Marking Schemes

Paper 1

1. (a) $(1+x)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n$	1M accept $(1+t)^{n-1} = \sum_{k=0}^{n-1} C_k^{n-1} t^k$
(i) $\frac{d}{dx}(1+x)^n = \frac{d}{dx}\left(C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n\right)$	1M
$n(1+x)^{n-1} = C_1^n + 2C_2^n x + 3C_3^n x^2 + \cdots + nC_n^n x^{n-1}$	1
Note that $nC_r^{n-1} = \frac{n(n-1)!}{r!(n-1-r)!} = \frac{(r+1)\,n!}{(r+1)!(n-r-1)!} = (r+1)\,C_{r+1}^n$	1M
for all $r = 0, 1, \ldots, n-1$.	
$n(1+x)^{n-1}$	
$= n(C_0^{n-1} + C_1^{n-1}x + C_2^{n-1}x^2 + \cdots + C_{n-1}^{n-1}x^{n-1})$	
$= nC_0^{n-1} + nC_1^{n-1}x + nC_2^{n-1}x^2 + \cdots + nC_{n-1}^{n-1}x^{n-1}$	
$= C_1^n + 2C_2^n x + 3C_3^n x^2 + \cdots + nC_n^n x^{n-1}$	1
(ii) $\int (1+x)^n dx = \int (C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n)\,dx$	1M
$\frac{(1+x)^{n+1}}{n+1} = C_0^n x + \frac{C_1^n}{2}x^2 + \frac{C_2^n}{3}x^3 + \cdots + \frac{C_n^n}{n+1}x^{n+1} + A$,	
where A is a real number independent of x .	
Putting $x = 0$, we have $A = \frac{1}{n+1}$.	
Thus, we have $\frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + C_0^n x + \frac{C_1^n}{2}x^2 + \frac{C_2^n}{3}x^3 + \cdots + \frac{C_n^n}{n+1}x^{n+1}$.	1
$\frac{d}{dx}\left(\frac{(1+x)^{n+1}}{n+1} - \left(C_0^n x + \frac{C_1^n}{2}x^2 + \frac{C_2^n}{3}x^3 + \cdots + \frac{C_n^n}{n+1}x^{n+1}\right)\right)$	1M
$= (1+x)^n - (C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n)$	
$= 0$	
So, we have $\frac{(1+x)^{n+1}}{n+1} = C_0^n x + \frac{C_1^n}{2}x^2 + \frac{C_2^n}{3}x^3 + \cdots + \frac{C_n^n}{n+1}x^{n+1} + A$,	
where A is a real number independent of x .	
Putting $x = 0$, we have $A = \frac{1}{n+1}$.	
Thus, we have $\frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + C_0^n x + \frac{C_1^n}{2}x^2 + \frac{C_2^n}{3}x^3 + \cdots + \frac{C_n^n}{n+1}x^{n+1}$.	1
$\int_0^x (1+t)^n dt = \int_0^x (C_0^n + C_1^n t + C_2^n t^2 + \cdots + C_n^n t^n)\,dt$	1M
$\left[\frac{(1+t)^{n+1}}{n+1}\right]_0^x = \left[C_0^n t + \frac{C_1^n}{2}t^2 + \frac{C_2^n}{3}t^3 + \cdots + \frac{C_n^n}{n+1}t^{n+1}\right]_0^x$	
$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0^n x + \frac{C_1^n}{2}x^2 + \frac{C_2^n}{3}x^3 + \cdots + \frac{C_n^n}{n+1}x^{n+1}$	
$\frac{(1+x)^{n+1}}{n+1} = \frac{1}{n+1} + C_0^n x + \frac{C_1^n}{2}x^2 + \frac{C_2^n}{3}x^3 + \cdots + \frac{C_n^n}{n+1}x^{n+1}$	1

Note that $\dfrac{C_r^{n+1}}{n+1}=\dfrac{(n+1)!}{(n+1)\,r!\,(n+1-r)!}=\dfrac{n!}{r(r-1)!(n-r+1)!}=\dfrac{C_{r-1}^n}{r}$	1M
for all $r=1,2,\ldots,n+1$.	
$\dfrac{(1+x)^{n+1}}{n+1}$	
$=\dfrac{C_0^{n+1}}{n+1}+\dfrac{C_1^{n+1}}{n+1}x+\dfrac{C_2^{n+1}}{n+1}x^2+\cdots+\dfrac{C_{n+1}^{n+1}}{n+1}x^{n+1}$	
$=\dfrac{1}{n+1}+C_0^n x+\dfrac{C_1^n}{2}x^2+\dfrac{C_2^n}{3}x^3+\cdots+\dfrac{C_n^n}{n+1}x^{n+1}$	1

(b) $n(1+x)^{n-1}\left(\dfrac{(1+x)^{n+1}}{n+1}\right)$	
$=(C_1^n+2C_2^n x+3C_3^n x^2+\cdots+nC_n^n x^{n-1})\left(\dfrac{1}{n+1}+C_0^n x+\dfrac{C_1^n}{2}x^2+\dfrac{C_2^n}{3}x^3+\cdots+\dfrac{C_n^n}{n+1}x^{n+1}\right)$	
$=(C_1^n+2C_2^n x+3C_3^n x^2+\cdots+nC_n^n x^{n-1})\left(\dfrac{1}{n+1}+C_n^n x+\dfrac{C_{n-1}^n}{2}x^2+\dfrac{C_{n-2}^n}{3}x^3+\cdots+\dfrac{C_0^n}{n+1}x^{n+1}\right)$	
Note that $n(1+x)^{n-1}\left(\dfrac{(1+x)^{n+1}}{n+1}\right)=\dfrac{n}{n+1}(1+x)^{2n}$.	
By comparing the coefficients of x^n in both sides, we have	1M
$\dfrac{(C_1^n)^2}{n}+\dfrac{2(C_2^n)^2}{n-1}+\dfrac{3(C_3^n)^2}{n-2}+\cdots+\dfrac{n(C_n^n)^2}{1}=\dfrac{n}{n+1}C_n^{2n}$	
$\dfrac{(C_1^n)^2}{n}+\dfrac{2(C_2^n)^2}{n-1}+\dfrac{3(C_3^n)^2}{n-2}+\cdots+\dfrac{n(C_n^n)^2}{1}=\dfrac{n(2n)!}{(n+1)(n!)^2}$	1
	----------(7)

2. (a) Let $f(x) = (x-1)(x-3)q_1(x) - 2x + 5$ for some polynomial $q_1(x)$.	1M for using Division Algorithm (either one)
Then, we have $f(1) = -2(1) + 5 = 3$.	1A
Alternative: Let $f(x) = (x-1)(x-3)q_1(x) - 2x + 5$ for some polynomial $q_1(x)$. Then, $f(x) = (x-1)(x-3)q_1(x) - 2(x-1) + 3 = (x-1)((x-3)q_1(x) - 2) + 3$. So, when $f(x)$ is divided by $x-1$, the remainder is 3 . Thus, we have $f(1) = 3$.	(either one) 1A
(b) (i) Let $f(x) = (x-1)(x-2)q_2(x) + kx + 8$ for some polynomial $q_2(x)$.	(either one)
By (a), we have $f(1) = 3$.	
So, we have $k + 8 = 3$.	1M for using (a)
Thus, we have $k = -5$.	1A
(ii) Note that the required remainder is $(f(2))^{2007}$.	
By (b)(i), we have $f(2) = 2k + 8 = -10 + 8 = -2$.	1M for attempting to find $f(2)$
So, we have $(f(2))^{2007} = (-2)^{2007} = -2^{2007}$.	1A accept $(-2)^{2007}$
Thus, the required remainder is -2^{2007} .	
	----------(6)

3. (a) (i) Let $\dfrac{1}{x(x+1)(x-1)} = \dfrac{C_1}{x} + \dfrac{C_2}{x+1} + \dfrac{C_3}{x-1}$. | 1A

$1 \equiv C_1(x+1)(x-1) + C_2 x(x-1) + C_3 x(x+1)$

Putting $x = 0, -1, 1$, we have $C_1 = -1$, $C_2 = \dfrac{1}{2}$, $C_3 = \dfrac{1}{2}$. | 1A for all correct

Thus, we have $\dfrac{1}{x(x+1)(x-1)} = \dfrac{-1}{x} + \dfrac{1}{2(x+1)} + \dfrac{1}{2(x-1)}$.

(ii) Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x(x+1)(x-1)}\right) = \dfrac{-3x^2+1}{x^2(x+1)^2(x-1)^2}$. | 1A or equivalent

By (a)(i), we have $\dfrac{1}{x(x+1)(x-1)} = \dfrac{-1}{x} + \dfrac{1}{2(x+1)} + \dfrac{1}{2(x-1)}$.

Differentiate both sides with respect to x , we have

$$\frac{-3x^2+1}{x^2(x+1)^2(x-1)^2} = \frac{1}{x^2} - \frac{1}{2(x+1)^2} - \frac{1}{2(x-1)^2}$$

$$\frac{3x^2-1}{x^2(x+1)^2(x-1)^2} = \frac{-1}{x^2} + \frac{1}{2(x+1)^2} + \frac{1}{2(x-1)^2}$$ 1A for correct partial fractions

Let $\dfrac{3x^2-1}{x^2(x+1)^2(x-1)^2} = \dfrac{D_1}{x} + \dfrac{D_2}{x^2} + \dfrac{D_3}{x+1} + \dfrac{D_4}{(x+1)^2} + \dfrac{D_5}{x-1} + \dfrac{D_6}{(x-1)^2}$. | 1A or equivalent

$3x^2 - 1 \equiv D_1x(x+1)^2(x-1)^2 + D_2(x+1)^2(x-1)^2 + D_3x^2(x+1)(x-1)^2$
$\quad + D_4x^2(x-1)^2 + D_5x^2(x+1)^2(x-1) + D_6x^2(x+1)^2$

So, $D_1 = 0$, $D_2 = -1$, $D_3 = 0$, $D_4 = \dfrac{1}{2}$, $D_5 = 0$ and $D_6 = \dfrac{1}{2}$. | 1A for all correct

Thus, we have $\dfrac{3x^2-1}{x^2(x+1)^2(x-1)^2} = \dfrac{-1}{x^2} + \dfrac{1}{2(x+1)^2} + \dfrac{1}{2(x-1)^2}$.

(b)
$$\sum_{k=2}^{n} \frac{3k^2-1}{k^2(k+1)^2(k-1)^2}$$

$$= \sum_{k=2}^{n}\left(\frac{-1}{k^2} + \frac{1}{2(k+1)^2} + \frac{1}{2(k-1)^2}\right) \quad \text{(by (a)(ii))}$$

$$= \sum_{k=2}^{n}\left(\left(\frac{1}{2(k+1)^2} - \frac{1}{2k^2}\right) + \left(\frac{1}{2(k-1)^2} - \frac{1}{2k^2}\right)\right)$$

$$= \sum_{k=2}^{n}\left(\frac{1}{2(k+1)^2} - \frac{1}{2k^2}\right) + \sum_{k=2}^{n}\left(\frac{1}{2(k-1)^2} - \frac{1}{2k^2}\right)$$ 1M

$$= \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{(k+1)^2} - \frac{1}{k^2}\right) + \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{(k-1)^2} - \frac{1}{k^2}\right)$$

$$= \frac{1}{2}\left(\frac{-1}{4} + \frac{1}{(n+1)^2}\right) + \frac{1}{2}\left(1 - \frac{1}{n^2}\right)$$ 1A or equivalent

$$= \frac{3}{8} - \frac{1}{2n^2} + \frac{1}{2(n+1)^2}$$

Thus, we have $\displaystyle\sum_{k=2}^{\infty} \frac{3k^2-1}{k^2(k+1)^2(k-1)^2} = \lim_{n\to\infty}\left(\frac{3}{8} - \frac{1}{2n^2} + \frac{1}{2(n+1)^2}\right) = \frac{3}{8}$. | 1A

----------(7)

Solution	Marks
4. (a) Note that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the matrix representing T.	1A can be absorbed
So, we have $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -5 \\ 12 \end{pmatrix} = \begin{pmatrix} -12 \\ -5 \end{pmatrix}$	
Thus, we have $-5\cos\theta - 12\sin\theta = -12$ and $-5\sin\theta + 12\cos\theta = -5$.	1M
Therefore, we have $\sin\theta = 1$ and $\cos\theta = 0$.	
Solving, with the help of $0 < \theta < 2\pi$, we have $\theta = \frac{\pi}{2}$.	1A
Let O be the origin.	
Note that (the slope of OP_1)(the slope of OP_2) $= \left(\frac{12}{-5}\right)\left(\frac{-5}{-12}\right) = -1$.	1M
So, we have $OP_1 \perp OP_2$.	1A can be absorbed
Since P_1 and P_2 are in the second quadrant and the third quadrant respectively and $0 < \theta < 2\pi$, we have $\theta = \frac{\pi}{2}$.	1A
(b) By the result of (a), we have $A^4 = I$.	1M
Thus, we have $A^{2007} = (A^4)^{501}A^3 = A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	1A
$A^{2007} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}^{2007}$	
$= \begin{pmatrix} \cos\frac{2007\pi}{2} & -\sin\frac{2007\pi}{2} \\ \sin\frac{2007\pi}{2} & \cos\frac{2007\pi}{2} \end{pmatrix}$	1M
$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	1A
(c) Note that $P_1 = (-5, 12)$, $P_2 = (-12, -5)$ and $\theta = \frac{\pi}{2}$.	
$P_n = \begin{cases} (-5, 12) & \text{if } n = 1, 5, 9, \ldots, \\ (-12, -5) & \text{if } n = 2, 6, 10, \ldots, \\ (5, -12) & \text{if } n = 3, 7, 11, \ldots, \\ (12, 5) & \text{if } n = 4, 8, 12, \ldots. \end{cases}$	1A for any two cases correct 1A for all being correct
Let $P_n = (x_n, y_n)$. Then, we have	
$x_n = -5\cos\frac{(n-1)\pi}{2} - 12\sin\frac{(n-1)\pi}{2}$ and	1A
$y_n = -5\sin\frac{(n-1)\pi}{2} + 12\cos\frac{(n-1)\pi}{2}$.	1A
	----------(7)

5. (a) Note that $M^2 = (P^{-1}QP)(P^{-1}QP) = P^{-1}Q^2P$. — 1A

$M^2 = \lambda M + \mu I$

$P^{-1}Q^2P = \lambda P^{-1}QP + \mu I$

$P(P^{-1}Q^2P)P^{-1} = P(\lambda P^{-1}QP + \mu I)P^{-1}$ — 1M

$Q^2 = \lambda Q + \mu I$

$$\begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix} = \lambda \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, we have $\alpha^2 = \lambda\alpha + \mu$ and $\beta^2 = \lambda\beta + \mu$. — 1M

Since $\alpha \neq \beta$, the roots of the equation $x^2 - \lambda x - \mu = 0$ are α and β .

Thus, we have $\lambda = \alpha + \beta$ and $\mu = -\alpha\beta$. — 1A for both correct

(b) By (a), we have $M^2 + \alpha\beta I = \lambda M + \mu I + \alpha\beta I = \lambda M = (\alpha + \beta)M$.

$\det(M^2 + \alpha\beta I)$

$= \det((\alpha + \beta)M)$

$= (\alpha + \beta)^2 \det M$ — 1M (for either)

$= (\alpha + \beta)^2 \det(P^{-1}QP)$

$= (\alpha + \beta)^2 (\det P^{-1})(\det Q)(\det P)$

$= (\alpha + \beta)^2 \left(\frac{1}{\det P}\right)(\det Q)(\det P)$

$= (\alpha + \beta)^2 \det Q$

$= \alpha\beta(\alpha + \beta)^2$ — 1

$\det(M^2 + \alpha\beta I)$

$= \det(P^{-1}Q^2P + \alpha\beta I)$

$= \det(P^{-1}Q^2P + P^{-1}(\alpha\beta I)P)$

$= \det(P^{-1}(Q^2 + \alpha\beta I)P)$

$= (\det P^{-1})(\det(Q^2 + \alpha\beta I))(\det P)$ — (for either)

$= \left(\frac{1}{\det P}\right)(\det(Q^2 + \alpha\beta I))(\det P)$

$= \det(Q^2 + \alpha\beta I)$

$= \begin{vmatrix} \alpha\beta + \alpha^2 & 0 \\ 0 & \alpha\beta + \beta^2 \end{vmatrix}$

$= (\alpha\beta + \alpha^2)(\alpha\beta + \beta^2)$

$= \alpha\beta(\alpha + \beta)^2$ — 1

----------(6)

6. (a) Note that $r^{p+q}-r^p-r^q+1=(r^p-1)(r^q-1)$. There are 2 cases.	
Case 1 : $0<r\le 1$ Since $r^p-1\le 0$ and $r^q-1\le 0$, we have $(r^p-1)(r^q-1)\ge 0$. Case 2 : $r>1$ Since $r^p-1\ge 0$ and $r^q-1\ge 0$, we have $(r^p-1)(r^q-1)\ge 0$.	1M for considering 2 cases
By combining the above 2 cases, we have $r^{p+q}-r^p-r^q+1\ge 0$.	1
(b) Let r be the common ratio of the geometric sequence. Note that $r>0$.	
(i) $a_1+a_n-a_k-a_{n-k+1}$ $=a_1+a_1r^{n-1}-a_1r^{k-1}-a_1r^{n-k}$ $=a_1(r^{n-1}-r^{n-k}-r^{k-1}+1)$ ≥ 0 (by putting $p=n-k$ and $q=k-1$ in (a)) Thus, we have $a_1+a_n\ge a_k+a_{n-k+1}$.	1
(ii) By (b)(i), we have $a_1+a_n\ge a_k+a_{n-k+1}$ for all $k=1,2,\dots,n$.	
So, we have $\sum_{k=1}^{n}(a_1+a_n)\ge\sum_{k=1}^{n}(a_k+a_{n-k+1})$.	1M
Therefore, we have $n(a_1+a_n)\ge 2\sum_{k=1}^{n}a_k$.	
Hence, we have $\frac{1}{2}(a_1+a_n)\ge\frac{1}{n}\sum_{k=1}^{n}a_k$.	1
Note that $\left(\prod_{k=1}^{n}a_k\right)^{\frac{1}{n}}=\left(a_1^{\,n}r^{1+2+\cdots+(n-1)}\right)^{\frac{1}{n}}=a_1r^{\frac{n-1}{2}}=\sqrt{a_1a_n}$.	
By A.M. $\ge$ G.M. on $\{a_1,a_2,\dots,a_n\}$, we have $\frac{1}{n}\sum_{k=1}^{n}a_k\ge\left(\prod_{k=1}^{n}a_k\right)^{\frac{1}{n}}$.	1
Hence, we have $\frac{1}{n}\sum_{k=1}^{n}a_k\ge\sqrt{a_1a_n}$.	1
By A.M. $\ge$ G.M. on $\{a_k,a_{n-k+1}\}$, we have $\frac{a_k+a_{n-k+1}}{2}\ge\sqrt{a_ka_{n-k+1}}$.	1
Note that $\sqrt{a_ka_{n-k+1}}=\sqrt{a_1r^{k-1}a_1r^{n-k}}=\sqrt{a_1^{\,2}r^{n-1}}=\sqrt{a_1a_n}$. So, we have $\frac{a_k+a_{n-k+1}}{2}\ge\sqrt{a_1a_n}$ for all $k=1,2,\dots,n$. Therefore, we have $\sum_{k=1}^{n}\frac{a_k+a_{n-k+1}}{2}\ge\sum_{k=1}^{n}\sqrt{a_1a_n}$. Simplifying, we have $\sum_{k=1}^{n}a_k\ge n\sqrt{a_1a_n}$.	
Hence, we have $\frac{1}{n}\sum_{k=1}^{n}a_k\ge\sqrt{a_1a_n}$.	1
Thus, we have $\frac{1}{2}(a_1+a_n)\ge\frac{1}{n}\sum_{k=1}^{n}a_k\ge\sqrt{a_1a_n}$.	
	----------(7)

7. (a) (i) (E) has a unique solution

$\Leftrightarrow \Delta \neq 0$

$\Leftrightarrow \Delta = \begin{vmatrix} 1 & -3 & 0 \\ 1 & 5 & a \\ 2 & a & -1 \end{vmatrix} \neq 0$ — 1M

$\Leftrightarrow -a^2 - 6a - 8 \neq 0$ — 1A

$\Leftrightarrow -(a+2)(a+4) \neq 0$

$\Leftrightarrow a \neq -2$ and $a \neq -4$ — 1A

$\Leftrightarrow a < -4$, $-4 < a < -2$ or $a > -2$

The augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 1 & 5 & a & b \\ 2 & a & -1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 8 & a & b-1 \\ 0 & a+6 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 8 & a & b-1 \\ 0 & 0 & (a+2)(a+4) & (a+6)(b-1) \end{array}\right)$$ — 1A

(E) has a unique solution

$\Leftrightarrow (a+2)(a+4) \neq 0$ — 1M

$\Leftrightarrow a \neq -2$ and $a \neq -4$ — 1A

$\Leftrightarrow a < -4$, $-4 < a < -2$ or $a > -2$

When (E) has a unique solution,

$$x = \frac{\begin{vmatrix} 1 & -3 & 0 \\ b & 5 & a \\ 2 & a & -1 \end{vmatrix}}{\Delta} = \frac{-(a^2+6a+3b+5)}{\Delta}$$ — 1M for Cramer's rule

$$= \frac{a^2+6a+3b+5}{(a+2)(a+4)}$$

$$y = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & b & a \\ 2 & 2 & -1 \end{vmatrix}}{\Delta} = \frac{1-b}{\Delta}$$

$$= \frac{b-1}{(a+2)(a+4)}$$

$$z = \frac{\begin{vmatrix} 1 & -3 & 1 \\ 1 & 5 & b \\ 2 & a & 2 \end{vmatrix}}{\Delta} = \frac{(6+a)(1-b)}{\Delta}$$

$$= \frac{(a+6)(b-1)}{(a+2)(a+4)}$$ — 1A + 1A (1A for any one, 1A for all)

When (E) has a unique solution, the augmented matrix of (E) becomes	
$\left(\begin{array}{ccc\|c} 1 & -3 & 0 & 1 \\ 0 & 8 & a & b-1 \\ 0 & 0 & 1 & \frac{(a+6)(b-1)}{(a+2)(a+4)} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & \frac{b-1}{(a+2)(a+4)} \\ 0 & 0 & 1 & \frac{(a+6)(b-1)}{(a+2)(a+4)} \end{array}\right)$	1M
$\sim \left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{a^2+6a+3b+5}{(a+2)(a+4)} \\ 0 & 1 & 0 & \frac{b-1}{(a+2)(a+4)} \\ 0 & 0 & 1 & \frac{(a+6)(b-1)}{(a+2)(a+4)} \end{array}\right)$	
$\therefore \quad x=\frac{a^2+6a+3b+5}{(a+2)(a+4)}$, $y=\frac{b-1}{(a+2)(a+4)}$, $z=\frac{(a+6)(b-1)}{(a+2)(a+4)}$.	1A + 1A (1A for any one, 1A for all)
(ii) When $a=-2$, the augmented matrix of (E) becomes	
$\left(\begin{array}{ccc\|c} 1 & -3 & 0 & 1 \\ 0 & 8 & -2 & b-1 \\ 0 & 0 & 0 & b-1 \end{array}\right)$	
(E) is consistent when $b=1$.	1A
Therefore, the solution set is $\{(1+3t,\ t,\ 4t): t\in \mathbf{R}\}$.	1A or equivalent
	---------(8)
(b) Putting $a=1$ and $b=16$ in (a), we have, by (a)(i), the solution of the first	1A
three equations of the system of linear equations is $x=4$, $y=1$ and $z=7$.	1M
Note that $4-1-7=-4\neq 3$.	
Thus, the system of linear equations is inconsistent.	1A f.t.
	---------(3)
(c) Putting $a=-2$ and $b=1$ in (a), we have, by (a)(ii), the solution of the first three equations of the system of linear equations is $x=1+3t$, $y=t$ and $z=4t$, where $t\in\mathbf{R}$.	1A
Putting $x=1+3t$, $y=t$ and $z=4t$ in $x-y-z=3$, we have $1-2t=3$.	1M
So, we have $t=-1$.	1A
Thus, the required solution is $x=-2$, $y=-1$ and $z=-4$.	1A
	---------(4)

8\. (a) $x^4 + px^3 + qx^2 + rx + \frac{r^2}{p^2} = 0$

$\Leftrightarrow \quad x^2 + \frac{r^2}{p^2x^2} + px + \frac{r}{x} + q = 0$ (since $x = 0$ is not a root)

$\Leftrightarrow \quad \left(x + \frac{r}{px}\right)^2 - \frac{2r}{p} + px + \frac{r}{x} + q = 0$ — 1M for completing the square

$\Leftrightarrow \quad \left(x + \frac{r}{px}\right)^2 + p\left(x + \frac{r}{px}\right) + \left(q - \frac{2r}{p}\right) = 0$ — 1

--------(2)

(b) (i) $(x+h)^4 + (x+h)^2 - 4(x+h) - 3 = 0$

$\Leftrightarrow \quad (x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4) + (x^2 + 2hx + h^2) - 4(x+h) - 3 = 0$ — 1A

$\Leftrightarrow \quad x^4 + 4hx^3 + (6h^2+1)x^2 + (4h^3 + 2h - 4)x + (h^4 + h^2 - 4h - 3) = 0$

So, $P = 4h$, $Q = 6h^2 + 1$, $R = 4h^3 + 2h - 4$ and $S = h^4 + h^2 - 4h - 3$. — 1A + 1A (1A for any one, 1A for all)

(ii) $P^2S = R^2$

$\Leftrightarrow \quad (4h)^2(h^4 + h^2 - 4h - 3) = (4h^3 + 2h - 4)^2$ — 1M for using (b)(i)

$\Leftrightarrow \quad (2h)^2(h^4 + h^2 - 4h - 3) = (2h^3 + h - 2)^2$

$\Leftrightarrow \quad 4h^6 + 4h^4 - 16h^3 - 12h^2 = 4h^6 + h^2 + 4 + 4h^4 - 4h - 8h^3$ — 1A

$\Leftrightarrow \quad 8h^3 + 13h^2 - 4h + 4 = 0$ — 1

--------(6)

(c) By (b)(ii), we have $8h^3 + 13h^2 - 4h + 4 = 0$.

Note that $8(-2)^3 + 13(-2)^2 - 4(-2) + 4 = -64 + 52 + 8 + 4 = 0$. — 1M for finding h

So, we have $(h+2)(8h^2 - 3h + 2) = 0$.

The real value of h is -2 . — 1A

By (b)(i), we have $P = -8$, $Q = 25$, $R = -40$ and $S = 25$.

So, when $y = x - 2$, (*) can be written as $x^4 - 8x^3 + 25x^2 - 40x + 25 = 0$.

$x^4 - 8x^3 + 25x^2 - 40x + 25 = 0$

$\Leftrightarrow \quad \left(x + \frac{5}{x}\right)^2 - 8\left(x + \frac{5}{x}\right) + (25 - 10) = 0$ — 1M for using (a)

$\Leftrightarrow \quad \left(x + \frac{5}{x}\right)^2 - 8\left(x + \frac{5}{x}\right) + 15 = 0$ — 1A

$\Leftrightarrow \quad x + \frac{5}{x} = 5$ or $x + \frac{5}{x} = 3$

$\Leftrightarrow \quad x^2 - 5x + 5 = 0$ or $x^2 - 3x + 5 = 0$ — 1M for forming a quadratic equation

$\Leftrightarrow \quad x = \frac{5 \pm \sqrt{5}}{2}$ or $x = \frac{3 \pm \sqrt{11}i}{2}$ — 1A for all the roots being correct

Thus, the roots of (*) are $\frac{1}{2} + \frac{\sqrt{5}}{2}$, $\frac{1}{2} - \frac{\sqrt{5}}{2}$, $\frac{-1}{2} + \frac{\sqrt{11}}{2}i$ and $\frac{-1}{2} - \frac{\sqrt{11}}{2}i$. — 1M

--------(7)

9. (a) Let $f(t)=\dfrac{(t+1)^{\alpha+1}}{t^{\alpha}}$ for all $t>0$.

Then, we have

$\dfrac{d}{dt}f(t)$

$=\dfrac{(t+1)^{\alpha}(t-\alpha)}{t^{\alpha+1}}$ — 1A

$\begin{cases}<0 & \text{if } 0<t<\alpha \\ =0 & \text{if } t=\alpha \\ >0 & \text{if } t>\alpha\end{cases}$ — 1M for testing + 1A

So, $f(t)$ attains its least value when $t=\alpha$. — 1A

Therefore, we have $f(t)\geq f(\alpha)$ for all $t>0$.

Thus, we have $\dfrac{(t+1)^{\alpha+1}}{t^{\alpha}}\geq\dfrac{(\alpha+1)^{\alpha+1}}{\alpha^{\alpha}}$ for all $t>0$. — 1

Let $f(t)=\dfrac{(t+1)^{\alpha+1}}{t^{\alpha}}$ for all $t>0$.	
Then, we have	
$\dfrac{d}{dt}f(t)=\dfrac{(t+1)^{\alpha}(t-\alpha)}{t^{\alpha+1}}$	1A
$\dfrac{d^2}{dt^2}f(t)=\dfrac{\alpha(\alpha+1)(t+1)^{\alpha-1}}{t^{\alpha+2}}$	
$\dfrac{d}{dt}f(t)=0 \Leftrightarrow t=\alpha$	
$\left.\dfrac{d^2}{dt^2}f(t)\right\rvert_{t=\alpha}=\dfrac{(\alpha+1)^{\alpha}}{\alpha^{\alpha+1}}>0$	1M for testing + 1A
Note that $f(t)$ has only one local minimum.	
So, $f(t)$ attains its least value when $t=\alpha$.	1A
Therefore, we have $f(t)\geq f(\alpha)$ for all $t>0$.	
Thus, we have $\dfrac{(t+1)^{\alpha+1}}{t^{\alpha}}\geq\dfrac{(\alpha+1)^{\alpha+1}}{\alpha^{\alpha}}$ for all $t>0$.	1

----------(5)

(b) (i) Note that $\dfrac{\alpha_1}{\alpha_2}>0$. Putting $\alpha=\dfrac{\alpha_1}{\alpha_2}$ in (a), we have

$$\frac{(t+1)^{\frac{\alpha_1}{\alpha_2}+1}}{t^{\frac{\alpha_1}{\alpha_2}}}\geq\frac{\left(\frac{\alpha_1}{\alpha_2}+1\right)^{\frac{\alpha_1}{\alpha_2}+1}}{\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{\alpha_1}{\alpha_2}}}$$ — 1M

$$\frac{(t+1)^{\alpha_1+\alpha_2}}{t^{\alpha_1}}\geq\frac{\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right)^{\alpha_1+\alpha_2}}{\left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_1}}$$

$$\frac{(t+1)^{\alpha_1+\alpha_2}}{t^{\alpha_1}}\geq\frac{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}$$ — 1

(b) (ii) Note that $\frac{\beta_1}{\beta_2} > 0$. Putting $t = \frac{\beta_1}{\beta_2}$ in (b)(i), we have 1A

$$\frac{\left(\frac{\beta_1}{\beta_2}+1\right)^{\alpha_1+\alpha_2}}{\left(\frac{\beta_1}{\beta_2}\right)^{\alpha_1}} \geq \frac{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}$$ 1M

$$\frac{(\beta_1+\beta_2)^{\alpha_1+\alpha_2}}{\beta_1^{\alpha_1}\beta_2^{\alpha_2}} \geq \frac{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}$$

$$\left(\frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}\right)^{\alpha_1+\alpha_2} \geq \left(\frac{\beta_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{\beta_2}{\alpha_2}\right)^{\alpha_2}$$ 1

---------(5)

(c) Since $\left(\frac{y_1}{x_1}\right)^{x_1} = \left(\frac{y_1}{x_1}\right)^{x_1}$, the statement is true for $n = 1$.

Assume that $\left(\frac{b_1+b_2+\cdots+b_k}{a_1+a_2+\cdots+a_k}\right)^{a_1+a_2+\cdots+a_k} \geq \left(\frac{b_1}{a_1}\right)^{a_1}\left(\frac{b_2}{a_2}\right)^{a_2}\cdots\left(\frac{b_k}{a_k}\right)^{a_k}$ 1M

for any positive real numbers $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$, where k is a positive integer.

Then, for any positive real numbers $x_1, x_2, \ldots, x_{k+1}$ and $y_1, y_2, \ldots, y_{k+1}$,

$$\left(\frac{y_1+y_2+\cdots+y_{k+1}}{x_1+x_2+\cdots+x_{k+1}}\right)^{x_1+x_2+\cdots+x_{k+1}}$$

$$= \left(\frac{(y_1+y_2+\cdots+y_k)+y_{k+1}}{(x_1+x_2+\cdots+x_k)+x_{k+1}}\right)^{(x_1+x_2+\cdots+x_k)+x_{k+1}}$$ 1M

$$\geq \left(\frac{y_1+y_2+\cdots+y_k}{x_1+x_2+\cdots+x_k}\right)^{x_1+x_2+\cdots+x_k}\left(\frac{y_{k+1}}{x_{k+1}}\right)^{x_{k+1}} \quad \text{(by (b)(ii))}$$ 1M

$$\geq \left(\frac{y_1}{x_1}\right)^{x_1}\left(\frac{y_2}{x_2}\right)^{x_2}\cdots\left(\frac{y_k}{x_k}\right)^{x_k}\left(\frac{y_{k+1}}{x_{k+1}}\right)^{x_{k+1}} \quad \text{(by induction assumption)}$$ 1M

$$= \left(\frac{y_1}{x_1}\right)^{x_1}\left(\frac{y_2}{x_2}\right)^{x_2}\cdots\left(\frac{y_{k+1}}{x_{k+1}}\right)^{x_{k+1}}$$ 1

Therefore, the statement is true for $n = k+1$ when it is true for $n = k$.
By mathematical induction, the statement is true for all positive integers n .

---------(5)

10. (a) $a_{n+4}-a_{n+2}$

$=\frac{5}{7}a_{n+3}+\frac{2}{7}a_{n+2}-a_{n+2}$ 1M

$=\frac{5}{7}a_{n+3}-\frac{5}{7}a_{n+2}$

$=\frac{5}{7}a_{n+3}-\left(a_{n+3}-\frac{2}{7}a_{n+1}\right)$

$=\frac{-2}{7}(a_{n+3}-a_{n+1})$ 1A

$=\left(\frac{-2}{7}\right)^2(a_{n+2}-a_n)$

$=\frac{4}{49}(a_{n+2}-a_n)$ 1

---------(3)

(b) (i) (1) $a_{2n+1}-a_{2n-1}$

$=\frac{4}{49}(a_{2n-1}-a_{2n-3})$ (by (a))

$=\left(\frac{4}{49}\right)^2(a_{2n-3}-a_{2n-5})$ 1M

$=\cdots$

$=\left(\frac{4}{49}\right)^{n-1}(a_3-a_1)$

Note that $a_3-a_1=\frac{5}{7}a_2+\frac{2}{7}a_1-a_1=\frac{5}{7}(a_2-a_1)\ge 0$.

Thus, we have $a_{2n+1}\ge a_{2n-1}$. 1

Note that $a_3-a_1=\frac{5}{7}a_2+\frac{2}{7}a_1-a_1=\frac{5}{7}(a_2-a_1)\ge 0$. So, the statement is true for $n=1$. Assume that $a_{2k+1}\ge a_{2k-1}$ for some positive integer k . $a_{2k+3}-a_{2k+1}$ $=\frac{4}{49}(a_{2k+1}-a_{2k-1})$ (by (a)) ≥ 0 (by induction assumption) By mathematical induction, we have $a_{2n+1}\ge a_{2n-1}$.	 1M 1

(2) $a_{2n+2}-a_{2n}$

$=\frac{4}{49}(a_{2n}-a_{2n-2})$ (by (a))

$=\left(\frac{4}{49}\right)^2(a_{2n-2}-a_{2n-4})$ 1M

$=\cdots$

$=\left(\frac{4}{49}\right)^{n-1}(a_4-a_2)$

Note that $a_4-a_2=\frac{5}{7}a_3+\frac{2}{7}a_2-a_2=\frac{5}{7}(a_3-a_2)$

$=\frac{5}{7}\left(\frac{5}{7}a_2+\frac{2}{7}a_1-a_2\right)=\frac{10}{49}(a_1-a_2)\le 0$.

Thus, we have $a_{2n+2}\le a_{2n}$. 1

Note that $a_4 - a_2 = \frac{5}{7}a_3 + \frac{2}{7}a_2 - a_2 = \frac{5}{7}(a_3 - a_2)$	
$= \frac{5}{7}\left(\frac{5}{7}a_2 + \frac{2}{7}a_1 - a_2\right) = \frac{10}{49}(a_1 - a_2) \le 0$.	
So, the statement is true for $n = 1$.	
Assume that $a_{2k+2} \le a_{2k}$ for some positive integer k .	
Then, we have	
$a_{2k+4} - a_{2k+2}$	
$= \frac{4}{49}(a_{2k+2} - a_{2k})$ (by (a))	1M
≤ 0 (by induction assumption)	1
By mathematical induction, we have $a_{2n+2} \le a_{2n}$.	

(3) $a_{2n} = \frac{5}{7}a_{2n-1} + \frac{2}{7}a_{2n-2}$

$a_{2n} \ge \frac{5}{7}a_{2n-1} + \frac{2}{7}a_{2n}$ (by (b)(i)(2)) 1A

$\left(1 - \frac{2}{7}\right)a_{2n} \ge \frac{5}{7}a_{2n-1}$

$\frac{5}{7}a_{2n} \ge \frac{5}{7}a_{2n-1}$

$a_{2n} \ge a_{2n-1}$ 1

$a_{2n} - a_{2n-1}$	
$= \frac{5}{7}a_{2n-1} + \frac{2}{7}a_{2n-2} - a_{2n-1}$	
$= \frac{-2}{7}(a_{2n-1} - a_{2n-2})$	
$= \frac{-2}{7}\left(\frac{5}{7}a_{2n-2} + \frac{2}{7}a_{2n-3} - a_{2n-2}\right)$	
$= \frac{4}{49}(a_{2n-2} - a_{2n-3})$	1A
$= \left(\frac{4}{49}\right)^2 (a_{2n-4} - a_{2n-5})$	
$= \cdots$	
$= \left(\frac{4}{49}\right)^{n-1} (a_2 - a_1) \ge 0$	
Thus, we have $a_{2n} \ge a_{2n-1}$.	1

Note that $a_2 \geq a_1$.	
So, the statement is true for $n = 1$.	
Assume that $a_{2k} \geq a_{2k-1}$ for some positive integer k .	
$a_{2k+2} - a_{2k+1}$	
$= \frac{5}{7}a_{2k+1} + \frac{2}{7}a_{2k} - a_{2k+1}$	
$= \frac{-2}{7}(a_{2k+1} - a_{2k})$	
$= \frac{-2}{7}\left(\frac{5}{7}a_{2k} + \frac{2}{7}a_{2k-1} - a_{2k}\right)$	
$= \frac{4}{49}(a_{2k} - a_{2k-1})$	1A
≥ 0 (by induction assumption)	1
By mathematical induction, we have $a_{2n} \geq a_{2n-1}$.	

(ii) Note that $a_1 \leq a_{2n-1} \leq a_{2n+1} \leq a_{2n+2} \leq a_{2n} \leq a_2$.	
So, $\{ a_1, a_3, a_5, \dots \}$ is increasing and bounded above by a_2 while	1A
$\{ a_2, a_4, a_6, \dots \}$ is decreasing and bounded below by a_1 .	1A
Thus, $\lim_{n\to\infty} a_{2n-1}$ and $\lim_{n\to\infty} a_{2n}$ both exist.	1
	---------(9)
) If $a_2 < a_1$, then $-a_2 > -a_1$.	1M
Define $b_n = -a_n$ for all $n = 1, 2, 3, \dots$.	1M
So, we have $b_2 > b_1$ and $b_{n+2} = \frac{5}{7}b_{n+1} + \frac{2}{7}b_n$ for all $n = 1, 2, 3, \dots$.	
By (b)(ii), $\lim_{n\to\infty} b_{2n-1}$ and $\lim_{n\to\infty} b_{2n}$ both exist.	
Thus, $\lim_{n\to\infty} a_{2n-1}$ and $\lim_{n\to\infty} a_{2n}$ both exist.	1A f.t.

Note that $a_3 - a_2 = \frac{5}{7}a_2 + \frac{2}{7}a_1 - a_2 = \frac{-2}{7}(a_2 - a_1) > 0$.	
So, we have $a_3 > a_2$.	1M
Define $b_n = a_{n+1}$ for all $n = 1, 2, 3, \dots$.	1M
So, we have $b_2 > b_1$ and $b_{n+2} = \frac{5}{7}b_{n+1} + \frac{2}{7}b_n$ for all $n = 1, 2, 3, \dots$.	
By (b)(ii), $\lim_{n\to\infty} b_{2n-1}$ and $\lim_{n\to\infty} b_{2n}$ both exist.	
So, $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ both exist.	
Thus, $\lim_{n\to\infty} a_{2n-1}$ and $\lim_{n\to\infty} a_{2n}$ both exist.	1A f.t.

$a_{2n+1} - a_{2n-1}$ $=\left(\frac{4}{49}\right)^{n-1}(a_3 - a_1)$ $=\left(\frac{4}{49}\right)^{n-1}\left(\frac{5}{7}\right)(a_2 - a_1)$ <0 Therefore, we have $a_{2n+1} < a_{2n-1}$. $a_{2n+2} - a_{2n}$ $=\left(\frac{4}{49}\right)^{n-1}(a_4 - a_2)$ $=\left(\frac{4}{49}\right)^{n-1}\left(\frac{10}{49}\right)(a_1 - a_2)$ >0 So, we have $a_{2n+2} > a_{2n}$. $a_{2n} = \frac{5}{7}a_{2n-1} + \frac{2}{7}a_{2n-2}$ $a_{2n} < \frac{5}{7}a_{2n-1} + \frac{2}{7}a_{2n}$ $\frac{5}{7}a_{2n} < \frac{5}{7}a_{2n-1}$ Hence, we have $a_{2n} < a_{2n-1}$. Note that $a_2 < a_{2n} < a_{2n+2} < a_{2n+1} < a_{2n-1} < a_1$.	
So, $\{a_1, a_3, a_5, \ldots\}$ is strictly decreasing and bounded below by a_2 while	1M
$\{a_2, a_4, a_6, \ldots\}$ is strictly increasing and bounded above by a_1 .	1M
Thus, $\lim_{n\to\infty} a_{2n-1}$ and $\lim_{n\to\infty} a_{2n}$ both exist.	1A f.t.
	---------(3)

11. (a) (i)

$$\frac{(1+i\tan\theta)^7-(1-i\tan\theta)^7}{(1+i\tan\theta)^7+(1-i\tan\theta)^7}$$

$$=\frac{\left(1+\frac{i\sin\theta}{\cos\theta}\right)^7-\left(1-\frac{i\sin\theta}{\cos\theta}\right)^7}{\left(1+\frac{i\sin\theta}{\cos\theta}\right)^7+\left(1-\frac{i\sin\theta}{\cos\theta}\right)^7}$$

$$=\frac{(\cos\theta+i\sin\theta)^7-(\cos\theta-i\sin\theta)^7}{(\cos\theta+i\sin\theta)^7+(\cos\theta-i\sin\theta)^7}$$ 1M

$$=\frac{(\cos\theta+i\sin\theta)^7-(\cos(-\theta)+i\sin(-\theta))^7}{(\cos\theta+i\sin\theta)^7+(\cos(-\theta)+i\sin(-\theta))^7}$$

$$=\frac{(\cos 7\theta+i\sin 7\theta)-(\cos(-7\theta)+i\sin(-7\theta))}{(\cos 7\theta+i\sin 7\theta)+(\cos(-7\theta)+i\sin(-7\theta))}$$ 1A

$$=\frac{(\cos 7\theta+i\sin 7\theta)-(\cos 7\theta-i\sin 7\theta)}{(\cos 7\theta+i\sin 7\theta)+(\cos 7\theta-i\sin 7\theta)}$$

$$=\frac{2i\sin 7\theta}{2\cos 7\theta}$$

$$=i\tan 7\theta$$ 1

(ii) Let $\tan\theta=t$.

Then, we have

$$(1+i\tan\theta)^7$$

$$=(1+it)^7$$

$$=1+7it-21t^2-35it^3+35t^4+21it^5-7t^6-it^7$$ 1M (for either one)

Also, we have

$$(1-i\tan\theta)^7$$

$$=(1-it)^7$$

$$=1-7it-21t^2+35it^3+35t^4-21it^5-7t^6+it^7$$

So, we have

$$(1+i\tan\theta)^7-(1-i\tan\theta)^7$$

$$=2i(7t-35t^3+21t^5-t^7)$$

Moreover, we have

$$(1+i\tan\theta)^7+(1-i\tan\theta)^7$$

$$=2(1-21t^2+35t^4-7t^6)$$ 1A (for both correct)

Therefore, we have

$$\frac{(1+i\tan\theta)^7-(1-i\tan\theta)^7}{(1+i\tan\theta)^7+(1-i\tan\theta)^7}$$

$$=\frac{2i(7t-35t^3+21t^5-t^7)}{2(1-21t^2+35t^4-7t^6)}$$

$$=\frac{i(7t-35t^3+21t^5-t^7)}{1-21t^2+35t^4-7t^6}$$

$$=\frac{i(7\tan\theta-35\tan^3\theta+21\tan^5\theta-\tan^7\theta)}{1-21\tan^2\theta+35\tan^4\theta-7\tan^6\theta}$$

By (a)(i), we have $\tan 7\theta=\dfrac{7\tan\theta-35\tan^3\theta+21\tan^5\theta-\tan^7\theta}{1-21\tan^2\theta+35\tan^4\theta-7\tan^6\theta}$.

Thus, we have $\tan 7\theta=\dfrac{\tan^7\theta-21\tan^5\theta+35\tan^3\theta-7\tan\theta}{7\tan^6\theta-35\tan^4\theta+21\tan^2\theta-1}$. 1

---------(6)

Let $\cos\theta = a$ and $\sin\theta = b$.

$(a+ib)^7 = a^7 + C_1^7 a^6(ib) + C_2^7 a^5(ib)^2 + C_3^7 a^4(ib)^3 + C_4^7 a^3(ib)^4 + C_5^7 a^2(ib)^5 + C_6^7 a(ib)^6 + (ib)^7$

$\cos 7\theta + i\sin 7\theta = a^7 + 7a^6(ib) - 21a^5b^2 - 35ia^4b^3 + 35a^3b^4 + 21ia^2b^5 - 7ab^6 - ib^7$

$\cos 7\theta + i\sin 7\theta = a^7 - 21a^5b^2 + 35a^3b^4 - 7ab^6 + i(7a^6b - 35a^4b^3 + 21a^2b^5 - b^7)$

By comparing the real and imaginary parts of both sides, we have

$$\begin{cases} \cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \\ \sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta \end{cases}$$ 1M + 1A

So, we have

$\tan 7\theta$

$= \dfrac{\sin 7\theta}{\cos 7\theta}$

$= \dfrac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta}$

$= \dfrac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$

$= \dfrac{\tan^7\theta - 21\tan^5\theta + 35\tan^3\theta - 7\tan\theta}{7\tan^6\theta - 35\tan^4\theta + 21\tan^2\theta - 1}$ 1

(b) Putting $x = \tan\theta$ in (a)(ii), we have $\tan 7\theta = \dfrac{x^7 - 21x^5 + 35x^3 - 7x}{7x^6 - 35x^4 + 21x^2 - 1}$. 1M for putting $x = \tan\theta$ in (a)(ii)

For $\tan 7\theta = 0$, we have $x^7 - 21x^5 + 35x^3 - 7x = 0$. 1M for setting $\tan 7\theta = 0$

Note that $\tan 7\theta = 0 \Leftrightarrow \tan\theta = \tan 0$, $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, … , $\tan\frac{6\pi}{7}$ 1A

Also note that $\tan 0$, $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$,…, $\tan\frac{6\pi}{7}$ are all distinct.

Further note that $x^7 - 21x^5 + 35x^3 - 7x = 0 \Leftrightarrow x(x^6 - 21x^4 + 35x^2 - 7) = 0$

Thus, the roots of the equation $x^6 - 21x^4 + 35x^2 - 7 = 0$ are

$\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, … , $\tan\frac{6\pi}{7}$. 1

---------(4)

(c) (i) By (b), we have

$$\left(\tan\frac{\pi}{7}\right)\left(\tan\frac{2\pi}{7}\right)\left(\tan\frac{3\pi}{7}\right)\left(\tan\frac{4\pi}{7}\right)\left(\tan\frac{5\pi}{7}\right)\left(\tan\frac{6\pi}{7}\right)=\frac{-7}{1}$$ 1M

$$\left(\tan\frac{\pi}{7}\right)\left(\tan\frac{2\pi}{7}\right)\left(\tan\frac{3\pi}{7}\right)\left(\tan\frac{4\pi}{7}\right)\left(\tan\frac{5\pi}{7}\right)\left(\tan\frac{6\pi}{7}\right)=-7$$

Note that $\tan\frac{6\pi}{7}=-\tan\frac{\pi}{7}$, $\tan\frac{5\pi}{7}=-\tan\frac{2\pi}{7}$ and $\tan\frac{4\pi}{7}=-\tan\frac{3\pi}{7}$. 1M

So, we have $-\left(\tan^2\frac{\pi}{7}\right)\left(\tan^2\frac{2\pi}{7}\right)\left(\tan^2\frac{3\pi}{7}\right)=-7$.

either one

Thus, we have $\left(\tan^2\frac{\pi}{7}\right)\left(\tan^2\frac{2\pi}{7}\right)\left(\tan^2\frac{3\pi}{7}\right)=7$. 1A f.t.

either one

(ii) By (b), we have

$$\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}+\tan^2\frac{4\pi}{7}+\tan^2\frac{5\pi}{7}+\tan^2\frac{6\pi}{7}$$

$$=\left(\tan\frac{\pi}{7}+\cdots+\tan\frac{6\pi}{7}\right)^2-2\left(\tan\frac{\pi}{7}\tan\frac{2\pi}{7}+\cdots+\tan\frac{5\pi}{7}\tan\frac{6\pi}{7}\right)$$

$$=0^2-2(-21)$$

$$=42$$ 1A f.t.

Note that $\tan^2\frac{6\pi}{7}=\tan^2\frac{\pi}{7}$, $\tan^2\frac{5\pi}{7}=\tan^2\frac{2\pi}{7}$ and $\tan^2\frac{4\pi}{7}=\tan^2\frac{3\pi}{7}$.

So, we have $2\left(\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}\right)=42$.

Thus, we have $\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}=21$. 1A f.t.

Note that $x^6-21x^4+35x^2-7=0 \Leftrightarrow (x^2)^3-21(x^2)^2+35x^2-7=0$.	1M
Also note that $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$ and $\tan^2\frac{3\pi}{7}$ are all distinct.	1M can be absorbed
Therefore, with the help of (b), the roots of the equation	
$x^3-21x^2+35x-7=0$ are $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$ and $\tan^2\frac{3\pi}{7}$.	1A f.t.
Thus, we have $\left(\tan^2\frac{\pi}{7}\right)\left(\tan^2\frac{2\pi}{7}\right)\left(\tan^2\frac{3\pi}{7}\right)=7$ and	1A f.t.
$\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}=21$.	1A f.t.

---------(5)

Paper 2

1. (a) $\lim_{x\to 0^+} x\ln x$

$= \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}}$

$= \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}}$ 1M

$= -\lim_{x\to 0^+} x$

$= 0$ 1

(b) (i) Since $f(x)$ is continuous at $x=0$, we have $\lim_{x\to 0^+} f(x) = f(0)$.

So, we have $\lim_{x\to 0^+} x^2 \ln x = 1+k$.

Note that $\lim_{x\to 0^+} x^2 \ln x = \left(\lim_{x\to 0^+} x\right)\left(\lim_{x\to 0^+} x\ln x\right) = 0$ (by (a)).

Therefore, we have $1+k=0$.

Thus, we have $k=-1$. 1A f.t.

(ii) By (b)(i), we have

$$f(x) = \begin{cases} \sin x + \cos 2x - 1 & \text{when } x \le 0 , \\ x^2 \ln x & \text{when } x > 0 . \end{cases}$$

$\lim_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0}$ 1M for one-sided derivative (either one)

$= \lim_{x\to 0^-} \dfrac{\sin x + \cos 2x - 1}{x}$

$= \lim_{x\to 0^-} \dfrac{\cos x - 2\sin 2x}{1}$

$= 1$ 1A

$\lim_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0^+} \dfrac{x^2\ln x - 0}{x}$

$= \lim_{x\to 0^+} x\ln x$

$= 0$ (by (a))

So, we have $\lim_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} \neq \lim_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0}$.

Thus, $f(x)$ is not differentiable at $x=0$. 1A f.t.

-----------(6)

2. (a) $f(x)=\dfrac{1}{\sqrt{4x-x^2}}$

$f(x)=(4x-x^2)^{\frac{-1}{2}}$

$f'(x)=\dfrac{-1}{2}(4x-x^2)^{\frac{-3}{2}}(4-2x)$

$f'(x)=\dfrac{x-2}{\sqrt{(4x-x^2)^3}}$

$(4x-x^2)f'(x)=\dfrac{x-2}{\sqrt{4x-x^2}}$

$(4x-x^2)f'(x)=(x-2)f(x)$ 1

Differentiate both sides n times with respect to x, we have

$(4x-x^2)f^{(n+1)}(x)+n(4-2x)f^{(n)}(x)+C_2^n(-2)f^{(n-1)}(x)=(x-2)f^{(n)}(x)+nf^{(n-1)}(x)$ 1M

$(4x-x^2)f^{(n+1)}(x)-2n(x-2)f^{(n)}(x)-(n^2-n)f^{(n-1)}(x)=(x-2)f^{(n)}(x)+nf^{(n-1)}(x)$

$(4x-x^2)f^{(n+1)}(x)=(2n+1)(x-2)f^{(n)}(x)+n^2f^{(n-1)}(x)$ 1

(b) Putting $x=2$ in (a), we have $4f^{(n+1)}(2)=n^2f^{(n-1)}(2)$.

So, we have $f^{(n+1)}(2)=\left(\dfrac{n}{2}\right)^2 f^{(n-1)}(2)$ for all positive integers n .

$f^{(7)}(2)$

$=3^2f^{(5)}(2)$

$=(3^2)(2^2)f^{(3)}(2)$ 1M (either one)

$=(3^2)(2^2)(1^2)f'(2)$

$=0$ 1A

$f^{(8)}(2)$

$=\left(\dfrac{7}{2}\right)^2 f^{(6)}(2)$

$=\left(\dfrac{7}{2}\right)^2\left(\dfrac{5}{2}\right)^2 f^{(4)}(2)$

$=\left(\dfrac{7}{2}\right)^2\left(\dfrac{5}{2}\right)^2\left(\dfrac{3}{2}\right)^2 f''(2)$

$=\left(\dfrac{7}{2}\right)^2\left(\dfrac{5}{2}\right)^2\left(\dfrac{3}{2}\right)^2\left(\dfrac{1}{2}\right)^2 f(2)$

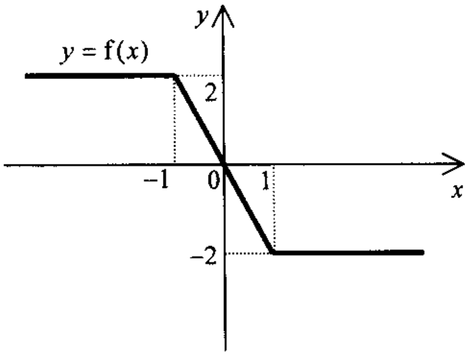
$=\left(\dfrac{7}{2}\right)^2\left(\dfrac{5}{2}\right)^2\left(\dfrac{3}{2}\right)^2\left(\dfrac{1}{2}\right)^2\left(\dfrac{1}{2}\right)$

$=\dfrac{11025}{512}$ 1A

----------(6)

3. (a) Since $f(x) = |x-1| - |x+1|$ for all $x \in \mathbf{R}$, we have

$$f(x) = \begin{cases} 2 & \text{if } x \le -1, \\ -2x & \text{if } -1 < x < 1, \\ -2 & \text{if } x \ge 1. \end{cases}$$



1M for the shape
1A for all correct

(b) Note that $f(x) \ne 3 \in \mathbf{R}$ for all $x \in \mathbf{R}$. 1M
Thus, f is not a surjective function. 1A f.t.

(c) (i)

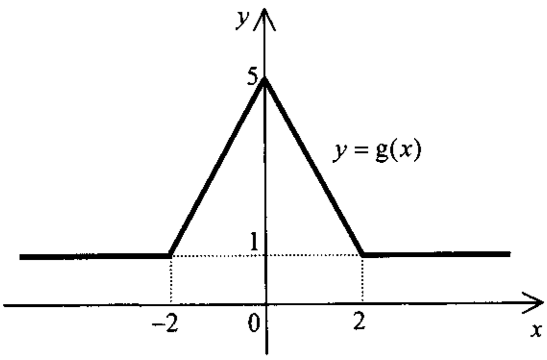
$g(x)$
$= f(x-1) - f(x+1) + 1$
$= (|x-2| - |x|) - (|x| - |x+2|) + 1$
$= |x+2| + |x-2| - 2|x| + 1$

$g(-x)$
$= |-x+2| + |-x-2| - 2|-x| + 1$
$= |x-2| + |x+2| - 2|x| + 1$
$= |x+2| + |x-2| - 2|x| + 1$

Thus, we have $g(x) = g(-x)$ for all $x \in \mathbf{R}$. 1
Hence, g is an even function.

(ii) Note that g is an even function and

$$g(x) = \begin{cases} 5-2x & \text{if } 0 \le x < 2, \\ 1 & \text{if } x \ge 2. \end{cases}$$



1M for the shape
1A for all correct

----------(7)

4. (a) $\int e^x \sin x \, dx$	
$= \int \sin x \, de^x$	
$= e^x \sin x - \int e^x \cos x \, dx$	1M for using integration by parts
$= e^x \sin x - \int \cos x \, de^x$	
$= e^x \sin x - e^x \cos x + \int e^x (-\sin x) \, dx$	
$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$	1A
So, we have $2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + \text{constant}$.	1M
Thus, we have $\int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + \text{constant}$.	1A pp−1 for omitting constant
(b) $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{\frac{k\pi}{n}} \sin \frac{k\pi}{n}$	
$= \frac{1}{\pi} \lim_{n \to \infty} \frac{\pi}{n} \sum_{k=1}^{n} e^{\frac{k\pi}{n}} \sin \frac{k\pi}{n}$	
$= \frac{1}{\pi} \int_0^{\pi} e^x \sin x \, dx$	1A
$= \frac{1}{\pi} \left[\frac{e^x}{2} (\sin x - \cos x) \right]_0^{\pi}$ (by (a))	1M for using the result of (a)
$= \frac{e^{\pi} + 1}{2\pi}$	1A
$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{\frac{k\pi}{n}} \sin \frac{k\pi}{n}$	
$= \int_0^1 e^{\pi x} \sin \pi x \, dx$	1A
$= \frac{1}{\pi} \int_0^{\pi} e^x \sin x \, dx$	
$= \frac{1}{\pi} \left[\frac{e^x}{2} (\sin x - \cos x) \right]_0^{\pi}$ (by (a))	1M for using the result of (a)
$= \frac{e^{\pi} + 1}{2\pi}$	1A
$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{\frac{k\pi}{n}} \sin \frac{k\pi}{n}$	
$= \int_0^1 e^{\pi x} \sin \pi x \, dx$	1A
$= \frac{1}{2\pi} \left[e^{\pi x} (\sin \pi x - \cos \pi x) \right]_0^1$ (by (a))	1M for using the result of (a)
$= \frac{e^{\pi} + 1}{2\pi}$	1A
	----------(7)

5. (a) $\int\left(\frac{(x-2)(x-5)}{x}\right)^2 dx$	
$= \int\left(x^2-14x+69-\frac{140}{x}+\frac{100}{x^2}\right)dx$	1M for division
$= \frac{x^3}{3}-7x^2+69x-140\ln\lvert x\rvert-\frac{100}{x}+\text{constant}$	1A accept without absolute value pp−1 for omitting constant
(b) Note that the x-intercepts of the curve are 0 and 3 . The required volume	
$= \pi\int_0^3\left(\frac{x(x-3)}{(x+2)}\right)^2 dx$	1M for lower and upper limits + 1A
$= \pi\int_2^5\left(\frac{(x-2)(x-5)}{x}\right)^2 dx$	1M for using integration by substitution
$= \pi\left[\frac{x^3}{3}-7x^2+69x-140\ln\lvert x\rvert-\frac{100}{x}\right]_2^5$ (by (a))	1M for using the result of (a)
$= \pi\left(129-140\ln(\frac{5}{2})\right)$	1A

Note that the x-intercepts of the curve are 0 and 3 . The required volume	
$= \pi\int_0^3\left(\frac{x(x-3)}{(x+2)}\right)^2 dx$	1M for lower and upper limits + 1A
$= \pi\int_0^3\left(x^2-10x+45-\frac{140}{x+2}+\frac{100}{(x+2)^2}\right)dx$	1M for $x^2+ax+\frac{b}{x+2}+\frac{c}{(x+2)^2}$
$= \pi\left[\frac{x^3}{3}-5x^2+45x-140\ln\lvert x+2\rvert-\frac{100}{x+2}\right]_0^3$	1M for $\frac{x^3}{3}+\frac{ax^2}{2}+b\ln\lvert x+2\rvert-\frac{c}{x+2}$
$= \pi\left(129-140\ln(\frac{5}{2})\right)$	1A

Note that the x-intercepts of the curve are 0 and 3 and the curve has a minimum point $(\sqrt{10}-2,\ 2\sqrt{10}-7)$ between $x=0$ and $x=3$. The required volume	
$= 2\pi\int_{2\sqrt{10}-7}^{0}\lvert y\rvert\sqrt{y^2+14y+9}\,dy$	1M for lower and upper limits + 1A
$= \pi\int_0^{2\sqrt{10}-7}\sqrt{y^2+14y+9}\,d(y^2+14y+9)-14\pi\int_0^{2\sqrt{10}-7}\sqrt{y^2+14y+9}\,dy$	1M for using substitution
$= \pi\left[\frac{2(y^2+14y+9)^{\frac{3}{2}}}{3}-7(y+7)\sqrt{y^2+14y+9}+280\ln\left\lvert y+7+\sqrt{y^2+14y+9}\right\rvert\right]_0^{2\sqrt{10}-7}$	1M for $\frac{2t^3}{3}+ast+b\ln\lvert c(s+t)\rvert$, $s=y+7$ and $t=\sqrt{y^2+14y+9}$
$= \pi\left(129-140\ln(\frac{5}{2})\right)$	1A

----------(7)

6. (a) (i) Case 1: $a = 0$	
The equation of the normal is $y = 0$.	
Case 2: $a \neq 0$	
Since $x = t^2 + 1$ and $y = 2t$, we have	
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}$ if $t \neq 0$.	1M
The equation of the normal is	
$y - 2a = -a(x - a^2 - 1)$	
$y - 2a = -ax + a^3 + a$	
$ax + y - a^3 - 3a = 0$	1A
Thus, by combining the above two cases, the equation of the normal is $ax + y - a^3 - 3a = 0$.	
(ii) AB is normal to Γ at A	
$\Leftrightarrow a(b^2 + 1) + 2b - a^3 - 3a = 0$	1M for using the result of (a)(i)
$\Leftrightarrow ab^2 + a + 2b - a^3 - 3a = 0$	
$\Leftrightarrow 2b - 2a + ab^2 - a^3 = 0$	
$\Leftrightarrow 2(b - a) + a(b - a)(b + a) = 0$	
$\Leftrightarrow (b - a)(a^2 + ab + 2) = 0$	
$\Leftrightarrow a^2 + ab + 2 = 0$ (since $a \neq b$)	1
Note that $a \neq -b$.	
Also note that the slope of the normal to Γ at A is $-a$.	
Further note that the slope of AB is $\frac{2}{a+b}$.	
AB is normal to Γ at A	
$\Leftrightarrow -a = \frac{2}{a+b}$	1M for equating slopes
$\Leftrightarrow -a - ab = 2$	
$\Leftrightarrow a^2 + ab + 2 = 0$	1
(b) Putting $b = -3$ in (a)(ii), we have	1M for observing $P = ((-3)^2 + 1, 2(-3))$
$a^2 - 3a + 2 = 0$	1M for using (a)(ii) with b substituted
$(a-1)(a-2) = 0$	
$a = 1$ or $a = 2$	
Thus, two required points are $(2, 2)$ and $(5, 4)$.	1A for both correct
	----------(7)

7. (a) $f'(x)$

$= \dfrac{(x-6)^2\left((x+1)^2+2(x+1)(x+15)\right)-2(x-6)(x+15)(x+1)^2}{(x-6)^4}$ — 1M for quotient rule or product rule (either one)

$= \dfrac{(x+1)(x^2-19x-216)}{(x-6)^3}$

$= \dfrac{(x+1)(x+8)(x-27)}{(x-6)^3}$ — 1A or equivalent (either one)

$f''(x)$

$= \dfrac{(x-6)^3\left((x+8)(x-27)+(x+1)(x-27)+(x+1)(x+8)\right)-3(x-6)^2(x+1)(x+8)(x-27)}{(x-6)^6}$

$= \dfrac{686(x+3)}{(x-6)^4}$ — 1A or equivalent

----------(3)

(b) Note that $f'(x)=0 \Leftrightarrow x=-8$, $x=-1$ or $x=27$.

Also note that $f''(x)=0 \Leftrightarrow x=-3$.

x	$(-\infty,-8)$	-8	$(-8,-3)$	-3	$(-3,-1)$	-1	$(-1,6)$	$(6,27)$	27	$(27,\infty)$
$f'(x)$	+	0	−	−	−	0	+	−	0	+
$f''(x)$	−	−	−	0	+	+	+	+	+	+
$f(x)$	↗	$\frac{7}{4}$	↘	$\frac{16}{27}$	↘	0	↗	↘	$\frac{224}{3}$	↗

(i) $f'(x)>0 \Leftrightarrow x<-8$, $-1<x<6$ or $x>27$ — 1A

(ii) $f''(x)>0 \Leftrightarrow -3<x<6$ or $x>6$ — 1A accept $(-3,\infty)$

----------(2)

(c) The relative minimum points are $(-1,\ 0)$ and $(27,\ \frac{224}{3})$. — 1A + 1A

The relative maximum point is $(-8,\ \frac{7}{4})$, — 1A

The point of inflexion is $(-3,\frac{16}{27})$. — 1A

----------(4)

(d) $\because \lim\limits_{x\to 6^-} f(x)=\lim\limits_{x\to 6^-}\dfrac{(x+15)(x+1)^2}{(x-6)^2}=+\infty$ and

$\lim\limits_{x\to 6^+} f(x)=\lim\limits_{x\to 6^+}\dfrac{(x+15)(x+1)^2}{(x-6)^2}=+\infty$

$\therefore$ the vertical asymptote is $x=6$. — 1A

(d) $\because \lim\limits_{x\to 6^-} f(x)=\lim\limits_{x\to 6^-}\dfrac{(x+15)(x+1)^2}{(x-6)^2}=+\infty$ and

$\lim\limits_{x\to 6^+} f(x)=\lim\limits_{x\to 6^+}\dfrac{(x+15)(x+1)^2}{(x-6)^2}=+\infty$

$\therefore$ the vertical asymptote is $x=6$. — 1A

$\because \quad \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(x+15)(x+1)^2}{x(x-6)^2} = \lim_{x\to\infty} \frac{\left(1+\frac{15}{x}\right)\left(1+\frac{1}{x}\right)^2}{\left(1-\frac{6}{x}\right)^2} = 1$

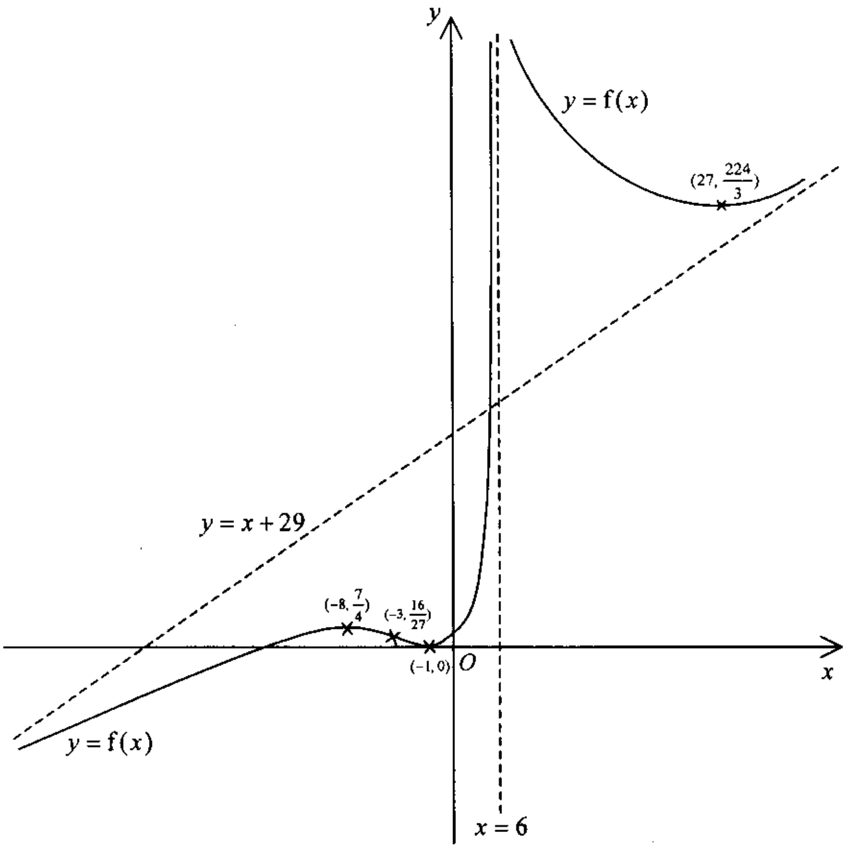
$\lim_{x\to\infty} (f(x)-x) = \lim_{x\to\infty} \frac{29x^2-5x+15}{(x-6)^2} = \lim_{x\to\infty} \frac{29-\frac{5}{x}+\frac{15}{x^2}}{\left(1-\frac{6}{x}\right)^2} = 29$ } 1M for finding oblique asymptote

$\therefore$ the oblique asymptote is $y = x+29$. 1A

$\because \quad \lim_{x\to 6^-} f(x) = \lim_{x\to 6^-} \frac{(x+15)(x+1)^2}{(x-6)^2} = \infty$ and	
$\lim_{x\to 6^+} f(x) = \lim_{x\to 6^+} \frac{(x+15)(x+1)^2}{(x-6)^2} = \infty$	
$\therefore$ the vertical asymptote is $x = 6$.	1A
$\because \quad f(x) = \frac{(x+15)(x+1)^2}{(x-6)^2} = x+29+\frac{343x-1029}{(x-6)^2}$	1M
$\therefore$ the oblique asymptote is $y = x+29$.	1A

----------(3)

(e)



1A for the extreme points and the point of inflexion
1A for the asymptotes
1A for all being correct

----------(3)

8. (a) $f(0) = e^0 + 3e^0 \int_0^0 e^{-8t} g(t)\,dt = 1 + 0 = 1$

$g(0) = e^0 - 3e^0 \int_0^0 e^{-2t} f(t)\,dt = 1 - 0 = 1$ — 1A for both correct

----------(1)

(b) $f(x) = e^{8x} + 3e^{8x} \int_0^x e^{-8t} g(t)\,dt$

By Fundamental Theorem of Calculus, $f'(x)$ exists and

$f'(x) = 8e^{8x} + 24e^{8x} \int_0^x e^{-8t} g(t)\,dt + 3e^{8x} e^{-8x} g(x)$ — 1M

$= 8f(x) + 3g(x)$ — 1

Thus, we have $f'(0) = 8f(0) + 3g(0) = 11$. — 1A

----------(3)

(c) $g(x) = e^{2x} - 3e^{2x} \int_0^x e^{-2t} f(t)\,dt$

By Fundamental Theorem of Calculus, $g'(x)$ exists and

$g'(x) = 2e^{2x} - 6e^{2x} \int_0^x e^{-2t} f(t)\,dt - 3e^{2x} e^{-2x} f(x)$

$= 2g(x) - 3f(x)$ — 1A

Note that $f'(x) = 8f(x) + 3g(x)$ (by (b)) and both $f(x)$ and $g(x)$ are differentiable.

Therefore, $f'(x)$ is differentiable and $f''(x) = 8f'(x) + 3g'(x)$. — 1A for correct $f''(x)$

So, we have $f''(x) = 8f'(x) - 9f(x) + 6g(x)$.

Hence, we have $f''(x) = 8f'(x) - 9f(x) + 2f'(x) - 16f(x)$ (by (b)) .

Thus, we have $f''(x) - 10f'(x) + 25f(x) = 0$. — 1M + 1

----------(4)

(d) Let $h(x) = e^{-5x} f(x)$ for all $x \in \mathbf{R}$. Then,

$h'(x) = e^{-5x} f'(x) - 5e^{-5x} f(x) = e^{-5x}\left(f'(x) - 5f(x)\right)$ — 1A

$h''(x) = -5e^{-5x}\left(f'(x) - 5f(x)\right) + e^{-5x}\left(f''(x) - 5f'(x)\right)$

$= e^{-5x}\left(f''(x) - 10f'(x) + 25f(x)\right)$

$= 0$ (by (c)) — 1A

Therefore, we have $h'(x) = a$.

So, we have $h(x) = ax + b$, where a and b are constants. — 1A

Note that $h(0) = e^0 f(0) = 1$ (by (a)) and

$h'(0) = e^0\left(f'(0) - 5f(0)\right) = 11 - 5 = 6$ (by (a) and (b)) . — 1M for using (a) and (b)

Hence, we have $a = 6$ and $b = 1$.

So, we have $h(x) = 6x + 1$.

Thus, we have $f(x) = (6x + 1)e^{5x}$. — 1A

----------(5)

(e) Note that $f'(x) = 5(6x+1)e^{5x} + 6e^{5x} = (30x+11)e^{5x}$.

By (b), we have

$$\begin{aligned} & g(x) \\ &= \frac{1}{3}\left(f'(x) - 8f(x)\right) \\ &= \frac{1}{3}\left((30x+11)e^{5x} - 8(6x+1)e^{5x}\right) \quad \text{1M} \\ &= (1-6x)e^{5x} \quad \text{1A} \end{aligned}$$

$$\begin{aligned} & g(x) \\ &= e^{2x} - 3e^{2x}\int_0^x (6t+1)e^{3t}\,dt \quad \text{1M} \\ &= e^{2x} - e^{2x}\int_0^x (6t+1)\,de^{3t} \\ &= e^{2x} - e^{2x}\left(\left[(6t+1)e^{3t}\right]_0^x - 6\int_0^x e^{3t}\,dt\right) \\ &= e^{2x} - (6x+1)e^{5x} + e^{2x} + e^{2x}(2e^{3x} - 2) \\ &= (1-6x)e^{5x} \quad \text{1A} \end{aligned}$$

Let $w(x) = e^{-5x}g(x)$ for all $x \in \mathbf{R}$. Then,

$w'(x) = e^{-5x}\left(g'(x) - 5g(x)\right)$

$w''(x) = e^{-5x}\left(g''(x) - 10g'(x) + 25g(x)\right)$ 1M

Note that $g'(x) = 2g(x) - 3f(x)$ (by the proof in (c)) and both $f(x)$ and $g(x)$ are differentiable.

Therefore, $g'(x)$ is differentiable and $g''(x) = 2g'(x) - 3f'(x)$.

Hence, we have $g''(x) = 2g'(x) - 24f(x) - 9g(x)$ (by (b)) .

So, we have $g''(x) = 2g'(x) + 8g'(x) - 16g(x) - 9g(x)$ (by the proof in (c)) .

Therefore, we have $g''(x) - 10g'(x) + 25g(x) = 0$.

So, we have $w''(x) = 0$.

Therefore, we have $w(x) = cx + d$, where c and d are constants.

Note that $w(0) = e^0 g(0) = 1$ (by (a))

and $w'(0) = e^0\left(g'(0) - 5g(0)\right) = -1 - 5 = -6$ (by the proofs in (c) and (a)) .

Hence, we have $c = -6$ and $d = 1$.

So, we have $w(x) = 1 - 6x$.

Thus, we have $g(x) = (1-6x)e^{5x}$. 1A

----------(2)

9. (a) (i)

$$\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}$$
$$=\frac{(x+2)(x^2-2x+2)-(x-2)(x^2+2x+2)}{(x^2+2x+2)(x^2-2x+2)}$$
$$=\frac{(x^3-2x+4)-(x^3-2x-4)}{(x^2+2)^2-4x^2}$$
$$=\frac{8}{x^4+4}$$ 1

Thus, we have $\frac{8}{x^4+4}=\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}$.

Note that $x^4+4=(x^2+2x+2)(x^2-2x+2)$.
Let $\frac{8}{x^4+4}=\frac{Ax+B}{x^2+2x+2}+\frac{Cx+D}{x^2-2x+2}$, where A , B , C and D are constants.
Therefore, we have $8\equiv(Ax+B)(x^2-2x+2)+(Cx+D)(x^2+2x+2)$.
So, $8\equiv(A+C)x^3+(B-2A+2C+D)x^2+(2A-2B+2C+2D)x+2B+2D$.
Hence, we have $A+C=0$, $B-2A+2C+D=0$, $2A-2B+2C+2D=0$ and $2B+2D=8$.
Solving, we have $A=1$, $B=2$, $C=-1$ and $D=2$.
Thus, we have $\frac{8}{x^4+4}=\frac{x+2}{x^2+2x+2}-\frac{x-2}{x^2-2x+2}$. 1

I_0
$$=\int_0^1\frac{dx}{x^4+4}$$
$$=\frac{1}{8}\int_0^1\frac{(x+2)dx}{x^2+2x+2}-\frac{1}{8}\int_0^1\frac{(x-2)dx}{x^2-2x+2}$$
$$=\frac{1}{8}\int_0^1\frac{(x+1)dx}{x^2+2x+2}+\frac{1}{8}\int_0^1\frac{dx}{(x+1)^2+1}-\frac{1}{8}\int_0^1\frac{(x-1)dx}{x^2-2x+2}+\frac{1}{8}\int_0^1\frac{dx}{(x-1)^2+1}$$ 1M
$$=\frac{1}{16}\left[\ln(x^2+2x+2)\right]_0^1+\frac{1}{8}\left[\tan^{-1}(x+1)\right]_0^1-\frac{1}{16}\left[\ln(x^2-2x+2)\right]_0^1+\frac{1}{8}\left[\tan^{-1}(x-1)\right]_0^1$$ 1M
$$=\frac{1}{16}\ln 5+\frac{1}{8}\tan^{-1}2$$ 1A

(ii) $I_{n+1}+4I_n$
$$=\int_0^1\frac{x^{4n+4}}{x^4+4}dx+4\int_0^1\frac{x^{4n}}{x^4+4}dx$$
$$=\int_0^1\frac{x^{4n+4}+4x^{4n}}{x^4+4}dx$$
$$=\int_0^1\frac{x^{4n}(x^4+4)}{x^4+4}dx$$
$$=\int_0^1 x^{4n}dx$$ 1A
$$=\left[\frac{x^{4n+1}}{4n+1}\right]_0^1$$
$$=\frac{1}{4n+1}$$ 1

Now, $I_1 = -4I_0 + 1 = (-4)^{0+1}I_0 + (-4)^0 \sum_{k=0}^{0} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.	
So, the statement is true for $n = 0$.	
Assume that $I_{r+1} = (-4)^{r+1}I_0 + (-4)^r \sum_{k=0}^{r} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$ for some non-negative integer r .	
Then, we have	
I_{r+2}	
$= -4I_{r+1} + \frac{1}{4(r+1)+1}$	
$= (-4)^{r+2}I_0 + (-4)^{r+1} \sum_{k=0}^{r} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k + \frac{1}{4(r+1)+1}$	1M
$= (-4)^{r+2}I_0 + (-4)^{r+1} \sum_{k=0}^{r+1} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$	
Therefore, the statement is true for $n = r+1$ if it is true for $n = r$.	
By mathematical induction, $I_{n+1} = (-4)^{n+1}I_0 + (-4)^n \sum_{k=0}^{n} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.	1
Now, we have $I_{k+1} + 4I_k = \frac{1}{4k+1}$.	
So, we have $\left(\frac{-1}{4}\right)^k I_{k+1} - \left(\frac{-1}{4}\right)^{k-1} I_k = \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.	
Then, we have $\sum_{k=0}^{n}\left(\left(\frac{-1}{4}\right)^k I_{k+1} - \left(\frac{-1}{4}\right)^{k-1} I_k\right) = \sum_{k=0}^{n} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.	1M
Therefore, we have $\left(\frac{-1}{4}\right)^n I_{n+1} + 4I_0 = \sum_{k=0}^{n} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.	
Thus, we have $I_{n+1} = (-4)^{n+1}I_0 + (-4)^n \sum_{k=0}^{n} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.	1
(iii) Note that $x^{4n+4} \le x^{4n}$ for all $x \in [0,1]$.	1M
So, we have $\frac{x^{4(n+1)}}{x^4+4} \le \frac{x^{4n}}{x^4+4}$ for all $x \in [0,1]$.	
Therefore, we have $\int_0^1 \frac{x^{4n+4}}{x^4+4}\,dx \le \int_0^1 \frac{x^{4n}}{x^4+4}\,dx$.	
Thus, we have $I_{n+1} \le I_n$.	1
Since $\lvert I_{n+1}\rvert = I_{n+1} \le I_0$ and $\lim_{n\to\infty}\left(\frac{-1}{4}\right)^n = 0$,	1M accept using $\lvert I_{n+1}\rvert \le 1$ or $I_{n+1} \ge 0$
we have $\lim_{n\to\infty}\left(\frac{-1}{4}\right)^n I_{n+1} = 0$.	1
	---------(12)

(b) By (a)(ii), we have $\sum_{k=0}^{n}\frac{1}{4k+1}\left(\frac{-1}{4}\right)^k=\left(\frac{-1}{4}\right)^n I_{n+1}+4I_0$.	1M for using (a)(ii)
So, we have	
$\sum_{k=0}^{\infty}\frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$	
$=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$	
$=\lim_{n\to\infty}\left(\left(\frac{-1}{4}\right)^n I_{n+1}+4I_0\right)$	
$=0+4I_0$ (by (a)(iii))	1M for using (a)(iii)
$=4I_0$	
$=\frac{1}{4}\ln 5+\frac{1}{2}\tan^{-1}2$ (by (a)(i))	1M for $4I_0$ with I_0 substituted
	---------(3)

10. (a) Let $x = a\sin\theta$. Then, we have $\frac{dx}{d\theta} = a\cos\theta$.	1M for a suitable substitution
$\int\sqrt{a^2 - x^2}dx$	
$= \int a^2\cos^2\theta \, d\theta$	1A
$= a^2 \int \frac{1+\cos 2\theta}{2} d\theta$	
$= a^2\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + \text{constant}$	
$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + \text{constant}$	1A or equivalent
	pp−1 for omitting constant
Let $x = a\cos\theta$. Then, we have $\frac{dx}{d\theta} = -a\sin\theta$.	1M for a suitable substitution
$\int\sqrt{a^2 - x^2}dx$	
$= -\int a^2\sin^2\theta \, d\theta$	1A
$= -a^2 \int \frac{1-\cos 2\theta}{2} d\theta$	
$= -a^2\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right) + \text{constant}$	
$= \frac{-a^2}{2}\cos^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + \text{constant}$	1A or equivalent
	pp−1 for omitting constant
	--------(3)
(b) Putting $y = mx + c$ in $y = \frac{b}{a}\sqrt{a^2 - x^2}$, we have	
$a^2(mx+c)^2 = b^2(a^2 - x^2)$	1M
$b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) - a^2b^2 = 0$	
$(a^2m^2 + b^2)x^2 + 2a^2cmx + a^2(c^2 - b^2) = 0$	1A or equivalent
The straight line $y = mx + c$ is a tangent to E	
$\Leftrightarrow (2a^2cm)^2 - 4a^2(a^2m^2 + b^2)(c^2 - b^2) = 0$ and	1M for discriminant = 0
the x-coordinate and the y-coordinate of the point of contact are positive.	
$\Leftrightarrow a^2c^2m^2 - (a^2m^2 + b^2)(c^2 - b^2) = 0$, $\frac{-a^2cm}{a^2m^2 + b^2} > 0$ and $\frac{-a^2cm^2}{a^2m^2 + b^2} + c > 0$	
$\Leftrightarrow a^2c^2m^2 - b^2c^2 + b^4 - a^2c^2m^2 + a^2b^2m^2 = 0$, $cm < 0$ and $\frac{b^2c}{a^2m^2 + b^2} > 0$	
$\Leftrightarrow b^2c^2 = b^4 + a^2b^2m^2$, $cm < 0$ and $c > 0$	
$\Leftrightarrow m < 0$, $c^2 = a^2m^2 + b^2$ and $c > 0$	
$\Leftrightarrow m < 0$ and $c = \sqrt{a^2m^2 + b^2}$	1

Solution	Marks
Since $y=\frac{b}{a}\sqrt{a^2-x^2}$, we have $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-bx}{a\sqrt{a^2-x^2}}$.	1M can be absorbed
Let $P(x_0\ ,\ \frac{b}{a}\sqrt{a^2-{x_0}^2})$ be a point on E .	
The equation of the tangent to E at P is	
$y-\frac{b}{a}\sqrt{a^2-{x_0}^2}=\frac{-bx_0(x-x_0)}{a\sqrt{a^2-{x_0}^2}}$	
$y=\left(\frac{-bx_0}{a\sqrt{a^2-{x_0}^2}}\right)x+\frac{ab}{\sqrt{a^2-{x_0}^2}}$	1M
"$\Rightarrow$" The straight line $y=mx+c$ is a tangent to E	
$\Rightarrow m=\frac{-bx_0}{a\sqrt{a^2-{x_0}^2}}$ and $c=\frac{ab}{\sqrt{a^2-{x_0}^2}}$, where $0<x_0<a$	
$\Rightarrow m<0$ and $c=\frac{ab}{\sqrt{a^2-\frac{a^4m^2}{a^2m^2+b^2}}}$	
$\Rightarrow m<0$ and $c=\sqrt{a^2m^2+b^2}$	1
"$\Leftarrow$" If $m<0$ and $c=\sqrt{a^2m^2+b^2}$,	
then put $x_0=\frac{-a^2m}{\sqrt{a^2m^2+b^2}}$. Note that $0<x_0<a$.	
So, we have $m=\frac{-bx_0}{a\sqrt{a^2-{x_0}^2}}$ and $c=\frac{ab}{\sqrt{a^2-{x_0}^2}}$.	1
Therefore, the straight line $y=mx+c$ is a tangent to E .	

-------(4)

(c) (i) Putting $y=\sqrt{27-3x^2}$ in $y=\sqrt{9-\frac{x^2}{3}}$ and simplifying, we have

$\frac{8x^2}{3}=18$. So, we have $x^2=\frac{27}{4}$. — 1M for a quadratic equation in x or y

Since $x>0$, we have $x=\frac{3\sqrt{3}}{2}$.

Hence, we have $y=\sqrt{27-\frac{81}{4}}=\frac{3\sqrt{3}}{2}$.

Thus, the point of intersection of E_1 and E_2 is $\left(\frac{3\sqrt{3}}{2},\frac{3\sqrt{3}}{2}\right)$. — 1A

(ii) (1) Let the equation of L be $y=mx+c$.

By (b), we have $c=\sqrt{9m^2+27}$ and $c=\sqrt{27m^2+9}$. — 1M for using (b)

So, we have $27m^2+9=9m^2+27$.

Therefore, we have $m^2=1$.

Since $m<0$ (by (b)), we have $m=-1$.

Hence, we have $c=6$.

Thus, the equation of L is $y=-x+6$. — 1A

(2) Note that L touches E_1 and E_2 at $\left(\frac{3}{2}, \frac{9}{2}\right)$ and $\left(\frac{9}{2}, \frac{3}{2}\right)$ respectively.

The required area

$$= \int_{\frac{3}{2}}^{\frac{3\sqrt{3}}{2}} \left(6 - x - \sqrt{27 - 3x^2}\right) dx + \int_{\frac{3\sqrt{3}}{2}}^{\frac{9}{2}} \left(6 - x - \sqrt{9 - \frac{x^2}{3}}\right) dx$$

1M for lower and upper limits + 1A

$$= \int_{\frac{3}{2}}^{\frac{9}{2}} (6 - x)\, dx - \int_{\frac{3}{2}}^{\frac{3\sqrt{3}}{2}} \sqrt{27 - 3x^2}\, dx - \int_{\frac{3\sqrt{3}}{2}}^{\frac{9}{2}} \sqrt{9 - \frac{x^2}{3}}\, dx$$

$$\int_{\frac{3}{2}}^{\frac{9}{2}} (6 - x)\, dx$$

$$= \left[6x - \frac{x^2}{2}\right]_{\frac{3}{2}}^{\frac{9}{2}}$$

$$= 9$$

$$\int_{\frac{3}{2}}^{\frac{3\sqrt{3}}{2}} \sqrt{27 - 3x^2}\, dx$$

$$= \sqrt{3} \int_{\frac{3}{2}}^{\frac{3\sqrt{3}}{2}} \sqrt{9 - x^2}\, dx$$

$$= \sqrt{3} \left[\frac{9}{2}\sin^{-1}\frac{x}{3} + \frac{x}{2}\sqrt{9 - x^2}\right]_{\frac{3}{2}}^{\frac{3\sqrt{3}}{2}}$$

1M for using (a) — either one

$$= \frac{3\sqrt{3}\pi}{4}$$

$$\int_{\frac{3\sqrt{3}}{2}}^{\frac{9}{2}} \sqrt{9 - \frac{x^2}{3}}\, dx$$

$$= \frac{1}{\sqrt{3}} \int_{\frac{3\sqrt{3}}{2}}^{\frac{9}{2}} \sqrt{27 - x^2}\, dx$$

$$= \frac{1}{\sqrt{3}} \left[\frac{27}{2}\sin^{-1}\frac{x}{3\sqrt{3}} + \frac{x}{2}\sqrt{27 - x^2}\right]_{\frac{3\sqrt{3}}{2}}^{\frac{9}{2}}$$

$$= \frac{3\sqrt{3}\pi}{4}$$

The required area

$$= 9 - \frac{3\sqrt{3}\pi}{2}$$

1A

-------(8)

11. (a) Let $f(x) = \tan^{-1} x$ for all $x \in [a, b]$.	1M can be absorbed
Then, we have $f'(x) = \dfrac{1}{1+x^2}$.	1A
By Mean Value Theorem, there exists $\xi \in (a,b)$ such that	
$f'(\xi) = \dfrac{\tan^{-1} b - \tan^{-1} a}{b-a}$.	1M
So, we have $\dfrac{1}{1+\xi^2} = \dfrac{\tan^{-1} b - \tan^{-1} a}{b-a}$.	
Since $a < \xi < b$, we have $\dfrac{1}{1+b^2} < \dfrac{1}{1+\xi^2} < \dfrac{1}{1+a^2}$.	
Thus, we have $\dfrac{1}{1+b^2} < \dfrac{\tan^{-1} b - \tan^{-1} a}{b-a} < \dfrac{1}{1+a^2}$.	1

Let $f(x) = \dfrac{1}{1+x^2}$ for all $x \in [a, b]$.	1M can be absorbed
Note that f is continuous and strictly decreasing on $[a, b]$.	
So, we have $\int_a^b f(b)\,dx < \int_a^b f(x)\,dx < \int_a^b f(a)\,dx$.	1M
Note that $\int_a^b f(x)\,dx = \int_a^b \dfrac{dx}{1+x^2} = \tan^{-1} b - \tan^{-1} a$.	1A
Also note that $\int_a^b f(b)\,dx = \dfrac{b-a}{1+b^2}$ and $\int_a^b f(a)\,dx = \dfrac{b-a}{1+a^2}$.	
Therefore, we have $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$.	
Thus, we have $\dfrac{1}{1+b^2} < \dfrac{\tan^{-1} b - \tan^{-1} a}{b-a} < \dfrac{1}{1+a^2}$.	1

--------(4)

(b) (i) Let $\tan \dfrac{3\pi}{8} = t$.	
Since $\tan \dfrac{3\pi}{4} = -1$, we have $\tan\left((2)\left(\dfrac{3\pi}{8}\right)\right) = -1$.	
Then, we have $\dfrac{2t}{1-t^2} = -1$.	1M for using $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
So, we have $t^2 - 2t - 1 = 0$.	1A
Hence, we have $t = \dfrac{2+\sqrt{8}}{2}$ or $t = \dfrac{2-\sqrt{8}}{2}$ (rejected since $t > 0$) .	
Therefore, we have $t = 1 + \sqrt{2}$.	
Thus, we have $\tan \dfrac{3\pi}{8} = 1 + \sqrt{2}$.	1

Let $\tan\theta = 1+\sqrt{2}$, where $0<\theta<\frac{\pi}{2}$.	
Then, we have	
$\tan 2\theta$	
$=\frac{2\tan\theta}{1-\tan^2\theta}$	
$=\frac{2(1+\sqrt{2})}{1-(1+\sqrt{2})^2}$	1M for using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
$=\frac{2(1+\sqrt{2})}{1-(1+2\sqrt{2}+2)}$	
$=\frac{2(1+\sqrt{2})}{-2(1+\sqrt{2})}$	
$=-1$	1A
Since $0<\theta<\frac{\pi}{2}$, we have $0<2\theta<\pi$.	
So, we have $2\theta=\frac{3\pi}{4}$.	
Hence, we have $\theta=\frac{3\pi}{8}$.	
Thus, we have $\tan\frac{3\pi}{8}=1+\sqrt{2}$.	1

(ii) $\tan\frac{\pi}{24}$

$=\tan\left(\frac{3\pi}{8}-\frac{\pi}{3}\right)$ 1M

$=\frac{\tan\frac{3\pi}{8}-\tan\frac{\pi}{3}}{1+\left(\tan\frac{3\pi}{8}\right)\left(\tan\frac{\pi}{3}\right)}$ 1M

$=\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}}$

$=\left(\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}}\right)\left(\frac{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}\right)$ 1M

$=\frac{(1+\sqrt{2}-\sqrt{3})(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)}{\sqrt{6}+\sqrt{2}-\sqrt{3}-2+3\sqrt{2}+\sqrt{6}-3-2\sqrt{3}+6+2\sqrt{3}-3\sqrt{2}-2\sqrt{6}}$

$=\frac{(1+\sqrt{2}-\sqrt{3})(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)}{1+\sqrt{2}-\sqrt{3}}$

$=\sqrt{6}+\sqrt{2}-\sqrt{3}-2$ 1

$\tan\frac{\pi}{24}$	
$= \tan\left(\frac{3\pi}{8} - \frac{\pi}{3}\right)$	1M
$= \dfrac{\tan\frac{3\pi}{8} - \tan\frac{\pi}{3}}{1 + \left(\tan\frac{3\pi}{8}\right)\left(\tan\frac{\pi}{3}\right)}$	1M
$= \dfrac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}}$	
$= \left(\dfrac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}}\right)\left(\dfrac{1+\sqrt{3}-\sqrt{6}}{1+\sqrt{3}-\sqrt{6}}\right)$	1M
$= \dfrac{4\sqrt{2}-2\sqrt{3}-2}{2\sqrt{3}-2}$	
$= \dfrac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$	
$= \dfrac{(2\sqrt{2}-\sqrt{3}-1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$	
$= \dfrac{2\sqrt{6}+2\sqrt{2}-2\sqrt{3}-4}{2}$	
$= \sqrt{6}+\sqrt{2}-\sqrt{3}-2$	1
	--------(7)
(c) Note that $\sqrt{6}+\sqrt{2}-\sqrt{3}-2 = (\sqrt{2}-1)(\sqrt{3}-\sqrt{2}) > 0$.	
Putting $a = 0$ and $b = \sqrt{6}+\sqrt{2}-\sqrt{3}-2$ in (a), we have	1A
$\dfrac{1}{1+(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)^2} < \dfrac{\tan^{-1}(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)}{\sqrt{6}+\sqrt{2}-\sqrt{3}-2} < 1$.	
By (b)(ii), we have $\dfrac{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}{1+(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)^2} < \dfrac{\pi}{24} < \sqrt{6}+\sqrt{2}-\sqrt{3}-2$.	1M for using (b)(ii)
Let $\alpha = \sqrt{6}+\sqrt{2}-\sqrt{3}-2$. Therefore, we have	
$\dfrac{1}{\alpha} = \left(\dfrac{1}{\sqrt{2}-1}\right)\left(\dfrac{1}{\sqrt{3}-\sqrt{2}}\right) = (\sqrt{2}+1)(\sqrt{3}+\sqrt{2}) = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$.	1M for using $\dfrac{1}{\sqrt{m}-\sqrt{n}} = \dfrac{\sqrt{m}+\sqrt{n}}{m-n}$
So, we have $\dfrac{1}{\alpha}+\alpha = \sqrt{6}+\sqrt{3}+\sqrt{2}+2+\sqrt{6}+\sqrt{2}-\sqrt{3}-2 = 2(\sqrt{6}+\sqrt{2})$.	
Then, we have $\dfrac{1}{\frac{1}{\alpha}+\alpha} = \dfrac{1}{2(\sqrt{6}+\sqrt{2})} = \dfrac{\sqrt{6}-\sqrt{2}}{2(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})} = \dfrac{\sqrt{6}-\sqrt{2}}{8}$.	
Hence, we have $\dfrac{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}{1+(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)^2} = \dfrac{\alpha}{1+\alpha^2} = \dfrac{1}{\frac{1}{\alpha}+\alpha} = \dfrac{\sqrt{6}-\sqrt{2}}{8}$.	
Thus, we have $3(\sqrt{6}-\sqrt{2}) < \pi < 24(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)$.	1

Note that $\sqrt{6}+\sqrt{2}-\sqrt{3}-2=(\sqrt{2}-1)(\sqrt{3}-\sqrt{2})>0$.	
Putting $a=0$ and $b=\sqrt{6}+\sqrt{2}-\sqrt{3}-2$ in (a), we have	1A
$\frac{1}{1+(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)^2}<\frac{\tan^{-1}(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)}{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}<1$.	
By (b)(ii), we have $\frac{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}{1+(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)^2}<\frac{\pi}{24}<\sqrt{6}+\sqrt{2}-\sqrt{3}-2$.	1M for using (b)(ii)
$\frac{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}{1+(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)^2}$	
$=\frac{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}{16+8\sqrt{3}-10\sqrt{2}-6\sqrt{6}}$	
$=\left(\frac{\sqrt{6}+\sqrt{2}-\sqrt{3}-2}{16+8\sqrt{3}-10\sqrt{2}-6\sqrt{6}}\right)\left(\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}\right)$	1M for attempting to multiply $(\sqrt{6}-\sqrt{2})$
$=\frac{(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)(\sqrt{6}-\sqrt{2})}{8\sqrt{6}+8\sqrt{2}-8\sqrt{3}-16}$	
$=\frac{(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)(\sqrt{6}-\sqrt{2})}{8(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)}$	
$=\frac{\sqrt{6}-\sqrt{2}}{8}$	
Thus, we have $3(\sqrt{6}-\sqrt{2})<\pi<24(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)$.	1

----------(4)