2008 AL Pure Mathematics

評卷參考 *
Marking Schemes

> These documents were prepared for markers' reference. They should not be regarded as sets of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret their contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. Marks may be deducted for poor presentation (*pp*). The symbol (*pp-1*) should be used to denote 1 mark deducted for *pp*. At most deduct 1 mark from Section A and 1 mark from Section B for *pp*. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

7. In the marking scheme, 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

* 此部分只設英文版本

Paper 1

Solution	Marks
1. (a) $(1+x)^m(1+x)^n=(1+x)^{m+n}$	
$(C_0^m+C_1^m x+\cdots+C_m^m x^m)(C_0^n+C_1^n x+\cdots+C_n^n x^n)=C_0^{m+n}+C_1^{m+n}x+\cdots+C_{m+n}^{m+n}x^{m+n}$	1A for either one correct
For each $r=m,m+1,\dots,n$, by comparing the coefficients of x^r in both	1M
sides, we have $C_0^m C_r^n+C_1^m C_{r-1}^n+\cdots+C_m^m C_{r-m}^n=C_r^{m+n}$.	1
Thus, we have $\sum_{k=0}^{m} C_k^m C_{r-k}^n=C_r^{m+n}$ for all $r=m,m+1,\dots,n$.	
(b) $\sum_{k=0}^{99} C_k^{99} C_{k+1}^{100}$	
$=\sum_{k=0}^{99} C_k^{99} C_{99-k}^{100}$	1M (either one)
$=C_{99}^{199}$ (by (a) with $m=99$, $n=100$ and $r=99$)	1A (either one)
$\sum_{k=0}^{99} C_k^{99} C_k^{101}$	
$=\sum_{k=0}^{99} C_k^{99} C_{101-k}^{101}$	(either one with 1M)
$=C_{101}^{200}$ (by (a) with $m=99$, $n=101$ and $r=101$)	(either one with 1A)
$\dfrac{\sum_{k=0}^{99} C_k^{99} C_{k+1}^{100}}{\sum_{k=0}^{99} C_k^{99} C_k^{101}}$ $=\dfrac{C_{99}^{199}}{C_{101}^{200}}$ $=\dfrac{\frac{199!}{99!100!}}{\frac{200!}{101!99!}}$ $=\left(\dfrac{199!}{99!100!}\right)\left(\dfrac{101!99!}{200!}\right)$	
$=\dfrac{101}{200}$	1A
	----------(6)

Solution	Marks
2. (a) Let $\frac{7x+9}{x(x+1)(x+3)}=\frac{C_1}{x}+\frac{C_2}{x+1}+\frac{C_3}{x+3}$.	1M can be absorbed
Then, we have $7x+9\equiv C_1(x+1)(x+3)+C_2\,x(x+3)+C_3\,x(x+1)$.	
Putting $x=0,-1,-3$, we have $C_1=3$, $C_2=-1$, $C_3=-2$.	1A for all correct
Thus, we have $\frac{7x+9}{x(x+1)(x+3)}=\frac{3}{x}-\frac{1}{x+1}-\frac{2}{x+3}$.	
(b) $\sum_{k=1}^{n}\frac{7k+9}{k(k+1)(k+3)}$	
$=\sum_{k=1}^{n}\left(\frac{3}{k}-\frac{1}{k+1}-\frac{2}{k+3}\right)$	
$=\sum_{k=1}^{n}\left(\frac{3}{k}-\frac{3}{k+1}\right)+\sum_{k=1}^{n}\left(\frac{2}{k+1}-\frac{2}{k+3}\right)$	1M
$=3\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)+2\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)$	
$=3\left(1-\frac{1}{n+1}\right)+2\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right)$	1A for either sum correct
$=\frac{14}{3}-\frac{3}{n+1}-\frac{2}{n+2}-\frac{2}{n+3}$	1A
(c) $\sum_{k=1}^{4}\frac{7k+9}{k(k+1)(k+3)}$	
$=\frac{14}{3}-\frac{3}{5}-\frac{2}{6}-\frac{2}{7}$	1M (either one)
$=\frac{362}{105}$	
$\sum_{k=1}^{\infty}\frac{7k+9}{k(k+1)(k+3)}$	
$=\lim_{n\to\infty}\left(\frac{14}{3}-\frac{3}{n+1}-\frac{2}{n+2}-\frac{2}{n+3}\right)$	
$=\frac{14}{3}$	
$\sum_{k=5}^{\infty}\frac{7k+9}{k(k+1)(k+3)}$	
$=\sum_{k=1}^{\infty}\frac{7k+9}{k(k+1)(k+3)}-\sum_{k=1}^{4}\frac{7k+9}{k(k+1)(k+3)}$	
$=\frac{14}{3}-\frac{362}{105}$	
$=\frac{128}{105}$	1A
	---------(7)

Solution	Marks
3. (a) $a^3+b^3+c^3$	
$=(-b-c)^3+b^3+c^3$	
$=-b^3-3b^2c-3bc^2-c^3+b^3+c^3$	1M
$=-3b^2c-3bc^2$	
$=3(-b-c)bc$	
$=3abc$	1
Note that $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.	1M
Since $a+b+c=0$, we have $a^3+b^3+c^3=3abc$.	1
(b) Note that $-(3x+6\sqrt{2}-7\sqrt{3})^3=(-3x-6\sqrt{2}+7\sqrt{3})^3$.	
Also note that $8(x+4\sqrt{2}-6\sqrt{3})^3=(2x+8\sqrt{2}-12\sqrt{3})^3$.	
So, we have	
$(x-2\sqrt{2}+5\sqrt{3})^3-(3x+6\sqrt{2}-7\sqrt{3})^3+8(x+4\sqrt{2}-6\sqrt{3})^3=0$	
$\Leftrightarrow (x-2\sqrt{2}+5\sqrt{3})^3+(-3x-6\sqrt{2}+7\sqrt{3})^3+(2x+8\sqrt{2}-12\sqrt{3})^3=0$	1M
Since $(x-2\sqrt{2}+5\sqrt{3})+(-3x-6\sqrt{2}+7\sqrt{3})+(2x+8\sqrt{2}-12\sqrt{3})=0$,	1M withhold 1M for no checking
we have	
$(x-2\sqrt{2}+5\sqrt{3})^3-(3x+6\sqrt{2}-7\sqrt{3})^3+8(x+4\sqrt{2}-6\sqrt{3})^3=0$	
$\Leftrightarrow 3(x-2\sqrt{2}+5\sqrt{3})(-3x-6\sqrt{2}+7\sqrt{3})(2x+8\sqrt{2}-12\sqrt{3})=0$	1M for using (a)
$\Leftrightarrow (x-2\sqrt{2}+5\sqrt{3})(-3x-6\sqrt{2}+7\sqrt{3})(2x+8\sqrt{2}-12\sqrt{3})=0$	
$\Leftrightarrow x=2\sqrt{2}-5\sqrt{3}$, $x=\dfrac{7\sqrt{3}-6\sqrt{2}}{3}$ or $x=6\sqrt{3}-4\sqrt{2}$	1A for all correct
Thus, the roots are $2\sqrt{2}-5\sqrt{3}$, $6\sqrt{3}-4\sqrt{2}$ and $\dfrac{7\sqrt{3}-6\sqrt{2}}{3}$.	
	----------(6)

Solution	Marks
4. (a) A	
$= \begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix}$	1A
$= \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$	
(b) (i) A^{-1}	
$= \begin{pmatrix} \cos(-120^\circ) & -\sin(-120^\circ) \\ \sin(-120^\circ) & \cos(-120^\circ) \end{pmatrix}$	1M for finding A^{-1}
$= \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$	
Let the coordinates of P be (a, b) .	
Then, we have $A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 2\sqrt{3} \end{pmatrix}$.	
So, we have	
$\begin{pmatrix} a \\ b \end{pmatrix}$	
$= A^{-1}\begin{pmatrix} 4 \\ 2\sqrt{3} \end{pmatrix}$	
$= \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 2\sqrt{3} \end{pmatrix}$	
$= \begin{pmatrix} 1 \\ -3\sqrt{3} \end{pmatrix}$	1A
Thus, the coordinates of P are $(1, -3\sqrt{3})$.	
(ii) T_n is the rotation in the Cartesian plane anticlockwise about the origin by $120n^\circ$.	1A for rotation 1A for the details
(iii) Let O be the origin and Q be the point $(3, 5\sqrt{3})$.	
Then, we have $OP = 2\sqrt{7}$ and $OQ = 2\sqrt{21}$.	
So, we have $OP \neq OQ$.	1M
Thus, there is no transformation T_n which transforms P to the point $(3, 5\sqrt{3})$.	1A f.t.

Solution	Marks
By (b)(i), $A^n = \begin{pmatrix} \cos 120n^\circ & -\sin 120n^\circ \\ \sin 120n^\circ & \cos 120n^\circ \end{pmatrix}$.	
Suppose that $\begin{pmatrix} \cos 120n^\circ & -\sin 120n^\circ \\ \sin 120n^\circ & \cos 120n^\circ \end{pmatrix}\begin{pmatrix} 1 \\ -3\sqrt{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 5\sqrt{3} \end{pmatrix}$ for some positive integer m.	
Then, we have $\begin{cases} \cos 120n^\circ + 3\sqrt{3}\sin 120n^\circ = 3 \\ \sin 120n^\circ - 3\sqrt{3}\cos 120n = 5\sqrt{3} \end{cases}$.	1M
So, we have $\sin 120n^\circ = \frac{\sqrt{3}}{2}$ and $\cos 120n^\circ = \frac{-3}{2}$ which is impossible.	
Thus, there is no transformation T_n which transforms P to the point $(3, 5\sqrt{3})$.	1A f.t.
	----------(7)
5. (a) $3\lvert z-2i\rvert = \lvert z+8-2i\rvert$	
$9(z-2i)(\bar{z}+2i) = (z+8-2i)(\bar{z}+8+2i)$	1M
$z\bar{z} + 2iz - 2i\bar{z} - z - \bar{z} - 4 = 0$	
$z\bar{z} + (-1+2i)z + (-1-2i)\bar{z} - 4 = 0$	
$z\bar{z} + (-1+2i)z + (-1-2i)\bar{z} + (-1-2i)(-1+2i) - (-1-2i)(-1+2i) - 4 = 0$	1M
$z\bar{z} + (-1+2i)z + (-1-2i)\bar{z} + (-1-2i)(-1+2i) - 5 - 4 = 0$	
$(z+(-1-2i))(\bar{z}+(-1+2i)) = 9$	
$\lvert z-(1+2i)\rvert = 3$	1A
Thus, S is represented by a circle on the Argand diagram.	
The centre and the radius of the circle are $1+2i$ and 3 respectively.	1A for both correct
Let $z = x+yi$ where $x, y \in \mathbf{R}$.	
$3\lvert z-2i\rvert = \lvert z+8-2i\rvert$	
$3\lvert x+(y-2)i\rvert = \lvert (x+8)+(y-2)i\rvert$	
$9x^2 + 9(y-2)^2 = (x+8)^2 + (y-2)^2$	1M
$x^2 + y^2 - 2x - 4y - 4 = 0$	
$(x-1)^2 + (y-2)^2 - 5 - 4 = 0$	1M
$(x-1)^2 + (y-2)^2 = 9$	1A
Thus, S is represented by a circle on the Argand diagram.	
The centre and the radius of the circle are $1+2i$ and 3 respectively.	1A for both correct
(b) The distance between P and the centre of the circle	
$= \sqrt{(-7-1)^2 + (17-2)^2}$	1M
$= 17$	
The longest distance between P and Q	
$= 17+3$	1M
$= 20$	1A
	----------(7)

Solution	Marks
6. (a) (i) By Cauchy-Schwarz's inequality, we have	
$((1)(\alpha)+(1)(\beta)+(1)(\gamma)+(1)(\delta))^2 \le (1^2+1^2+1^2+1^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)$	1M
$(\alpha+\beta+\gamma+\delta)^2 \le 4(\alpha^2+\beta^2+\gamma^2+\delta^2)$	1
Since $(x-y)^2 \ge 0$, $2xy \le x^2+y^2$ for any real numbers x and y .	
Hence, we have $2\alpha\beta \le \alpha^2+\beta^2$, $2\alpha\gamma \le \alpha^2+\gamma^2$, $2\alpha\delta \le \alpha^2+\delta^2$,	1M for either
$2\beta\gamma \le \beta^2+\gamma^2$, $2\beta\delta \le \beta^2+\delta^2$ and $2\gamma\delta \le \gamma^2+\delta^2$.	
So, we have $2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) \le 3(\alpha^2+\beta^2+\gamma^2+\delta^2)$.	
$(\alpha+\beta+\gamma+\delta)^2$	
$=(\alpha^2+\beta^2+\gamma^2+\delta^2)+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	
$\le(\alpha^2+\beta^2+\gamma^2+\delta^2)+3(\alpha^2+\beta^2+\gamma^2+\delta^2)$	
$=4(\alpha^2+\beta^2+\gamma^2+\delta^2)$	1
(ii) Note that α, β, γ and δ are positive real numbers.	
By Cauchy-Schwarz's inequality, we have	
$(\alpha^{\frac{1}{2}}\alpha^{\frac{3}{2}}+\beta^{\frac{1}{2}}\beta^{\frac{3}{2}}+\gamma^{\frac{1}{2}}\gamma^{\frac{3}{2}}+\delta^{\frac{1}{2}}\delta^{\frac{3}{2}})^2$	1M
$\le((\alpha^{\frac{1}{2}})^2+(\beta^{\frac{1}{2}})^2+(\gamma^{\frac{1}{2}})^2+(\delta^{\frac{1}{2}})^2)((\alpha^{\frac{3}{2}})^2+(\beta^{\frac{3}{2}})^2+(\gamma^{\frac{3}{2}})^2+(\delta^{\frac{3}{2}})^2)$	
Thus, we have	
$(\alpha^2+\beta^2+\gamma^2+\delta^2)^2 \le (\alpha+\beta+\gamma+\delta)(\alpha^3+\beta^3+\gamma^3+\delta^3)$	1
Note that $ab(a-b)^2 \ge 0$ for any positive real numbers a and b .	
So, we have $2a^2b^2 \le a^3b+ab^3$ for any positive real numbers a and b .	
Since α, β, γ and δ are positive real numbers, we have	
$2\alpha^2\beta^2 \le \alpha^3\beta+\alpha\beta^3$, $2\alpha^2\gamma^2 \le \alpha^3\gamma+\alpha\gamma^3$, $2\alpha^2\delta^2 \le \alpha^3\delta+\alpha\delta^3$,	1M for either
$2\beta^2\gamma^2 \le \beta^3\gamma+\beta\gamma^3$, $2\beta^2\delta^2 \le \beta^3\delta+\beta\delta^3$ and $2\gamma^2\delta^2 \le \gamma^3\delta+\gamma\delta^3$.	
Hence, we have	
$2(\alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2)$	
$\le\alpha^3\beta+\alpha\beta^3+\alpha^3\gamma+\alpha\gamma^3+\alpha^3\delta+\alpha\delta^3+\beta^3\gamma+\beta\gamma^3+\beta^3\delta+\beta\delta^3+\gamma^3\delta+\gamma\delta^3$	
$=(\alpha+\beta+\gamma+\delta)(\alpha^3+\beta^3+\gamma^3+\delta^3)-(\alpha^4+\beta^4+\gamma^4+\delta^4)$	
Thus, we have	
$(\alpha^2+\beta^2+\gamma^2+\delta^2)^2$	
$=(\alpha^4+\beta^4+\gamma^4+\delta^4)+2(\alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2)$	
$\le(\alpha+\beta+\gamma+\delta)(\alpha^3+\beta^3+\gamma^3+\delta^3)$	1
(b) Note that α, β, γ and δ are positive real numbers.	
By (a)(i) and (a)(ii), we have	
$(\alpha+\beta+\gamma+\delta)^2(\alpha^2+\beta^2+\gamma^2+\delta^2)^2$	1M
$\le 4(\alpha^2+\beta^2+\gamma^2+\delta^2)(\alpha+\beta+\gamma+\delta)(\alpha^3+\beta^3+\gamma^3+\delta^3)$	
So, we have $(\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2) \le 4(\alpha^3+\beta^3+\gamma^3+\delta^3)$.	
Thus, we have $\dfrac{\alpha+\beta+\gamma+\delta}{4} \le \dfrac{\alpha^3+\beta^3+\gamma^3+\delta^3}{\alpha^2+\beta^2+\gamma^2+\delta^2}$.	1
The equality in (b) holds	
if and only if the equalities in (a)(i) and (a)(ii) both hold	
if and only if $\alpha=\beta=\gamma=\delta$	1A
	----------(7)

Solution	Marks
7. (a) (i) (E) has a unique solution	
$\Leftrightarrow \quad \Delta \neq 0$	
$\Leftrightarrow \quad \Delta = \begin{vmatrix} 1 & a+2 & a+1 \\ 1 & -3 & -1 \\ 3 & -2 & a-1 \end{vmatrix} \neq 0$	1M
$\Leftrightarrow \quad -3(a-1)-3(a+2)-2(a+1)+9(a+1)-2-(a+2)(a-1) \neq 0$	1A
$\Leftrightarrow \quad 4-a^2 \neq 0$	
$\Leftrightarrow \quad a^2 \neq 4$	1
The augmented matrix of (E) is	
$\left(\begin{array}{ccc\|c} 1 & a+2 & a+1 & 1 \\ 1 & -3 & -1 & b \\ 3 & -2 & a-1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & a+2 & a+1 & 1 \\ 0 & -a-5 & -a-2 & b-1 \\ 0 & -3a-8 & -2a-4 & -2 \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & a+2 & a+1 & 1 \\ 0 & -a-5 & -a-2 & b-1 \\ 0 & -a+2 & 0 & -2b \end{array}\right)$	1A
(E) has unique solution	
$\Leftrightarrow \quad -a+2 \neq 0$ and $-a-2 \neq 0$	1M
$\Leftrightarrow \quad (-a+2)(-a-2) \neq 0$	
$\Leftrightarrow \quad a^2 \neq 4$	1
When (E) has a unique solution,	
$x = \dfrac{\begin{vmatrix} 1 & a+2 & a+1 \\ b & -3 & -1 \\ 1 & -2 & a-1 \end{vmatrix}}{4-a^2} = \dfrac{-a^2b-3ab-a+2}{4-a^2}$	1M for Cramer's Rule
$y = \dfrac{\begin{vmatrix} 1 & 1 & a+1 \\ 1 & b & -1 \\ 3 & 1 & a-1 \end{vmatrix}}{4-a^2} = \dfrac{2b}{a-2}$	
$z = \dfrac{\begin{vmatrix} 1 & a+2 & 1 \\ 1 & -3 & b \\ 3 & -2 & 1 \end{vmatrix}}{4-a^2} = \dfrac{3ab-a+8b+2}{4-a^2}$	1A + 1A (1A for any one, 1A for all)

Solution	Marks

When (E) has a unique solution, the augmented matrix of (E)

$$\sim\left(\begin{array}{ccc:c} 1 & a+2 & a+1 & 1 \\ 0 & -a-5 & -a-2 & b-1 \\ 0 & 1 & 0 & \frac{2b}{a-2} \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & a+2 & a+1 & 1 \\ 0 & 0 & 1 & \frac{3ab-a+8b+2}{4-a^2} \\ 0 & 1 & 0 & \frac{2b}{a-2} \end{array}\right)$$

1M

$$\sim\left(\begin{array}{ccc:c} 1 & 0 & 0 & \frac{-a^2b-3ab-a+2}{4-a^2} \\ 0 & 0 & 1 & \frac{3ab-a+8b+2}{4-a^2} \\ 0 & 1 & 0 & \frac{2b}{a-2} \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 0 & 0 & \frac{-a^2b-3ab-a+2}{4-a^2} \\ 0 & 1 & 0 & \frac{2b}{a-2} \\ 0 & 0 & 1 & \frac{3ab-a+8b+2}{4-a^2} \end{array}\right)$$

Thus, $x=\frac{-a^2b-3ab-a+2}{4-a^2}$, $y=\frac{2b}{a-2}$ and $z=\frac{3ab-a+8b+2}{4-a^2}$. — 1A + 1A (1A for any one, 1A for all)

(ii) (1) When $a=2$, the augmented matrix of (E) is

$$\left(\begin{array}{ccc:c} 1 & 4 & 3 & 1 \\ 1 & -3 & -1 & b \\ 3 & -2 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 4 & 3 & 1 \\ 0 & -7 & -4 & b-1 \\ 0 & 0 & 0 & -2b \end{array}\right)$$

1M (either one: for (1) or (2))

So, (E) is consistent when $b=0$. — 1A

Thus, the solution set $\left\{\left(\frac{1+5t}{4},\ t,\ \frac{1-7t}{4}\right):\ t\in\mathbf{R}\right\}$. — 1A or equivalent

(2) When $a=-2$, the augmented matrix of (E) is

$$\left(\begin{array}{ccc:c} 1 & 0 & -1 & 1 \\ 1 & -3 & -1 & b \\ 3 & -2 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 0 & -1 & 1 \\ 0 & -3 & 0 & b-1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

So, (E) is consistent when $b-1=-3$.

Therefore, (E) is consistent when $b=-2$. — 1A

Thus, the solution set is $\{(1+t,\ 1,\ t):\ t\in\mathbf{R}\}$. — 1A or equivalent

----------(11)

(b) Putting $x=\frac{1+5t}{4}$, $y=t$ and $z=\frac{1-7t}{4}$ in $2x^2+15y^2-10z^2$, we have — 1M for using the result of (a)(ii)(1)

$$2x^2+15y^2-10z^2$$
$$=\frac{-25}{2}t^2+10t-\frac{1}{2}$$
$$=\frac{-25}{2}\left(t-\frac{2}{5}\right)^2+\frac{3}{2}$$

1M + 1A or equivalent

Thus, the greatest value is $\frac{3}{2}$. — 1A

----------(4)

Solution	Marks
8. (a) (i) M^2	
$=\begin{pmatrix} p & q \\ r & -p \end{pmatrix}\begin{pmatrix} p & q \\ r & -p \end{pmatrix}$	
$=\begin{pmatrix} p^2+qr & 0 \\ 0 & p^2+qr \end{pmatrix}$	1A
$=(p^2+qr)I$	
(ii) By (a)(i), we have $M^2=I$.	1A
$(M+sI)^2$ $=M^2+2sM+s^2I$ $=2sM+(s^2+1)I$	
$\frac{(s+1)^2-(s-1)^2}{2}M+\frac{(s+1)^2+(s-1)^2}{2}I$ $=2sM+(s^2+1)I$	
So, we have $(M+sI)^2=\frac{(s+1)^2-(s-1)^2}{2}M+\frac{(s+1)^2+(s-1)^2}{2}I$.	1
Therefore, the statement is true for $n=1$.	
Define $a=(s+1)^2$ and $b=(s-1)^2$.	
Assume that $(M+sI)^{2k}=\frac{a^k-b^k}{2}M+\frac{a^k+b^k}{2}I$ for some $k\in\mathbf{N}$.	1M
Then, we have $(M+sI)^{2(k+1)}$	
$=(M+sI)^{2k}(M+sI)^2$	
$=\left(\frac{a^k-b^k}{2}M+\frac{a^k+b^k}{2}I\right)(M+sI)^2$	1M for using induction assumption
$=\left(\frac{a^k-b^k}{2}M+\frac{a^k+b^k}{2}I\right)\left(\frac{a-b}{2}M+\frac{a+b}{2}I\right)$	1M for using the result of $n=1$
$=\left(\frac{a^k-b^k}{2}\right)\left(\frac{a-b}{2}\right)M^2+\left(\frac{a^{k+1}-b^{k+1}}{2}\right)M+\left(\frac{a^k+b^k}{2}\right)\left(\frac{a+b}{2}\right)I$	
$=\left(\frac{a^k-b^k}{2}\right)\left(\frac{a-b}{2}\right)I+\left(\frac{a^{k+1}-b^{k+1}}{2}\right)M+\left(\frac{a^k+b^k}{2}\right)\left(\frac{a+b}{2}\right)I$	
$=\left(\frac{a^{k+1}-b^{k+1}}{2}\right)M+\left[\left(\frac{a^k-b^k}{2}\right)\left(\frac{a-b}{2}\right)+\left(\frac{a^k+b^k}{2}\right)\left(\frac{a+b}{2}\right)\right]I$	
$=\left(\frac{a^{k+1}-b^{k+1}}{2}\right)M+\left(\frac{a^{k+1}+b^{k+1}}{2}\right)I$	1
$=\frac{(s+1)^{2(k+1)}-(s-1)^{2(k+1)}}{2}M+\frac{(s+1)^{2(k+1)}+(s-1)^{2(k+1)}}{2}I$	
Hence, the statement is true for $n=k+1$ when it is true for $n=k$.	
By mathematical induction,	
$(M+sI)^{2n}=\frac{(s+1)^{2n}-(s-1)^{2n}}{2}M+\frac{(s+1)^{2n}+(s-1)^{2n}}{2}I$.	
	----------(7)

Solution	Marks
(b) (i) Let $\begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} + \mu I$.	
Then, we have $\begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \alpha+\mu & \beta \\ \gamma & -\alpha+\mu \end{pmatrix}$.	
So, we have $\alpha+\mu=6$, $\beta=3$, $\gamma=-1$ and $-\alpha+\mu=2$.	1M can be absorbed
Therefore, we have $\alpha=2$, $\beta=3$, $\gamma=-1$ and $\mu=4$.	1A for all correct
Thus, we have $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} + 4I$.	
(ii) By (b)(i), we have $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} + 4I$.	
Let $M = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$.	
Note that $2^2 + (-1)(3) = 4 - 3 = 1$.	1M withhold 1M for no checking
Thus, we have	
A^{800}	
$= \frac{(4+1)^{800} - (4-1)^{800}}{2} M + \frac{(4+1)^{800} + (4-1)^{800}}{2} I$ (by (a)(ii))	1M for using (a)(ii)
$= \frac{5^{800} - 3^{800}}{2} M + \frac{5^{800} + 3^{800}}{2} I$	
$= \frac{5^{800} - 3^{800}}{2} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} + \frac{5^{800} + 3^{800}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$= \begin{pmatrix} \frac{3(5^{800}) - 3^{800}}{2} & \frac{3(5^{800}) - 3^{801}}{2} \\ \frac{3^{800} - 5^{800}}{2} & \frac{3^{801} - 5^{800}}{2} \end{pmatrix}$	1A
(iii) $(A^{-1})^{800}$	
$= (A^{800})^{-1}$	1A
$= \frac{1}{\det(A^{800})} \begin{pmatrix} \frac{3^{801} - 5^{800}}{2} & \frac{3^{801} - 3(5^{800})}{2} \\ \frac{5^{800} - 3^{800}}{2} & \frac{3(5^{800}) - 3^{800}}{2} \end{pmatrix}$ (by (b)(ii))	1M for finding the inverse of A^{800}
Note that $\det A = 12 + 3 = 15$.	
So, we have $\det(A^{800}) = (\det A)^{800} = 15^{800}$.	
Thus, we have $(A^{-1})^{800} = \frac{1}{15^{800}} \begin{pmatrix} \frac{3^{801} - 5^{800}}{2} & \frac{3^{801} - 3(5^{800})}{2} \\ \frac{5^{800} - 3^{800}}{2} & \frac{3(5^{800}) - 3^{800}}{2} \end{pmatrix}$.	1A
	----------(8)

Solution	Marks
9. (a) $x_{n+1}-y_{n+1}$	
$=\dfrac{x_n^2y_n+x_ny_n^2}{x_n^2+y_n^2}-\dfrac{x_n^2+y_n^2}{x_n+y_n}$	
$=\dfrac{(x_n^2y_n+x_ny_n^2)(x_n+y_n)-(x_n^2+y_n^2)^2}{(x_n^2+y_n^2)(x_n+y_n)}$	
$=\dfrac{x_n^3y_n+2x_n^2y_n^2+x_ny_n^3-x_n^4-2x_n^2y_n^2-y_n^4}{(x_n^2+y_n^2)(x_n+y_n)}$	1M
$=\dfrac{-(x_n^4-x_n^3y_n-x_ny_n^3+y_n^4)}{(x_n^2+y_n^2)(x_n+y_n)}$	
$=\dfrac{-(x_n^3-y_n^3)(x_n-y_n)}{(x_n+y_n)(x_n^2+y_n^2)}$	1
	----------(2)
(b) Note that $x_n>0$ and $y_n>0$ for all $n=1,2,3,\ldots$.	1A withhold 1A if omitted
(i) $x_{n+1}-y_{n+1}$	
$=\dfrac{-(x_n^3-y_n^3)(x_n-y_n)}{(x_n+y_n)(x_n^2+y_n^2)}$	
$=\dfrac{-(x_n^2+x_ny_n+y_n^2)(x_n-y_n)^2}{(x_n+y_n)(x_n^2+y_n^2)}$	1M
Note that $x_n^2+x_ny_n+y_n^2>0$ and $x_n+y_n>0$.	
So, we have $x_{n+1}\le y_{n+1}$.	1
Since $x_1<y_1$, we have $x_n\le y_n$.	
$\dfrac{x_{n+1}}{y_{n+1}}$	
$=\dfrac{(x_n+y_n)(x_n^2y_n+x_ny_n^2)}{(x_n^2+y_n^2)^2}$	
$=\dfrac{(x_n+y_n)^2x_ny_n}{(x_n^2+y_n^2)^2}$	
$=\dfrac{(x_n^2+y_n^2+2x_ny_n)x_ny_n}{(x_n^2+y_n^2)^2}$	
$\le\dfrac{(x_n^2+y_n^2+x_n^2+y_n^2)x_ny_n}{(x_n^2+y_n^2)^2}$	1M (either one)
$=\dfrac{2(x_n^2+y_n^2)x_ny_n}{(x_n^2+y_n^2)^2}$	
$=\dfrac{2x_ny_n}{x_n^2+y_n^2}$	
$\le\dfrac{x_n^2+y_n^2}{x_n^2+y_n^2}$	
$=1$	
So, we have $x_{n+1}\le y_{n+1}$.	1
Since $x_1<y_1$, we have $x_n\le y_n$.	

Solution	Marks
(ii) $x_{n+1}-x_n$	
$=\dfrac{x_n^2 y_n + x_n y_n^2}{x_n^2+y_n^2}-x_n$	
$=\dfrac{-x_n^2(x_n-y_n)}{x_n^2+y_n^2}$	
≥ 0	1
Thus, we have $x_{n+1}\geq x_n$.	
(iii) $y_{n+1}-y_n$	
$=\dfrac{x_n^2+y_n^2}{x_n+y_n}-y_n$	
$=\dfrac{x_n(x_n-y_n)}{x_n+y_n}$	
≤ 0	1
Thus, we have $y_{n+1}\leq y_n$.	
	----------(5)
(c) By (b), we have $2\leq x_n\leq x_{n+1}\leq y_{n+1}\leq y_n\leq 8$ for all $n=1,2,3,\ldots$.	
$\{x_n\}$ is monotonic increasing and bounded above by 8 .	1A
$\{y_n\}$ is monotonic decreasing and bounded below by 2 .	1A
Therefore, $\{x_n\}$ and $\{y_n\}$ converge.	
Let $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$.	
Then, we have $B=\dfrac{A^2+B^2}{A+B}$.	1M
So, we have $AB+B^2=A^2+B^2$.	
Hence, we have $(A-B)A=0$.	
Note that $A=\lim_{n\to\infty} x_n \geq 2>0$.	
Therefore, we have $A=B$.	1
Thus, $\{x_n\}$ and $\{y_n\}$ converge to the same limit.	
	----------(4)
(d) $x_n y_n$	
$=\left(\dfrac{x_{n-1}^2 y_{n-1}+x_{n-1}y_{n-1}^2}{x_{n-1}^2+y_{n-1}^2}\right)\left(\dfrac{x_{n-1}^2+y_{n-1}^2}{x_{n-1}+y_{n-1}}\right)$	
$=x_{n-1}y_{n-1}\left(\dfrac{x_{n-1}+y_{n-1}}{x_{n-1}^2+y_{n-1}^2}\right)\left(\dfrac{x_{n-1}^2+y_{n-1}^2}{x_{n-1}+y_{n-1}}\right)$	
$=x_{n-1}y_{n-1}$	
So, $x_n y_n$ is independent of n .	1
Since $x_1 y_1=16$, we have $x_n y_n=16$ for all $n=1,2,3,\ldots$.	1A
Therefore, we have $\lim_{n\to\infty}(x_n y_n)=16$.	
By (c), we have $\left(\lim_{n\to\infty} x_n\right)^2=16$.	1M for using (c)
Since $\lim_{n\to\infty} x_n>0$, we have $\lim_{n\to\infty} x_n=4$.	1A f.t.
	----------(4)

Solution	Marks
10. (a) Let $p(x)=a_0x^k+a_1x^{k-1}+\cdots+a_k$, where $a_0\neq 0$ and $k\geq 1$.	
Then, we have $p(x-1)=a_0(x-1)^k+a_1(x-1)^{k-1}+\cdots+a_k$.	
Since $k\geq 1$ and $a_0k\neq 0$, the degree and the leading coefficient of $p(x)-p(x-1)$ are $k-1$ and a_0k respectively.	
By noting that $p(x)-p(x-1)=x^{100}$, we have $k-1=100$ and $a_0k=1$.	1M can be absorbed
Thus, we have $k=101$ and the coefficient of x^k in $p(x)$ is $\frac{1}{101}$.	1A + 1A
	----------(3)
(b) $p(1)=p(1-1)+1$	
$p(1)=p(0)+1$	
$1=p(0)+1$	
$p(0)=0$	1A f.t.
$p(0)=p(0-1)+0$	
$0=p(-1)+0$	
$p(-1)=0$	1A f.t.
	----------(2)
(c) Since $p(x)=p(x-1)+x^{100}$, we have $p(-x)=p(-x-1)+(-x)^{100}$.	1M
So, we have $p(-x)=p(-x-1)+x^{100}$.	
Hence, we have $p(x)-p(x-1)=p(-x)-p(-x-1)$.	
Thus, we have $p(x)+p(-x-1)=p(x-1)+p(-x)$ for all $x\in\mathbf{R}$.	1
$p(n)+p(-n-1)$	
$=p(n-1)+p(-n)$	
$=p(n-2)+p(-n+1)$	1M
$=\cdots$	
$=p(0)+p(-1)$	
$=0+0$ (by (b))	
$=0$	1
Thus, we have $p(n)+p(-n-1)=0$ for all $n\in\mathbf{N}$.	
	----------(4)
(d) Let $g(x)=p(x)+p(-x-1)$ for all $x\in\mathbf{R}$.	
By (a), the degree of $p(x)$ is 101 .	
So, the degree of $g(x)$ is not greater than 101 .	1M
By (c), $g(x)$ has infinitely many zeros.	1M
Therefore, we have $g(x)=0$ for all $x\in\mathbf{R}$.	1
Thus, we have $p(x)+p(-x-1)=0$ for all $x\in\mathbf{R}$.	
	----------(3)
(e) Putting $x=\frac{-1}{2}$ in (d), we have $p\left(\frac{-1}{2}\right)+p\left(\frac{1}{2}-1\right)=0$.	1M
So, we have $p\left(\frac{-1}{2}\right)=0$.	1A f.t.
With the help of (b), x , $x+1$ and $2x+1$ are factors of $p(x)$.	
Thus, $p(x)$ is divisible by $x(x+1)(2x+1)$.	1
	----------(3)

Solution	Marks
11. (a) (i) Suppose that $\sqrt{10}$ is a rational number.	
So, we have $\sqrt{10}=\frac{p}{q}$, where p and q are relatively prime integers.	1M
Hence, we have $10q^2=p^2$.	
Then, p^2 is an even number and hence p is also an even number.	1M (for either correct)
Therefore, $p=2h$ for some integer h .	
Hence, we have $5q^2=2h^2$.	
So, q^2 is an even number and hence q is also an even number.	
It leads to a contradiction since p and q are relatively prime.	
Thus, we have $\sqrt{10}$ is an irrational number.	1
Suppose that $\sqrt{10}$ is a rational number.	
So, we have $\sqrt{10}=\frac{p}{q}$, where p and q are relatively prime integers.	1M
Hence, we have $10q^2=p^2$.	
Then, p^2 is divisible by 5 and hence p is divisible by 5 .	1M (for either correct)
Therefore, $p=5k$ for some integer k .	
Hence, we have $2q^2=5k^2$.	
So, q^2 is divisible by 5 and hence q is divisible by 5 .	
It leads to a contradiction since p and q are relatively prime.	
Thus, we have $\sqrt{10}$ is an irrational number.	1
Suppose that $\sqrt{10}$ is a rational number.	
So, we have $\sqrt{10}=\frac{p}{q}$, where p and q are relatively prime integers.	1M
Hence, we have $10q^2=p^2$.	
Then, p^2 is divisible by 10 and hence p is divisible by 10 .	1M (for either correct)
Therefore, $p=10l$ for some integer l .	
Hence, we have $q^2=10l^2$.	
So, q^2 is divisible by 10 and hence q is divisible by 10 .	
It leads to a contradiction since p and q are relatively prime.	
Thus, we have $\sqrt{10}$ is an irrational number.	1
(ii) Assume that $n\neq 0$.	
Then, we have $\sqrt{10}=\frac{-m}{n}$.	
By (a)(i), $\sqrt{10}$ is an irrational number but $\frac{-m}{n}$ is a rational number.	
So, it is impossible.	
Therefore, we have $n=0$.	
Thus, we have $m=n=0$.	1
	----------(4)

Solution	Marks
(b) [Existence] Putting $a_1 = 3$ and $b_1 = 1$, we have $3+\sqrt{10} = a_1 + b_1\sqrt{10}$ and $3-\sqrt{10} = a_1 - b_1\sqrt{10}$.	1A for either one
Assume that there exists a pair of positive integers a_k and b_k such that $(3+\sqrt{10})^k = a_k + b_k\sqrt{10}$ and $(3-\sqrt{10})^k = a_k - b_k\sqrt{10}$ for some $k \in \mathbf{N}$.	
Then, we have $(3\pm\sqrt{10})^{k+1}$ $=(3\pm\sqrt{10})^k(3\pm\sqrt{10})$ $=(a_k \pm b_k\sqrt{10})(3\pm\sqrt{10})$ $=(3a_k+10b_k)\pm(3b_k+a_k)\sqrt{10}$	1M for using assumption
Putting $a_{k+1} = 3a_k + 10b_k$ and $b_{k+1} = 3b_k + a_k$, we have $(3\pm\sqrt{10})^{k+1} = a_{k+1} \pm b_{k+1}\sqrt{10}$, where a_{k+1} and b_{k+1} are positive integers.	
So, the statement is true for $n = k+1$ when it is true for $n = k$.	
By mathematical induction, there exists a pair of positive integers a_n and b_n such that $(3+\sqrt{10})^n = a_n + b_n\sqrt{10}$ and $(3-\sqrt{10})^n = a_n - b_n\sqrt{10}$.	1 + 1
$(3\pm\sqrt{10})^n$ $=\sum_{k=0}^{n} C_k^n (3^{n-k})(\pm\sqrt{10})^k$	1A for binomial expansion
$=\begin{cases} \sum_{k=0}^{\frac{n}{2}} C_{2k}^n (3^{n-2k})(\pm\sqrt{10})^{2k} + \sum_{k=0}^{\frac{n}{2}-1} C_{2k+1}^n (3^{n-2k-1})(\pm\sqrt{10})^{2k+1} & \text{when } n \text{ is even} \\ \sum_{k=0}^{\frac{n-1}{2}} C_{2k}^n (3^{n-2k})(\pm\sqrt{10})^{2k} + \sum_{k=0}^{\frac{n-1}{2}} C_{2k+1}^n (3^{n-2k-1})(\pm\sqrt{10})^{2k+1} & \text{when } n \text{ is odd} \end{cases}$	1M for considering two cases
$=\begin{cases} \sum_{k=0}^{\frac{n}{2}} C_{2k}^n (3^{n-2k})(10)^k \pm \left(\sum_{k=0}^{\frac{n}{2}-1} C_{2k+1}^n (3^{n-2k-1})(10)^k \right)\sqrt{10} & \text{when } n \text{ is even} \\ \sum_{k=0}^{\frac{n-1}{2}} C_{2k}^n (3^{n-2k})(10)^k \pm \left(\sum_{k=0}^{\frac{n-1}{2}} C_{2k+1}^n (3^{n-2k-1})(10)^k \right)\sqrt{10} & \text{when } n \text{ is odd} \end{cases}$	
Note that C_{2k}^n and C_{2k+1}^n are positive integers. Thus, there exists a pair of positive integers a_n and b_n such that $(3+\sqrt{10})^n = a_n + b_n\sqrt{10}$ and $(3-\sqrt{10})^n = a_n - b_n\sqrt{10}$.	1 + 1

Solution	Marks
$(3\pm\sqrt{10})^n$	
$=\sum_{k=0}^{n} C_k^n(3^{n-k})(\pm\sqrt{10})^k$	1A for binomial expansion
$=\sum_{k=0}^{\left[\frac{n}{2}\right]} C_{2k}^n(3^{n-2k})(\pm\sqrt{10})^{2k}+\sum_{k=0}^{\left[\frac{n-1}{2}\right]} C_{2k+1}^n(3^{n-2k-1})(\pm\sqrt{10})^{2k+1}$	1M for considering two cases
$=\sum_{k=0}^{\left[\frac{n}{2}\right]} C_{2k}^n(3^{n-2k})(10)^k \pm\left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]} C_{2k+1}^n(3^{n-2k-1})(10)^k\right)\sqrt{10}$	
Note that C_{2k}^n and C_{2k+1}^n are positive integers.	
Thus, there exists a pair of positive integers a_n and b_n such that	
$(3+\sqrt{10})^n=a_n+b_n\sqrt{10}$ and $(3-\sqrt{10})^n=a_n-b_n\sqrt{10}$.	1 + 1
[Uniqueness]	
Assume that there are integers a_n , b_n , c_n and d_n such that	
$(3\pm\sqrt{10})^n=a_n\pm b_n\sqrt{10}$ and $(3\pm\sqrt{10})^n=c_n\pm d_n\sqrt{10}$.	
So, we have $a_n\pm b_n\sqrt{10}=c_n\pm d_n\sqrt{10}$.	
Then, we have $(a_n-c_n)\pm(b_n-d_n)\sqrt{10}=0$.	1M
By (a)(ii), we have $a_n-c_n=\pm(b_n-d_n)=0$.	
Thus, we have $a_n=c_n$ and $b_n=d_n$.	1
By combining the above results, for any positive integer n , there exists a unique pair of positive integers a_n and b_n such that	
$(3+\sqrt{10})^n=a_n+b_n\sqrt{10}$ and $(3-\sqrt{10})^n=a_n-b_n\sqrt{10}$.	
	----------(6)
(c) For every positive integer n , define $y_n=(3-\sqrt{10})^n$.	
By (b), we have $x_{2n-1}+y_{2n-1}=2a_{2n-1}$ and $x_{2n}+y_{2n}=2a_{2n}$.	1A for either one
Since $-1<3-\sqrt{10}<0$, we have	1M
$-1<y_{2n-1}<0$ and $0<y_{2n}<1$.	1M for either one
Hence, we have $2a_{2n-1}<x_{2n-1}<2a_{2n-1}+1$ and $2a_{2n}-1<x_{2n}<2a_{2n}$.	
By (b), a_{2n-1} and a_{2n} are positive integers.	
Therefore, the greatest integer less than x_{2n-1} is $2a_{2n-1}$ and the greatest integer less than x_{2n} is $2a_{2n}-1$.	
Thus, the greatest integer less than x_{2n-1} is an even number and the greatest integer less than x_{2n} is an odd number.	1 + 1
	----------(5)

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Marking Schemes

Paper 2

Solution	Marks
1. (a) Note that $-1 \le \cos\frac{1}{x} \le 1$ for all $x \ne 0$.	1M
So, we have $-\lvert x\rvert \le x\cos\frac{1}{x} \le \lvert x\rvert$ for all $x \ne 0$.	
Since $\lim_{x\to 0}(-\lvert x\rvert) = 0 = \lim_{x\to 0}\lvert x\rvert$, we have	1M for using Sandwich Theorem
$\lim_{x\to 0} x\cos\frac{1}{x} = 0$ (by Sandwich Theorem).	1
Note that $\left\lvert \cos\frac{1}{x} \right\rvert \le 1$ for all $x \ne 0$.	1M
Note also that $\lim_{x\to 0} x = 0$.	1M
Thus, we have $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.	1
(b) $\lim_{x\to 0}\frac{x^2\cos\frac{1}{x}}{\tan x}$	
$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\left(x\cos\frac{1}{x}\right)(\cos x)$	1M
$= \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\left(\lim_{x\to 0} x\cos\frac{1}{x}\right)\left(\lim_{x\to 0}\cos x\right)$	
$= (1)(0)(1)$	1M for using the result of (a)
$= 0$	1A
$\lim_{x\to 0}\frac{x}{\tan x}$	
$= \lim_{x\to 0}\frac{1}{\sec^2 x}$	
$= 1$	
$\lim_{x\to 0}\frac{x^2\cos\frac{1}{x}}{\tan x}$	
$= \lim_{x\to 0}\left(\frac{x}{\tan x}\right)\left(x\cos\frac{1}{x}\right)$	1M
$= \left(\lim_{x\to 0}\frac{x}{\tan x}\right)\left(\lim_{x\to 0} x\cos\frac{1}{x}\right)$	
$= (1)(0)$	1M for using the result of (a)
$= 0$	1A
	----------(6)

Solution	Marks
2. (a) Since $f(x)$ is continuous at $x=0$, $\lim_{x\to 0^-} f(x)$ exists.	
So, $\lim_{x\to 0^-} \frac{2+a\sqrt{1-x}}{x}$ exists.	1M
Therefore, we have $\lim_{x\to 0^-}(2+a\sqrt{1-x})=0$.	
Hence, we have $2+a=0$.	
Thus, we have $a=-2$	1
(b) $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$	
$=\lim_{x\to 0^+} \frac{1+b\tan\frac{x}{8}-1}{x-0}$	1M (either one)
$=\lim_{x\to 0^+} \frac{b\tan\frac{x}{8}}{x}$	
$=\lim_{x\to 0^+} \frac{\frac{b}{8}\sec^2\frac{x}{8}}{1}$	
$=\frac{b}{8}$	1A
$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$	
$=\lim_{x\to 0^-} \frac{\frac{2+a\sqrt{1-x}}{x}-1}{x-0}$	
$=\lim_{x\to 0^-} \frac{2-2\sqrt{1-x}-x}{x^2}$	
$=\lim_{x\to 0^-} \frac{\frac{1}{\sqrt{1-x}}-1}{2x}$	
$=\lim_{x\to 0^-} \frac{(1-x)^{\frac{-3}{2}}}{4}$	
$=\frac{1}{4}$	1A
Since $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$, we have $\frac{b}{8}=\frac{1}{4}$.	1M
Thus, we have $b=2$.	1A
	----------(7)

Solution	Marks
3. (a) Let $u = \sin x$.	
$\frac{d}{dx}\int_0^{\sin x} f(t)\,dt$	
$= \left(\frac{d}{du}\int_0^{u} f(t)\,dt\right)\left(\frac{du}{dx}\right)$	1M for chain rule
$= f(u)\frac{du}{dx}$	1M for $\frac{d}{du}\int_0^{u} f(t)\,dt = f(u)$
$= \cos x\, f(\sin x)$	1A
(b) $g'(x)$	
$= \cos x \frac{\cos^2(\sin x)}{2+\sin x}$	1M (either one)
$g'(\pi)$	
$= \cos \pi \frac{\cos^2(\sin \pi)}{2+\sin \pi}$	
$= \frac{-1}{2}$	1A
(c) Note that $h(x) = \cos x \int_0^{\sin x} \frac{\cos t}{2+t}\,dt$.	
$h'(x)$	
$= -\sin x \int_0^{\sin x} \frac{\cos t}{2+t}\,dt + \cos x \frac{d}{dx}\int_0^{\sin x} \frac{\cos t}{2+t}\,dt$	1M for product rule (either one)
$= -\sin x \int_0^{\sin x} \frac{\cos t}{2+t}\,dt + \frac{\cos^2 x \cos(\sin x)}{2+\sin x}$	
$h'(\pi)$	
$= -\sin \pi \int_0^{\sin \pi} \frac{\cos t}{2+t}\,dt + \frac{\cos^2 \pi \cos(\sin \pi)}{2+\sin \pi}$	
$= \frac{1}{2}$	1A
	----------(7)

Solution	Marks
4. (a) (i) $\int \frac{1}{1+4x^2}\,dx$	
$= \frac{1}{4}\int \frac{1}{(\frac{1}{2})^2 + x^2}\,dx$	
$= \frac{1}{2}\tan^{-1}(2x) + \text{constant}$	1A pp−1 for omitting constant
(ii) $\int \ln(1+4x^2)\,dx$	
$= x\ln(1+4x^2) - \int \frac{8x^2}{1+4x^2}\,dx$	1M for integration by parts
$= x\ln(1+4x^2) - 2\int \left(1 - \frac{1}{1+4x^2}\right)dx$	1M for division
$= x\ln(1+4x^2) - 2\int dx + 2\int \frac{1}{1+4x^2}\,dx$	
$= x\ln(1+4x^2) - 2x + \tan^{-1}(2x) + \text{constant}$	1A pp−1 for omitting constant
(b) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=n+1}^{2n} \ln\left(1+\frac{4k^2}{n^2}\right)$	
$= \int_1^2 \ln(1+4x^2)\,dx$	1A
$= \left[x\ln(1+4x^2) - 2x + \tan^{-1}(2x)\right]_1^2$ (by (a)(ii))	1M for using (a)(ii)
$= 2\ln 17 - 4 + \tan^{-1}4 - \ln 5 + 2 - \tan^{-1}2$	
$= 2\ln 17 - \ln 5 + \tan^{-1}4 - \tan^{-1}2 - 2$	1A
$= \ln\frac{17^2}{5} + \tan^{-1}\left(\frac{4-2}{1+(4)(2)}\right) - 2$	
$= \ln\frac{289}{5} + \tan^{-1}\frac{2}{9} - 2$	
	----------(7)

	Solution	Marks
5. (a)	$\int y\sqrt{1-y^2}\,\mathrm{d}y$	
	$=\frac{-1}{2}\int\sqrt{1-y^2}\,\mathrm{d}(1-y^2)$	1M
	$=\frac{-1}{3}(1-y^2)^{\frac{3}{2}}+\text{constant}$	1A pp−1 for omitting constant
	Putting $t=1-y^2$, we have $\frac{\mathrm{d}t}{\mathrm{d}y}=-2y$.	1M
	$\int y\sqrt{1-y^2}\,\mathrm{d}y$	
	$=\frac{-1}{2}\int\sqrt{t}\,\mathrm{d}t$	
	$=\frac{-1}{3}t^{\frac{3}{2}}+C$	
	$=\frac{-1}{3}(1-y^2)^{\frac{3}{2}}+\text{constant}$	1A pp−1 for omitting constant
	Putting $t=\sqrt{1-y^2}$, we have $\frac{\mathrm{d}t}{\mathrm{d}y}=-y(1-y^2)^{\frac{-1}{2}}$.	1M
	$\int y\sqrt{1-y^2}\,\mathrm{d}y$	
	$=-\int t^2\,\mathrm{d}t$	
	$=\frac{-1}{3}t^3+C$	
	$=\frac{-1}{3}(1-y^2)^{\frac{3}{2}}+\text{constant}$	1A pp−1 for omitting constant
	Let $y=\sin\theta$. We have $\frac{\mathrm{d}y}{\mathrm{d}\theta}=\cos\theta$.	1M
	$\int y\sqrt{1-y^2}\,\mathrm{d}y$	
	$=\int\sin\theta\cos^2\theta\,\mathrm{d}\theta$	
	$=-\int\cos^2\theta\,\mathrm{d}\cos\theta$	
	$=\frac{-\cos^3\theta}{3}+C$	
	$=\frac{-1}{3}(1-y^2)^{\frac{3}{2}}+\text{constant}$	1A pp−1 for omitting constant
	Note that $\frac{\mathrm{d}}{\mathrm{d}y}(1-y^2)^{\frac{3}{2}}=-3y(1-y^2)^{\frac{1}{2}}$.	1M
	$\int y\sqrt{1-y^2}\,\mathrm{d}y$	
	$=\frac{-1}{3}\int\mathrm{d}(1-y^2)^{\frac{3}{2}}$	
	$=\frac{-1}{3}(1-y^2)^{\frac{3}{2}}+\text{constant}$	1A pp−1 for omitting constant

Solution	Marks
(b) Putting $x=0$ in $x=y^{\frac{1}{2}}(1-y^2)^{\frac{1}{4}}$, we have $y=0$ or $y=1$.	
The required volume	
$=\int_0^1 \pi x^2 \mathrm{d}y$	1M
$=\pi\int_0^1 y\sqrt{1-y^2}\mathrm{d}y$	1A
$=\pi\left[\frac{-1}{3}\left(1-y^2\right)^{\frac{3}{2}}\right]_0^1$	1M for using (a)
$=\frac{\pi}{3}$	1A
Since $x=y^{\frac{1}{2}}(1-y^2)^{\frac{1}{4}}$, we have $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1-2y^2}{2y^{\frac{1}{2}}(1-y^2)^{\frac{3}{4}}}$.	
So, x attains its greatest value when $y=\frac{1}{\sqrt{2}}$.	
The greatest value of x is $\frac{1}{\sqrt{2}}$.	
Solving $x^4+y^4-y^2=0$, we have $y^2=\frac{1\pm\sqrt{1-4x^4}}{2}$.	
Let $y_1=\sqrt{\frac{1+\sqrt{1-4x^4}}{2}}$ and $y_2=\sqrt{\frac{1-\sqrt{1-4x^4}}{2}}$.	
The required volume	
$=\int_0^{\frac{1}{\sqrt{2}}} 2\pi x(y_1-y_2)\mathrm{d}x$	1M
$=2\pi\int_0^{\frac{1}{\sqrt{2}}} x\left(\sqrt{\frac{1+\sqrt{1-4x^4}}{2}}-\sqrt{\frac{1-\sqrt{1-4x^4}}{2}}\right)\mathrm{d}x$	
Let $2x^2=\sin\theta$. Then, we have $4x\frac{\mathrm{d}x}{\mathrm{d}\theta}=\cos\theta$.	1M for suitable substitution
The required volume	
$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos\theta\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\mathrm{d}\theta$	1A
$=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\left(\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}+\sin\frac{\theta}{2}-\sin\frac{3\theta}{2}\right)\mathrm{d}\theta$	
$=\frac{\pi}{4}\left[\frac{2}{3}\sin\frac{3\theta}{2}+2\sin\frac{\theta}{2}-2\cos\frac{\theta}{2}+\frac{2}{3}\cos\frac{3\theta}{2}\right]_0^{\frac{\pi}{2}}$	
$=\frac{\pi}{4}\left(\frac{\sqrt{2}}{3}+\sqrt{2}-\sqrt{2}-\frac{\sqrt{2}}{3}+2-\frac{2}{3}\right)$	
$=\frac{\pi}{3}$	1A
	----------(6)

Solution	Marks
6. (a) Putting $x = 3$ in $\frac{x^2}{25}+\frac{y^2}{16}=1$, we have $\frac{9}{25}+\frac{y^2}{16}=1$.	
Therefore, we have $y=\frac{16}{5}$ or $y=\frac{-16}{5}$ (rejected) .	
Thus, the coordinates of P are $\left(3, \frac{16}{5}\right)$.	1A
(b) (i) Differentiating both sides of $\frac{x^2}{25}+\frac{y^2}{16}=1$ with respect to x , we have	
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-16x}{25y}$ for any $y \neq 0$.	1M
So, we have $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_P=\frac{-3}{5}$.	
Hence, the slope of L_2 is $\frac{5}{3}$.	1M
Thus, the equation of L_2 is	
$y-0=\frac{5}{3}(x+3)$	
$5x-3y+15=0$	1A
The equation of the tangent to E at P is	
$\frac{3x}{25}+\frac{(\frac{16}{5})y}{16}=1$	1M
$y=\frac{-3x}{5}+5$	
So, the slope of the tangent to E at P $\frac{-3}{5}$.	
Hence, the slope of L_2 is $\frac{5}{3}$.	1M
Thus, the equation of L_2 is	
$y-0=\frac{5}{3}(x+3)$	
$5x-3y+15=0$	1A
Differentiating $y=\frac{4}{5}\sqrt{25-x^2}$ with respect to x , we have	
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4}{5}(25-x^2)^{\frac{-1}{2}}(-x)$.	1M
So, we have $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_P=\frac{-3}{5}$.	
Hence, the slope of L_2 is $\frac{5}{3}$.	1M
Thus, the equation of L_2 is	
$y-0=\frac{5}{3}(x+3)$	
$5x-3y+15=0$	1A

Solution	Marks
(ii) Putting $x = 3$ in $5x-3y+15=0$, we have $y = 10$. Therefore, the coordinates of R are $(3,10)$.	1M
Since $PQ=\sqrt{(3+3)^2+\left(\frac{16}{5}\right)^2}=\frac{34}{5}$ and $PR=10-\frac{16}{5}=\frac{34}{5}$,	1M for comparing PQ with PR
we have $PQ=PR$. Thus, ΔPQR is isosceles.	1A f.t.
Note that the slope of QR is $\frac{5}{3}$. Also note that the slope of PQ is $\frac{8}{15}$.	1M (for both)
Let $\angle PQR=\theta$ and $\angle PRQ=\phi$, where $0<\theta,\phi<\pi$.	
Since $\tan\theta=\dfrac{\frac{5}{3}-\frac{8}{15}}{1+\left(\frac{5}{3}\right)\left(\frac{8}{15}\right)}=\frac{9}{15}=\frac{3}{5}$ and $\tan\phi=\dfrac{1}{\frac{5}{3}}=\frac{3}{5}$,	1M for comparing $\tan\theta$ with $\tan\phi$
we have $\tan\theta = \tan\phi$. So, we have $\theta = \phi$. Hence, we have $\angle PQR=\angle PRQ$. Thus, ΔPQR is isosceles.	1A f.t.
	----------(7)

Solution	Marks

7. (a) $f'(x) = 4(x-3)^2 e^{2x}$ — 1A or equivalent

$f''(x) = 8(x-2)(x-3)\, e^{2x}$ — 1A or equivalent

----------(2)

(b) Note that $f'(x) = 0 \Leftrightarrow x = 3$. Also note that $f''(x) = 0 \Leftrightarrow x = 2$ or $x = 3$.

Further note that $f(x) = \left[2\left(x - \frac{7}{2}\right)^2 + \frac{1}{2}\right] e^{2x} > 0$ for all $x \in \mathbf{R}$.

x	$(-\infty, 2)$	2	$(2, 3)$	3	$(3, \infty)$
$f'(x)$	+	+	+	0	+
$f''(x)$	+	0	−	0	+
$f(x)$	+	$5e^4$	+	e^6	+

(i) $f(x) > 0 \Leftrightarrow x \in \mathbf{R}$ — 1A

(ii) $f'(x) > 0 \Leftrightarrow x \neq 3$ — 1A accept ($x < 3$ or $x > 3$)

(iii) $f''(x) > 0 \Leftrightarrow x < 2$ or $x > 3$ — 1A

------------(3)

(c) From the table in (b), the points of inflexion are $(2, 5e^4)$ and $(3, e^6)$. — 1A + 1A

------------(2)

(d)

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left((2x^2 - 14x + 25)\, e^{2x}\right)$$

$$= \lim_{x \to -\infty} \frac{2x^2 - 14x + 25}{e^{-2x}}$$

$$= \lim_{x \to -\infty} \frac{4x - 14}{-2e^{-2x}}$$ — 1M can be absorbed

$$= \lim_{x \to -\infty} \frac{4}{4e^{-2x}}$$

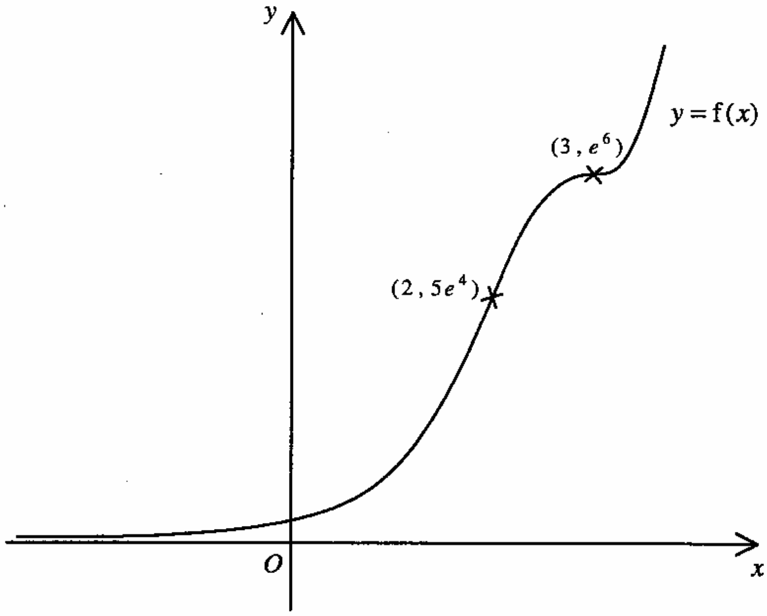
$$= 0$$

The horizontal asymptote is $y = 0$. — 1A

Also, there are no vertical asymptotes and oblique asymptotes.

------------(2)

(e)



1A for the points of inflexion and the asymptote

1A for all being correct

----------(2)

Solution	Marks
(f) The required area	
$=3e^6-\int_0^3 \mathrm{f}(x)\mathrm{d}x$	1A pp−1 for omitting dx
$\int_0^3 \mathrm{f}(x)\mathrm{d}x$	
$=\int_0^3 (2x^2-14x+25)\,e^{2x}\mathrm{d}x$	
$=\int_0^3 (2x^2-14x+25)\,\mathrm{d}\left(\frac{1}{2}e^{2x}\right)$	
$=\left[(2x^2-14x+25)\left(\frac{1}{2}e^{2x}\right)\right]_0^3-\int_0^3(4x-14)\left(\frac{1}{2}e^{2x}\right)\mathrm{d}x$	1M for integration by parts
$=\left(\frac{1}{2}e^6-\frac{25}{2}\right)-\int_0^3(2x-7)\,\mathrm{d}\left(\frac{1}{2}e^{2x}\right)$	1A
$=\left(\frac{1}{2}e^6-\frac{25}{2}\right)-\left[(2x-7)\left(\frac{1}{2}e^{2x}\right)\right]_0^3+\int_0^3 e^{2x}\mathrm{d}x$	
$=\left(\frac{1}{2}e^6-\frac{25}{2}\right)-\left(\frac{7}{2}-\frac{1}{2}e^6\right)+\left[\frac{1}{2}e^{2x}\right]_0^3$	
$=e^6-\frac{32}{2}+\left(\frac{1}{2}e^6-\frac{1}{2}\right)$	
$=\frac{3}{2}e^6-\frac{33}{2}$	
The required area	
$=3e^6-\left(\frac{3}{2}e^6-\frac{33}{2}\right)$	
$=\frac{3}{2}e^6+\frac{33}{2}$	1A
	----------(4)

Solution	Marks
8. (a) (i) I_{n+2}	
$=\int_0^{\pi} x\sin^{n+2}x\,dx$	
$=\int_0^{\pi} x\sin^{n+1}x\,d(-\cos x)$	
$=\left[-x\cos x\sin^{n+1}x\right]_0^{\pi}+\int_0^{\pi}\cos x(\sin^{n+1}x+(n+1)x\sin^n x\cos x)\,dx$	1M
$=\int_0^{\pi}(\cos x\sin^{n+1}x\,dx+(n+1)x\sin^n x\cos^2 x)\,dx$	1A
$=\int_0^{\pi}\cos x\sin^{n+1}x\,dx+(n+1)\int_0^{\pi}x\sin^n x(1-\sin^2 x)\,dx$	1M for using $\cos^2 x=1-\sin^2 x$
$=\int_0^{\pi}\sin^{n+1}x\,d\sin x+(n+1)\int_0^{\pi}(x\sin^n x-x\sin^{n+2}x)\,dx$	
$=\left[\dfrac{\sin^{n+2}x}{n+2}\right]_0^{\pi}+(n+1)\int_0^{\pi}x\sin^n x\,dx-(n+1)\int_0^{\pi}x\sin^{n+2}x\,dx$	
$=(n+1)I_n-(n+1)I_{n+2}$	
So, we have $(n+2)I_{n+2}=(n+1)I_n$.	
Thus, we have $I_{n+2}=\dfrac{n+1}{n+2}I_n$.	1
I_{n+2}	
$=\int_0^{\pi} x\sin^{n+2}x\,dx$	
$=\int_0^{\pi} x\sin^n x(1-\cos^2 x)\,dx$	1M for using $\sin^2 x=1-\cos^2 x$
$=\int_0^{\pi} x\sin^n x\,dx-\int_0^{\pi}x\sin^n x\cos^2 x\,dx$	
$=I_n-\int_0^{\pi}x\sin^n x\cos^2 x\,dx$	
$=I_n-\int_0^{\pi}x\cos x\,d\left(\dfrac{\sin^{n+1}x}{n+1}\right)$	
$=I_n-\left[\dfrac{x\cos x\sin^{n+1}x}{n+1}\right]_0^{\pi}+\dfrac{1}{n+1}\int_0^{\pi}(\sin^{n+1}x\cos x-x\sin^{n+2}x)\,dx$	1M
$=I_n+\dfrac{1}{n+1}\int_0^{\pi}\sin^{n+1}x\cos x\,dx-\dfrac{1}{n+1}\int_0^{\pi}x\sin^{n+2}x\,dx$	1A
$=I_n+\dfrac{1}{n+1}\int_0^{\pi}\sin^{n+1}x\,d\sin x-\dfrac{1}{n+1}I_{n+2}$	
$=I_n+\dfrac{1}{(n+1)(n+2)}\left[\sin^{n+2}x\right]_0^{\pi}-\dfrac{1}{n+1}I_{n+2}$	
$=I_n-\dfrac{1}{n+1}I_{n+2}$	
So, we have $\dfrac{n+2}{n+1}I_{n+2}=I_n$.	
Thus, we have $I_{n+2}=\dfrac{n+1}{n+2}I_n$.	1

Solution	Marks
(ii) Note that $0 \le \sin x \le 1$ for all $x \in [0, \pi]$.	
So, $x\sin^{2n+2} x \le x\sin^{2n+1} x \le x\sin^{2n} x$ for all $x \in [0, \pi]$.	1M
Hence, $\int_0^{\pi} x\sin^{2n+2} x\,dx \le \int_0^{\pi} x\sin^{2n+1} x\,dx \le \int_0^{\pi} x\sin^{2n} x\,dx$.	
Therefore, we have $I_{2n+2} \le I_{2n+1} \le I_{2n}$.	1
Note also that $x\sin^n x > 0$ for all $x \in (0, \pi)$.	
So, we have $\int_0^{\pi} x\sin^n x\,dx > 0$.	
Therefore, we have $I_n > 0$ for all non-negative integers n .	1
Thus, we have $0 < I_{2n+2} \le I_{2n+1} \le I_{2n}$.	
(iii) By (a)(i) and (a)(ii), we have $\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$.	1M for using (a)(i) and (a)(ii)
Since $\lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{2+\frac{2}{n}} = \frac{2}{2} = 1$,	
we have $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.	1
(iv) I_0 $= \int_0^{\pi} x\,dx$ $= \left[\frac{x^2}{2}\right]_0^{\pi}$ $= \frac{\pi^2}{2}$	1A (for either)
I_1 $= \int_0^{\pi} x\sin x\,dx$ $= [-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x\,dx$ $= \pi$	
I_{2n} $= \frac{2n-1}{2n} I_{2n-2}$ $= \frac{(2n-1)(2n-3)}{(2n)(2n-2)} I_{2n-4}$ $= \frac{(2n-1)(2n-3)\cdots(1)}{(2n)(2n-2)\cdots(2)} I_0$ $= \left(\frac{(2n-1)(2n-3)\cdots(1)}{(2n)(2n-2)\cdots(2)}\right)\left(\frac{(2n)(2n-2)\cdots(2)}{(2n)(2n-2)\cdots(2)}\right) I_0$ $= \left(\frac{(2n)!}{(2n)^2(2n-2)^2\cdots(2)^2}\right)\frac{\pi^2}{2}$	
$= \frac{(2n)!\,\pi^2}{2^{2n+1}(n!)^2}$ (which is also true for $n = 0$)	1

Solution	Marks
I_{2n+1}	
$=\dfrac{2n}{2n+1}I_{2n-1}$	
$=\dfrac{(2n)(2n-2)}{(2n+1)(2n-1)}I_{2n-3}$	
$=\dfrac{(2n)(2n-2)\cdots(2)}{(2n+1)(2n-1)\cdots(3)}I_1$	
$=\left(\dfrac{(2n)(2n-2)\cdots(2)}{(2n+1)(2n-1)\cdots(3)}\right)\left(\dfrac{(2n)(2n-2)\cdots(2)}{(2n)(2n-2)\cdots(2)}\right)I_1$	
$=\dfrac{(2n)^2(2n-2)^2\cdots(2)^2\pi}{(2n+1)!}$	
$=\dfrac{2^{2n}(n!)^2\pi}{(2n+1)!}$ (which is also true for $n=0$)	1
I_0	
$=\int_0^{\pi} x\,\mathrm{d}x$	
$=\left[\dfrac{x^2}{2}\right]_0^{\pi}$	
$=\dfrac{\pi^2}{2}$	1A (for either)
$=\dfrac{0!\pi^2}{2^1(0!)^2}$	
I_1	
$=\int_0^{\pi} x\sin x\,\mathrm{d}x$	
$=[-x\cos x]_0^{\pi}+\int_0^{\pi}\cos x\,\mathrm{d}x$	
$=\pi$	
$=\dfrac{2^0(0!)^2\pi}{1!}$	
So, the statement is true for $n=0$.	
Assume that $I_{2k}=\dfrac{(2k)!\pi^2}{2^{2k+1}(k!)^2}$ and $I_{2k+1}=\dfrac{2^{2k}(k!)^2\pi}{(2k+1)!}$ for some non-negative integer k .	
I_{2k+2}	
$=\dfrac{2k+1}{2k+2}I_{2k}$	
$=\left(\dfrac{2k+1}{2k+2}\right)\left(\dfrac{(2k)!\pi^2}{2^{2k+1}(k!)^2}\right)$	
$=\dfrac{(2k+2)(2k+1)(2k)!\pi^2}{(2k+2)^2(2^{2k+1})(k!)^2}$	
$=\dfrac{(2k+2)!\pi^2}{2^{2k+3}((k+1)!)^2}$	1

Solution	Marks
I_{2k+3} $=\frac{2k+2}{2k+3}I_{2k+1}$ $=\left(\frac{2k+2}{2k+3}\right)\left(\frac{2^{2k}(k!)^2\pi}{(2k+1)!}\right)$ $=\frac{(2k+2)(2k+2)(2^{2k}(k!)^2\pi)}{(2k+3)(2k+2)(2k+1)!}$ $=\frac{2^{2k+2}((k+1)!)^2\pi}{(2k+3)!}$ By induction, the statement is true for all non-negative integers n.	1
	----------(12)
(b) $\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{\sqrt{2n+1}\,(2n)!}$	
$=\lim_{n\to\infty}\sqrt{\frac{(2^{2n}(n!)^2)(2^{2n}(n!)^2)}{(2n+1)((2n)!)^2}}$	1M for taking square root
$=\lim_{n\to\infty}\sqrt{\frac{(2^{2n}(n!)^2)(2^{2n}(n!)^2)}{(2n+1)!(2n)!}}$	
$=\lim_{n\to\infty}\sqrt{\frac{\pi^2 I_{2n+1}}{2\pi I_{2n}}}$	1M for considering $\frac{I_{2n+1}}{I_{2n}}$
$=\sqrt{\frac{\pi}{2}}\lim_{n\to\infty}\sqrt{\frac{I_{2n+1}}{I_{2n}}}$	
$=\sqrt{\frac{\pi}{2}}$	1A
Note that $\frac{I_{2n+1}}{I_{2n}}=\frac{\frac{2^{2n}(n!)^2\pi}{(2n+1)!}}{\frac{(2n)!\pi^2}{2^{2n+1}(n!)^2}}$.	1M for considering $\frac{I_{2n+1}}{I_{2n}}$
Since $\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}}=1$, we have $\lim_{n\to\infty}\frac{2^{4n+1}(n!)^4}{(2n+1)!(2n)!\pi}=1$.	
So, we have $\lim_{n\to\infty}\frac{2^{4n}(n!)^4}{(2n+1)!(2n)!}=\frac{\pi}{2}$.	
Therefore, we have $\lim_{n\to\infty}\sqrt{\frac{2^{4n}(n!)^4}{(2n+1)!(2n)!}}=\sqrt{\frac{\pi}{2}}$.	1M for taking square root
Thus, we have $\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{\sqrt{2n+1}\,(2n)!}=\sqrt{\frac{\pi}{2}}$.	1A
	----------(3)

Solution	Marks
9. (a) (i) Let $u=-x$. Then, we have $\frac{du}{dx}=-1$.	
$\int_{-1}^{0} f(x)dx$	
$=\int_{1}^{0} -f(-u)du$	
$=\int_{0}^{1} f(-u)du$	
$=\int_{0}^{1} f(-x)dx$	1
Thus, we have $\int_{-1}^{0} f(x)dx=\int_{0}^{1} f(-x)dx$.	
(ii) $\int_{-1}^{1} f(x)dx$	
$=\int_{-1}^{0} f(x)dx+\int_{0}^{1} f(x)dx$	1M
$=\int_{0}^{1} f(-x)dx+\int_{0}^{1} f(x)dx$ (by (a)(i))	
$=-\int_{0}^{1} f(x)dx+\int_{0}^{1} f(x)dx$	
$=0$	1 ----------(3)
(b) (i) $\int_{-1}^{1} x^2 g(x)dx$	
$=\int_{-1}^{0} x^2 g(x)dx+\int_{0}^{1} x^2 g(x)dx$	
$=\int_{0}^{1} (-x)^2 g(-x)dx+\int_{0}^{1} x^2 g(x)dx$ (by (a)(i))	1M for using (a)(i)
$=\int_{0}^{1} x^2 g(-x)dx+\int_{0}^{1} x^2 g(x)dx$	
$=\int_{0}^{1} x^2\left(g(-x)+g(x)\right)dx$	
$=\int_{0}^{1} x^2 dx$	1A
$=\left[\frac{x^3}{3}\right]_0^1$	
$=\frac{1}{3}$	1
Let $h(x)=g(x)-\frac{1}{2}$ for all $x\in\mathbf{R}$. $h(-x)=g(-x)-\frac{1}{2}=1-g(x)-\frac{1}{2}=\frac{1}{2}-g(x)=-h(x)$ for all $x\in\mathbf{R}$. So, $h(x)$ is an odd function. Therefore, $g(x)=h(x)+\frac{1}{2}$, where $h(x)$ is an odd function.	

Solution	Marks
$\int_{-1}^{1} x^2 \mathrm{g}(x)\,\mathrm{d}x$	
$= \int_{-1}^{1} x^2 \left(\mathrm{h}(x) + \frac{1}{2}\right)\mathrm{d}x$	
$= \int_{-1}^{1} x^2 \mathrm{h}(x)\,\mathrm{d}x + \frac{1}{2}\int_{-1}^{1} x^2\,\mathrm{d}x$	
Since x^2 is an even function, $x^2\mathrm{h}(x)$ is an odd function.	
By (a)(ii), we have $\int_{-1}^{1} x^2 \mathrm{h}(x)\,\mathrm{d}x = 0$.	1M for using (a)(ii)
$\int_{-1}^{1} x^2 \mathrm{g}(x)\,\mathrm{d}x$	
$= \frac{1}{2}\int_{-1}^{1} x^2\,\mathrm{d}x$	1A
$= \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1}$	
$= \frac{1}{3}$	1
(ii) $\mathrm{g}(-x) + \mathrm{g}(x) = 1$ for all $x \in \mathbf{R}$	
Differentiating both sides $(2n-1)$ times with respect to x , we have	
$-\mathrm{g}^{(2n-1)}(-x) + \mathrm{g}^{(2n-1)}(x) = 0$ for all $x \in \mathbf{R}$	
$\mathrm{g}^{(2n-1)}(-x) = \mathrm{g}^{(2n-1)}(x)$ for all $x \in \mathbf{R}$	
Thus, $\mathrm{g}^{(2n-1)}(x)$ is an even function	1
$\mathrm{g}(-x) + \mathrm{g}(x) = 1$ for all $x \in \mathbf{R}$	
Differentiating both sides $2n$ times with respect to x , we have	
$\mathrm{g}^{(2n)}(-x) + \mathrm{g}^{(2n)}(x) = 0$ for all $x \in \mathbf{R}$	
$\mathrm{g}^{(2n)}(-x) = -\mathrm{g}^{(2n)}(x)$ for all $x \in \mathbf{R}$	
Thus, $\mathrm{g}^{(2n)}(x)$ is an odd function.	1
To prove the latter result, we have two cases.	
Case 1: n is an odd number.	
Since $\mathrm{g}^{(n)}(x)$ is an even function and x^{n+2} is an odd function, $x^{n+2}\mathrm{g}^{(n)}(x)$ is an odd function.	
By (a)(ii), we have $\int_{-1}^{1} x^{n+2}\mathrm{g}^{(n)}(x)\,\mathrm{d}x = 0$.	1
Case 2: n is an even number.	
Since $\mathrm{g}^{(n)}(x)$ is an odd function and x^{n+2} is an even function, $x^{n+2}\mathrm{g}^{(n)}(x)$ is an odd function.	
By (a)(ii), we have $\int_{-1}^{1} x^{n+2}\mathrm{g}^{(n)}(x)\,\mathrm{d}x = 0$.	1
Thus, $\int_{-1}^{1} x^{n+2}\mathrm{g}^{(n)}(x)\,\mathrm{d}x = 0$ for any positive integer n .	
	----------(7)

Solution	Marks
(c) $g(-x)+g(x)=\dfrac{1}{1+e^{-x}}+\dfrac{1}{1+e^{x}}=\dfrac{e^{x}}{e^{x}+1}+\dfrac{1}{1+e^{x}}=\dfrac{e^{x}+1}{e^{x}+1}=1$ for all $x\in\mathbf{R}$	1M withhold 1M for no checking
Note that g has derivatives of any order.	
$G(x)=x^3 g(x)$	
$G'''(x)=x^3 g'''(x)+3(3x^2)g''(x)+3(6x)g'(x)+6g(x)$	1M
$x^2 G'''(x)=x^5 g'''(x)+9x^4 g''(x)+18x^3 g'(x)+6x^2 g(x)$	1A
$\int_{-1}^{1} x^2 G'''(x)\,dx$	
$=\int_{-1}^{1}\left(x^5 g'''(x)+9x^4 g''(x)+18x^3 g'(x)+6x^2 g(x)\right)dx$	
$=\int_{-1}^{1} x^5 g'''(x)\,dx+9\int_{-1}^{1} x^4 g''(x)\,dx+18\int_{-1}^{1} x^3 g'(x)\,dx+6\int_{-1}^{1} x^2 g(x)\,dx$	
$=6\int_{-1}^{1} x^2 g(x)\,dx$ (by (b)(ii))	1M for using (b)(ii)
$=\dfrac{6}{3}$ (by (b)(i))	
$=2$	1A
$g(x)+g(-x)=\dfrac{1}{1+e^{x}}+\dfrac{1}{1+e^{-x}}=\dfrac{1}{1+e^{x}}+\dfrac{e^{x}}{e^{x}+1}=\dfrac{1+e^{x}}{1+e^{x}}=1$ for all $x\in\mathbf{R}$	1M withhold 1M for no checking
Hence, we have $G(x)-G(-x)=x^3(g(x)+g(-x))=x^3$.	
Therefore, we have $G'(x)+G'(-x)=3x^2$ and $G''(x)-G''(-x)=6x$.	1M
So, we have $G''(1)-G''(-1)=6$, $G'(1)+G'(-1)=3$ and $G(1)-G(-1)=1$.	
$\int_{-1}^{1} x^2 G'''(x)\,dx$	
$=\int_{-1}^{1} x^2\, dG''(x)$	
$=\left[x^2 G''(x)\right]_{-1}^{1}-2\int_{-1}^{1} xG''(x)\,dx$	1M for integration by parts
$=G''(1)-G''(-1)-2\int_{-1}^{1} x\,dG'(x)$	
$=G''(1)-G''(-1)-2\left[xG'(x)\right]_{-1}^{1}+2\int_{-1}^{1} G'(x)\,dx$	
$=G''(1)-G''(-1)-2(G'(1)+G'(-1))+2\left[G(x)\right]_{-1}^{1}$	
$=G''(1)-G''(-1)-2(G'(1)+G'(-1))+2(G(1)-G(-1))$	1A
$=6-6+2$	
$=2$	1A
	----------(5)

Solution	Marks
10. (a) $f'(x)=4^{-x^2}$	1A ----------(1)
(b) (i) Since $(t-1)^2\ge 0$, we have $t^2-2t+1\ge 0$.	1M
So, we have $-t^2\le 1-2t$.	
For any $x\in\mathbf{R}^+$, we have $4^{-t^2}\le 4^{1-2t}$ for all $t\in[0,x]$.	1M
Hence, we have $\int_0^x 4^{-t^2}\,dt\le\int_0^x 4^{1-2t}\,dt$.	
Thus, we have $f(x)\le\int_0^x 4^{1-2t}\,dt$.	1
(ii) Let $e^y=4^{1-2t}$.	
Then, we have $y=(1-2t)\ln 4$ and $\frac{dy}{dt}=-2\ln 4$.	
$\int_0^x 4^{1-2t}\,dt$	
$=\frac{-1}{2\ln 4}\int_{\ln 4}^{(1-2x)\ln 4}e^y\,dy$	1M
$=\frac{1}{2\ln 4}\int_{(1-2x)\ln 4}^{\ln 4}e^y\,dy$	
$=\frac{1}{2\ln 4}\left[e^y\right]_{(1-2x)\ln 4}^{\ln 4}$	1M
$=\frac{1}{2\ln 4}\left(4-4^{1-2x}\right)$	1A
$=\frac{1}{\ln 2}\left(1-4^{-2x}\right)$	
Let $e^y=4^{-2t}$.	
Then, we have $y=-2t\ln 4$ and $\frac{dy}{dt}=-2\ln 4$.	
$\int_0^x 4^{1-2t}\,dt$	
$=4\int_0^x 4^{-2t}\,dt$	
$=\frac{-4}{2\ln 4}\int_0^{-2x\ln 4}e^y\,dy$	1M
$=\frac{4}{2\ln 4}\int_{-2x\ln 4}^{0}e^y\,dy$	
$=\frac{2}{\ln 4}\left[e^y\right]_{-2x\ln 4}^{0}$	1M
$=\frac{2}{\ln 4}\left(1-4^{-2x}\right)$	1A
$=\frac{1}{\ln 2}\left(1-4^{-2x}\right)$	

Solution	Marks
$\int_0^x 4^{1-2t}\,dt$	
$= 4\int_0^x 4^{-2t}\,dt$	
$= -2\int_0^x 4^{-2t}\,d(-2t)$	1M
$= -2\left[\dfrac{4^{-2t}}{\ln 4}\right]_0^x$	1M
$= -2\left(\dfrac{4^{-2x}-1}{\ln 4}\right)$	1A
$= \dfrac{1}{\ln 2}\left(1-4^{-2x}\right)$	
	----------(6)
(c) (i) By (a), $f'(x) > 0$ for all $x \in \mathbf{R}^+$.	1M
So, f is a strictly increasing function.	
Thus, f is an injective function.	1A f.t.
$f(x_1) = f(x_2)$	
$\int_0^{x_1} 4^{-t^2}\,dt = \int_0^{x_2} 4^{-t^2}\,dt$	
$\int_{x_2}^{x_1} 4^{-t^2}\,dt = 0$	1M
Note that 4^{-t^2} is continuous on $\mathbf{R}^+$ and $4^{-t^2} > 0$ for all $t > 0$.	
Therefore, we have $x_1 = x_2$.	
Thus, f is an injective function.	1A f.t.
(ii) By (b)(i) and (b)(ii), we have $f(x) \le \dfrac{1}{\ln 2}\left(1-4^{-2x}\right)$ for all $x \in \mathbf{R}^+$.	
So, we have $f(x) < \dfrac{1}{\ln 2}$ for all $x \in \mathbf{R}^+$.	1M
Therefore, we have $f(x) \ne \dfrac{1}{\ln 2}$ for all $x \in \mathbf{R}^+$.	
By noting that $\dfrac{1}{\ln 2} \in \mathbf{R}^+$, f is not a surjective function.	1A f.t.
	----------(4)
(d) Let $v = \log_4 x$.	
Then, we have $4^v = x$.	1M
So, we have $g(v) = f(4^v)$ for all $v \in \mathbf{R}$ and g is differentiable in $\mathbf{R}$.	
$g'(v)$	
$= f'(4^v)\dfrac{d4^v}{dv}$	1M for chain rule
$= f'(4^v)\,4^v \ln 4$	1A
> 0 (since $f'(x) > 0$ for all $x \in \mathbf{R}^+$ (by (a)))	
Hence, $g'(v) > 0$ for all $v \in \mathbf{R}$.	
Therefore, $g'(x) \ne 0$ for all $x \in \mathbf{R}$.	
Thus, the graph of $y = g(x)$ does not have any extreme point.	1A f.t.

Solution	Marks
$g(\log_4 x) = f(x)$	
Let $v = \log_4 x$.	
Then, we have $4^v = x$.	1M
Note that g is differentiable in $\mathbf{R}$.	
$\frac{d}{dx} g(\log_4 x) = \frac{d}{dx} f(x)$	
$g'(v)\frac{dv}{dx} = f'(x)$	1M for chain rule
$\frac{g'(v)}{x \ln 4} = 4^{-x^2}$	
$g'(v)$	
$= x 4^{-x^2} \ln 4$	1A
$= 4^v 4^{-(4^v)^2} \ln 4$	
> 0	
Hence, $g'(v) > 0$ for all $v \in \mathbf{R}$.	
Therefore, $g'(x) \neq 0$ for all $x \in \mathbf{R}$.	
Thus, the graph of $y = g(x)$ does not have any extreme point.	1A f.t.
	----------(4)

Solution	Marks
11. (a) The equation of the tangent to H at P is	
$\dfrac{\left(\frac{3}{t}\right)x+(3t)y}{2}=9$	1M
$x+t^2y-6t=0$	1A
So, the equation of the tangent to H at R is $t^2x+y+6t=0$.	1A
Solving $x+t^2y-6t=0$ and $t^2x+y+6t=0$,	
the coordinates of Q are $\left(\dfrac{6t}{1-t^2},\dfrac{-6t}{1-t^2}\right)$.	1A
Differentiating both sides of $xy=9$ with respect to x,	
we have $x\dfrac{dy}{dx}+y=0$.	
Hence, we have $\dfrac{dy}{dx}=\dfrac{-y}{x}$ for any $x\neq 0$.	
So, we have $\left.\dfrac{dy}{dx}\right\|_P=\dfrac{-1}{t^2}$.	
The equation of the tangent to H at P is	
$y-\dfrac{3}{t}=\dfrac{-1}{t^2}(x-3t)$	1M
$x+t^2y-6t=0$	1A
So, the equation of the tangent to H at R is $t^2x+y+6t=0$.	1A
Solving $x+t^2y-6t=0$ and $t^2x+y+6t=0$,	
the coordinates of Q are $\left(\dfrac{6t}{1-t^2},\dfrac{-6t}{1-t^2}\right)$.	1A
	------------(4)
(b) (i) By (a), the slope of the tangent to H at P is $\dfrac{-1}{t^2}$.	
So, the slope of the normal to H at P is t^2 .	1M
Thus, the equation of the normal to H at P is	
$y-\dfrac{3}{t}=t^2(x-3t)$	
$t^3x-ty-3t^4+3=0$	1A
(ii) By (b)(i), the equation of the normal to H at R is $tx-t^3y-3t^4+3=0$.	1A
Solving $t^3x-ty-3t^4+3=0$ and $tx-t^3y-3t^4+3=0$,	
the coordinates of S are $\left(\dfrac{3(t^2-1)}{t},\dfrac{3(1-t^2)}{t}\right)$.	1A
Since the slope of QS is -1 , the equation of the straight line passing through Q and S is $x+y=0$.	
Note that $(0,0)$ satisfies the equation $x+y=0$.	1M accept comparing slopes
Thus, the origin lies on the straight line passing through Q and S .	1
	------------(6)

Solution	Marks
(c) (i) Since $\angle QPS = 90^\circ$ and $\angle QRS = 90^\circ$, we have	1M
$\angle QPS + \angle QRS = 180^\circ$.	
Thus, P , Q , R and S are concyclic .	1
(ii) Note that QS is a diameter of the circle passing through P, Q, R and S.	1A
For the circle passing through P, Q, R and S centred at the origin, the mid-point of QS is the origin.	
So, we have $\frac{1}{2}\left(\frac{3(t^2-1)}{t}+\frac{6t}{1-t^2}\right)=0$.	1M
Simplifying, we have $t^4 - 4t^2 + 1 = 0$.	
Therefore, we have $t^2 = 2 \pm \sqrt{3}$.	
Since $2 \pm \sqrt{3} > 0$, there is a value of t such that the origin is the centre of the circle passing through P, Q, R and S.	1A f.t.
	------------(5)