2008-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2008

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(C) answer book.

4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Denote the coefficient of x^k in the expansion of $(1+x)^n$ by C_k^n .

   (a) Let m be a non-negative integer with $m \le n$. Using $(1+x)^m(1+x)^n = (1+x)^{m+n}$, prove that $\sum_{k=0}^{m} C_k^m C_{r-k}^n = C_r^{m+n}$ for all $r = m, m+1, \ldots, n$.

   (b) Using (a), or otherwise, evaluate $\dfrac{\sum_{k=0}^{99} C_k^{99} C_{k+1}^{100}}{\sum_{k=0}^{99} C_k^{99} C_k^{101}}$.

(6 marks)

2. (a) Resolve $\dfrac{7x+9}{x(x+1)(x+3)}$ into partial fractions.

   (b) Express $\sum_{k=1}^{n} \dfrac{7k+9}{k(k+1)(k+3)}$ in the form $A + \dfrac{B}{n+1} + \dfrac{C}{n+2} + \dfrac{D}{n+3}$, where A , B , C and D are constants.

   (c) Evaluate $\sum_{k=5}^{\infty} \dfrac{7k+9}{k(k+1)(k+3)}$.

(7 marks)

3. (a) Let a , b and c be three numbers such that $a+b+c=0$. Prove that $a^3+b^3+c^3=3abc$.

   (b) Using (a), solve the equation $(x-2\sqrt{2}+5\sqrt{3})^3 - (3x+6\sqrt{2}-7\sqrt{3})^3 + 8(x+4\sqrt{2}-6\sqrt{3})^3 = 0$.

(6 marks)

4. Let A be the matrix representing the rotation in the Cartesian plane anticlockwise about the origin by $120°$.

   (a) Write down the matrix A .

   (b) For every positive integer n , let T_n be the transformation represented by A^n . It is given that T_1 transforms the point P to the point $(4, 2\sqrt{3})$.

      (i) Find the coordinates of P .

      (ii) Describe the geometric meaning of the transformation T_n .

      (iii) Is there a transformation T_n which transforms P to the point $(3, 5\sqrt{3})$? Explain your answer.

(7 marks)

5. Let $S = \{ z \in \mathbf{C} : 3|z-2i| = |z+8-2i| \}$.

   (a) Prove that S is represented by a circle on the Argand diagram. Also find the centre and the radius of the circle.

   (b) On the Argand diagram, P is the point representing the complex number $-7+17i$ and Q is a point representing any $z \in S$. Find the longest distance between P and Q .

(7 marks)

6. Let α , β , γ and δ be positive real numbers.

   (a) Using Cauchy-Schwarz's inequality, or otherwise, prove that

      (i) $(\alpha+\beta+\gamma+\delta)^2 \le 4(\alpha^2+\beta^2+\gamma^2+\delta^2)$,

      (ii) $(\alpha^2+\beta^2+\gamma^2+\delta^2)^2 \le (\alpha+\beta+\gamma+\delta)(\alpha^3+\beta^3+\gamma^3+\delta^3)$.

   (b) Using (a), prove that $\dfrac{\alpha+\beta+\gamma+\delta}{4} \le \dfrac{\alpha^3+\beta^3+\gamma^3+\delta^3}{\alpha^2+\beta^2+\gamma^2+\delta^2}$.

   Write down a necessary and sufficient condition for the equality to hold.

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. (a) Consider the system of linear equations in x, y, z

$$(E): \begin{cases} x + (a+2)y + (a+1)z = 1 \\ x - 3y - z = b \\ 3x - 2y + (a-1)z = 1 \end{cases}, \text{ where } a, b \in \mathbf{R} .$$

(i) Prove that (E) has a unique solution if and only if $a^2 \neq 4$. Solve (E) when (E) has a unique solution.

(ii) For each of the following cases, find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b .

(1) $a = 2$,

(2) $a = -2$.

(11 marks)

(b) Find the greatest value of $2x^2 + 15y^2 - 10z^2$, where x, y and z are real numbers satisfying

$$\begin{cases} x + 4y + 3z = 1 \\ x - 3y - z = 0 \\ 3x - 2y + z = 1 \end{cases}.$$

(4 marks)

8. Denote the 2×2 identity matrix by I.

(a) Let $M = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}$, where p, q and r are real numbers.

(i) Evaluate M^2 .

(ii) Let s be a real number. Prove that if $p^2 + qr = 1$, then for any positive integer n ,

$$(M + sI)^{2n} = \frac{(s+1)^{2n} - (s-1)^{2n}}{2} M + \frac{(s+1)^{2n} + (s-1)^{2n}}{2} I .$$

(7 marks)

(b) Let $A = \begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix}$.

(i) Express A in the form $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} + \mu I$, where α, β, γ and μ are real numbers.

(ii) Evaluate A^{800} .

(iii) Using the result of (b)(ii), or otherwise, evaluate $(A^{-1})^{800}$.

(8 marks)

9. Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers, where $x_1 = 2$, $y_1 = 8$ and

$$x_{n+1} = \frac{x_n^2 y_n + x_n y_n^2}{x_n^2 + y_n^2} \text{ and } y_{n+1} = \frac{x_n^2 + y_n^2}{x_n + y_n} \text{ for all } n = 1, 2, 3, \dots .$$

(a) Prove that $x_{n+1} - y_{n+1} = \dfrac{-(x_n^3 - y_n^3)(x_n - y_n)}{(x_n + y_n)(x_n^2 + y_n^2)}$.

(2 marks)

(b) Prove that

(i) $x_n \leq y_n$,

(ii) $x_{n+1} \geq x_n$,

(iii) $y_{n+1} \leq y_n$.

(5 marks)

(c) Prove that $\{x_n\}$ and $\{y_n\}$ converge to the same limit.

(4 marks)

(d) Prove that $x_n y_n$ is independent of n .

Hence, or otherwise, find $\lim_{n \to \infty} x_n$.

(4 marks)

10. Let $\mathrm{p}(x)$ be a polynomial of degree k with real coefficients satisfying the following conditions:

(1) $\mathrm{p}(x) = \mathrm{p}(x-1) + x^{100}$ for all $x \in \mathbf{R}$;

(2) $\mathrm{p}(1) = 1$.

(a) Find k and the coefficient of x^k in $\mathrm{p}(x)$.

(3 marks)

(b) Find $\mathrm{p}(0)$ and $\mathrm{p}(-1)$.

(2 marks)

(c) Prove that $\mathrm{p}(x) + \mathrm{p}(-x-1) = \mathrm{p}(x-1) + \mathrm{p}(-x)$ for all $x \in \mathbf{R}$.

Hence, prove that $\mathrm{p}(n) + \mathrm{p}(-n-1) = 0$ for all $n \in \mathbf{N}$.

(4 marks)

(d) Prove that $\mathrm{p}(x) + \mathrm{p}(-x-1) = 0$ for all $x \in \mathbf{R}$.

(3 marks)

(e) Prove that $\mathrm{p}(x)$ is divisible by $x(x+1)(2x+1)$.

(3 marks)

11. (a) (i) Prove that $\sqrt{10}$ is an irrational number.

(ii) Using (a)(i), prove that if m and n are integers such that $m+n\sqrt{10}=0$, then $m=n=0$.

(4 marks)

(b) Prove that for any positive integer n, there exists a unique pair of positive integers a_n and b_n such that $(3+\sqrt{10})^n = a_n + b_n\sqrt{10}$ and $(3-\sqrt{10})^n = a_n - b_n\sqrt{10}$.

(6 marks)

(c) For every positive integer n, define $x_n = (3+\sqrt{10})^n$. Prove that for any positive integer n, the greatest integer less than x_{2n-1} is an even number and the greatest integer less than x_{2n} is an odd number.

(5 marks)

END OF PAPER

2008-AL
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PAPER 2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
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PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(E) answer book.
3. Answer any FOUR questions in Section B, using the AL(C) answer book.
4. Unless otherwise specified, all working must be clearly shown.

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FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

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$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Prove that $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

   (b) Evaluate $\lim_{x\to 0} \frac{x^2\cos\frac{1}{x}}{\tan x}$.

(6 marks)

2. Let a and b be real constants and $\mathrm{f}: \mathbf{R} \to \mathbf{R}$ be defined by

$$\mathrm{f}(x) = \begin{cases} \dfrac{2 + a\sqrt{1-x}}{x} & \text{when } x < 0 , \\ 1 + b\tan\dfrac{x}{8} & \text{when } x \geq 0 . \end{cases}$$

It is given that $\mathrm{f}(x)$ is continuous at $x = 0$.

   (a) Prove that $a = -2$.

   (b) Furthermore, $\mathrm{f}(x)$ is differentiable at $x = 0$. Find b .

(7 marks)

3. (a) Let $\mathrm{f}: [-1, 1] \to \mathbf{R}$ be a continuous function. Find $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\sin x} \mathrm{f}(t)\,\mathrm{d}t$.

   (b) Let $\mathrm{g}(x) = \int_0^{\sin x} \frac{\cos^2 t}{2+t}\,\mathrm{d}t$. Evaluate $\mathrm{g}'(\pi)$.

   (c) Let $\mathrm{h}(x) = \int_0^{\sin x} \frac{\cos x \cos t}{2+t}\,\mathrm{d}t$. Evaluate $\mathrm{h}'(\pi)$.

(7 marks)

4. (a) (i) Find $\int \frac{1}{1+4x^2} dx$.

(ii) Using the result of (a)(i), or otherwise, find $\int \ln(1+4x^2) dx$.

(b) Evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{k=n+1}^{2n} \ln\left(1+\frac{4k^2}{n^2}\right)$.

(7 marks)

5. (a) Find $\int y\sqrt{1-y^2} dy$.

(b) Let D be the region bounded by the curve $x = y^{\frac{1}{2}}(1-y^2)^{\frac{1}{4}}$ and the y-axis. Find the volume of the solid of revolution generated by revolving D about the y-axis.

(6 marks)

6. Let P be the point in the first quadrant at which the ellipse $E: \frac{x^2}{25}+\frac{y^2}{16}=1$ and the straight line $L_1: x=3$ intersect.

(a) Find the coordinates of P .

(b) Let L_2 be the straight line passing through $Q(-3, 0)$ and perpendicular to the tangent to E at P .

(i) Find the equation of L_2 .

(ii) Let R be the point of intersection of L_1 and L_2 . Is ΔPQR isosceles? Explain your answer.

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(C) answer book.

7. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = (2x^2 - 14x + 25)e^{2x}$.

(a) Find $f'(x)$ and $f''(x)$.

(2 marks)

(b) Solve each of the following inequalities:

(i) $f(x) > 0$,

(ii) $f'(x) > 0$,

(iii) $f''(x) > 0$.

(3 marks)

(c) Find the point(s) of inflexion of the graph of $y = f(x)$.

(2 marks)

(d) Find the asymptote(s) of the graph of $y = f(x)$.

(2 marks)

(e) Sketch the graph of $y = f(x)$.

(2 marks)

(f) Find the area of the region bounded by the graph of $y = f(x)$, the straight line $y = e^6$ and the y-axis.

(4 marks)

8. (a) For every non-negative integer n , define $I_n = \int_0^{\pi} x \sin^n x \, dx$. Prove that

(i) $I_{n+2} = \frac{n+1}{n+2} I_n$,

(ii) $0 < I_{2n+2} \le I_{2n+1} \le I_{2n}$,

(iii) $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$,

(iv) $I_{2n} = \frac{(2n)!\,\pi^2}{2^{2n+1}(n!)^2}$ and $I_{2n+1} = \frac{2^{2n}(n!)^2\,\pi}{(2n+1)!}$.

(12 marks)

(b) Using (a), or otherwise, evaluate $\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{\sqrt{2n+1}\,(2n)!}$.

(3 marks)

9. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function.

(i) Using integration by substitution, prove that $\int_{-1}^{0} f(x)\,dx = \int_{0}^{1} f(-x)\,dx$.

(ii) Using (a)(i), prove that if $f(x)$ is an odd function, then $\int_{-1}^{1} f(x)\,dx = 0$.

(3 marks)

(b) Let $g : \mathbf{R} \to \mathbf{R}$ be a function with derivatives of any order. Suppose that $g(-x) + g(x) = 1$ for all $x \in \mathbf{R}$.

(i) Prove that $\int_{-1}^{1} x^2 g(x)\,dx = \frac{1}{3}$.

(ii) Prove that $g^{(2n-1)}(x)$ is an even function and $g^{(2n)}(x)$ is an odd function for any positive integer n . Hence, prove that $\int_{-1}^{1} x^{n+2} g^{(n)}(x)\,dx = 0$ for any positive integer n .

(7 marks)

(c) Let $g : \mathbf{R} \to \mathbf{R}$ and $G : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = \frac{1}{1+e^x}$ and $G(x) = x^3 g(x)$. Using (b), or otherwise, evaluate $\int_{-1}^{1} x^2 G'''(x)\,dx$.

(5 marks)

10. Let $f : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by $f(x) = \int_{0}^{x} 4^{-t^2}\,dt$, where $\mathbf{R}^+$ is the set of positive real numbers.

(a) Find $f'(x)$.

(1 mark)

(b) (i) Prove that $f(x) \le \int_{0}^{x} 4^{1-2t}\,dt$.

(ii) Using the substitution $e^y = 4^{1-2t}$, or otherwise, find $\int_{0}^{x} 4^{1-2t}\,dt$.

(6 marks)

(c) (i) Is f an injective function? Explain your answer.

(ii) Is f a surjective function? Explain your answer.

(4 marks)

(d) Let $g : \mathbf{R} \to \mathbf{R}^+$ be a function such that $g(\log_4 x) = f(x)$ for all $x \in \mathbf{R}^+$.
Does the graph of $y = g(x)$ have an extreme point? Explain your answer.

(4 marks)

11. It is given that $P\left(3t, \frac{3}{t}\right)$ and $R\left(\frac{-3}{t}, -3t\right)$ are two distinct points on the hyperbola $H : xy = 9$, where $t \ne -1, 0$ and 1 . The tangents to H at P and R intersect at the point Q while the normals to H at P and R intersect at the point S .

(a) Find the coordinates of Q .

(4 marks)

(b) (i) Find the equation of the normal to H at P .

(ii) Prove that the origin lies on the straight line passing through Q and S .

(6 marks)

(c) (i) Prove that P , Q , R and S are concyclic.

(ii) Is there a value of t such that the origin is the centre of the circle passing through P , Q , R and S ? Explain your answer.

(5 marks)

END OF PAPER