2007-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2007

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(E) answer book.
3. Answer any FOUR questions in Section B, using the AL(A) answer book.
4. Unless otherwise specified, all working must be clearly shown.

Not to be taken away before the end of the examination session

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

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SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. For any positive integer n, denote the coefficient of x^k in the expansion of $(1+x)^n$ by C_k^n.

(a) Prove that

(i) $n(1+x)^{n-1} = C_1^n + 2C_2^n x + 3C_3^n x^2 + \cdots + nC_n^n x^{n-1}$,

(ii) $\dfrac{(1+x)^{n+1}}{n+1} = \dfrac{1}{n+1} + C_0^n x + \dfrac{C_1^n}{2}x^2 + \dfrac{C_2^n}{3}x^3 + \cdots + \dfrac{C_n^n}{n+1}x^{n+1}$.

(b) Using $(1+x)^{n-1}(1+x)^{n+1} = (1+x)^{2n}$, prove that

$$\frac{(C_1^n)^2}{n} + \frac{2(C_2^n)^2}{n-1} + \frac{3(C_3^n)^2}{n-2} + \cdots + \frac{n(C_n^n)^2}{1} = \frac{n(2n)!}{(n+1)(n!)^2}.$$

(7 marks)

2. Let $f(x)$ be a polynomial with real coefficients. When $f(x)$ is divided by $(x-1)(x-3)$, the remainder is $-2x+5$.

(a) Find $f(1)$.

(b) When $f(x)$ is divided by $(x-1)(x-2)$, the remainder is $kx+8$, where k is a real constant.

(i) Find k.

(ii) Find the remainder when $(f(x))^{2007}$ is divided by $x-2$.

(6 marks)

3. (a) (i) Resolve $\dfrac{1}{x(x+1)(x-1)}$ into partial fractions.

(ii) Using differentiation, or otherwise, resolve $\dfrac{3x^2-1}{x^2(x+1)^2(x-1)^2}$ into partial fractions.

(b) Evaluate $\displaystyle\sum_{k=2}^{\infty} \frac{3k^2-1}{k^2(k+1)^2(k-1)^2}$.

(7 marks)

4. Let T be the rotation in the Cartesian plane anticlockwise about the origin by an angle θ, where $0<\theta<2\pi$. It is given that $P_1, P_2, P_3, \ldots$ are the points in the Cartesian plane, where $P_1 = (-5, 12)$, $P_2 = (-12, -5)$ and T transforms P_k to P_{k+1} for each positive integer k.

(a) Find θ.

(b) Let A be the matrix representing T. Find A^{2007}.

(c) Write down the coordinates of P_n for all positive integers n.

(7 marks)

5. Let P be a non-singular 2×2 real matrix and $Q = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where α and β are two distinct real numbers. Define $M = P^{-1}QP$ and denote the 2×2 identity matrix by I.

(a) Find real numbers λ and μ, in terms of α and β, such that $M^2 = \lambda M + \mu I$.

(b) Prove that $\det(M^2 + \alpha\beta I) = \alpha\beta(\alpha+\beta)^2$.

(6 marks)

6. (a) Let r be a positive real number. Using $r^{p+q} - r^p - r^q + 1 = (r^p - 1)(r^q - 1)$, or otherwise, prove that $r^{p+q} - r^p - r^q + 1 \ge 0$ for any non-negative integers p and q.

(b) For each positive integer n, let a_n be the nth term of a geometric sequence of positive real numbers.

(i) Using (a), or otherwise, prove that $a_1 + a_n \ge a_k + a_{n-k+1}$ for all $k = 1, 2, \ldots, n$.

(ii) Using (b)(i) and A.M. $\ge$ G.M., or otherwise, prove that

$$\frac{1}{2}(a_1 + a_n) \ge \frac{1}{n}\sum_{k=1}^{n} a_k \ge \sqrt{a_1 a_n}.$$

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(A) answer book.

7. (a) Consider the system of linear equations in x, y, z

$$(E):\begin{cases} x - 3y = 1 \\ x + 5y + az = b \\ 2x + ay - z = 2 \end{cases}, \text{ where } a, b \in \mathbf{R} .$$

(i) Find the range of values of a for which (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) Suppose that $a = -2$. Find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b.

(8 marks)

(b) Is the system of linear equations

$$\begin{cases} x - 3y = 1 \\ x + 5y + z = 16 \\ 2x + y - z = 2 \\ x - y - z = 3 \end{cases}$$

consistent? Explain your answer.

(3 marks)

(c) Solve the system of linear equations

$$\begin{cases} x - 3y = 1 \\ x + 5y - 2z = 1 \\ 2x - 2y - z = 2 \\ x - y - z = 3 \end{cases}.$$

(4 marks)

8. (a) Let $p, q, r \in \mathbf{R}$ with $pr \neq 0$. Prove that the equation $x^4 + px^3 + qx^2 + rx + \frac{r^2}{p^2} = 0$ can be written as $\left(x + \frac{r}{px}\right)^2 + p\left(x + \frac{r}{px}\right) + \left(q - \frac{2r}{p}\right) = 0$.

(2 marks)

(b) Consider the equation

$$y^4 + y^2 - 4y - 3 = 0 \qquad \ldots\ldots\ldots\ldots\ldots (*) .$$

When $y = x + h$, (*) can be written as $x^4 + Px^3 + Qx^2 + Rx + S = 0$.

(i) Express P, Q, R and S in terms of h.

(ii) Prove that $P^2 S = R^2$ if and only if $8h^3 + 13h^2 - 4h + 4 = 0$.

(6 marks)

(c) Using (a) and (b), solve (*) in (b).

(7 marks)

9. (a) Let α be a positive real number.
By differentiating $f(t) = \frac{(t+1)^{\alpha+1}}{t^\alpha}$, prove that $\frac{(t+1)^{\alpha+1}}{t^\alpha} \geq \frac{(\alpha+1)^{\alpha+1}}{\alpha^\alpha}$ for all $t > 0$.

(5 marks)

(b) Let α_1 and α_2 be positive real numbers.

(i) By putting $\alpha = \frac{\alpha_1}{\alpha_2}$ in (a), prove that $\frac{(t+1)^{\alpha_1+\alpha_2}}{t^{\alpha_1}} \geq \frac{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}}$ for all $t > 0$.

(ii) Using (b)(i), prove that $\left(\frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}\right)^{\alpha_1+\alpha_2} \geq \left(\frac{\beta_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{\beta_2}{\alpha_2}\right)^{\alpha_2}$ for any positive real numbers β_1 and β_2.

(5 marks)

(c) Using mathematical induction, prove that

$$\left(\frac{y_1+y_2+\cdots+y_n}{x_1+x_2+\cdots+x_n}\right)^{x_1+x_2+\cdots+x_n} \geq \left(\frac{y_1}{x_1}\right)^{x_1}\left(\frac{y_2}{x_2}\right)^{x_2}\cdots\left(\frac{y_n}{x_n}\right)^{x_n}$$

for any positive real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$.

(5 marks)

10. Let a_1 and a_2 be real numbers. For each $n = 1, 2, 3, \ldots$, define $a_{n+2} = \frac{5}{7}a_{n+1} + \frac{2}{7}a_n$.

(a) Prove that $a_{n+4} - a_{n+2} = \frac{4}{49}(a_{n+2} - a_n)$.

(3 marks)

(b) Suppose that $a_2 \geq a_1$.

(i) Prove that

(1) $a_{2n+1} \geq a_{2n-1}$,

(2) $a_{2n+2} \leq a_{2n}$,

(3) $a_{2n} \geq a_{2n-1}$.

(ii) Prove that $\lim_{n\to\infty} a_{2n-1}$ and $\lim_{n\to\infty} a_{2n}$ both exist.

(9 marks)

(c) Suppose that $a_2 < a_1$. Do $\lim_{n\to\infty} a_{2n-1}$ and $\lim_{n\to\infty} a_{2n}$ exist? Explain your answer.

(3 marks)

11\. (a) Let θ be a real number such that $\cos 7\theta \neq 0$.

(i) Prove that $i\tan 7\theta = \frac{(1+i\tan\theta)^7 - (1-i\tan\theta)^7}{(1+i\tan\theta)^7 + (1-i\tan\theta)^7}$.

(ii) Using (a)(i), or otherwise, prove that

$$\tan 7\theta = \frac{\tan^7\theta - 21\tan^5\theta + 35\tan^3\theta - 7\tan\theta}{7\tan^6\theta - 35\tan^4\theta + 21\tan^2\theta - 1} .$$

(6 marks)

(b) Using (a)(ii), prove that the roots of the equation $x^6 - 21x^4 + 35x^2 - 7 = 0$ are $\tan\frac{\pi}{7}, \tan\frac{2\pi}{7}, \ldots, \tan\frac{6\pi}{7}$.

(4 marks)

(c) Using (b) and relations between coefficients and roots, evaluate

(i) $\left(\tan^2\frac{\pi}{7}\right)\left(\tan^2\frac{2\pi}{7}\right)\left(\tan^2\frac{3\pi}{7}\right)$,

(ii) $\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7}$.

(5 marks)

END OF PAPER

2007-AL
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PAPER 2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
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PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(E) answer book.

3. Answer any FOUR questions in Section B, using the AL(A) answer book.

4. Unless otherwise specified, all working must be clearly shown.

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$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

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$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Prove that $\lim_{x \to 0^+} x \ln x = 0$.

   (b) Let k be a real constant and $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \sin x + \cos 2x + k & \text{when } x \le 0 , \\ x^2 \ln x & \text{when } x > 0 . \end{cases}$$

   It is given that $f(x)$ is continuous at $x = 0$.

   (i) Find k .

   (ii) Is $f(x)$ differentiable at $x = 0$? Explain your answer.

(6 marks)

2. Define $f(x) = \dfrac{1}{\sqrt{4x - x^2}}$ for all $x \in (0, 4)$.

   (a) Prove that $(4x - x^2) f'(x) = (x-2) f(x)$.

   Hence prove that $(4x - x^2) f^{(n+1)}(x) = (2n+1)(x-2) f^{(n)}(x) + n^2 f^{(n-1)}(x)$ for all positive integers n , where $f^{(0)} = f$.

   (b) Using (a), or otherwise, evaluate $f^{(7)}(2)$ and $f^{(8)}(2)$.

(6 marks)

3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = |x-1| - |x+1|$.

   (a) Sketch the graph of $y = f(x)$.

   (b) Is f a surjective function? Explain your answer.

   (c) Let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = f(x-1) - f(x+1) + 1$.

   (i) Prove that g is an even function.

   (ii) Sketch the graph of $y = g(x)$.

(7 marks)

4. (a) Using integration by parts, find $\int e^x \sin x \, dx$.

(b) Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{\frac{k\pi}{n}} \sin\frac{k\pi}{n}$.

(7 marks)

5. (a) Find $\int \left(\frac{(x-2)(x-5)}{x}\right)^2 dx$.

(b) Let D be the region bounded by the curve $y=\frac{x(x-3)}{x+2}$ and the x-axis. Find the volume of the solid of revolution generated by revolving D about the x-axis.

(7 marks)

6. Consider the curve $\Gamma:\begin{cases} x=t^2+1 \\ y=2t \end{cases}$, where $t\in\mathbf{R}$.

(a) Let $A(a^2+1, 2a)$ and $B(b^2+1, 2b)$ be two distinct points on Γ .

(i) Find the equation of the normal to Γ at A .

(ii) Prove that AB is normal to Γ at A if and only if $a^2+ab+2=0$.

(b) It is given that $P(10,-6)$ is a point on Γ . Find two points on Γ , other than P , at which the normals to Γ intersect at P .

(7 marks)

SECTION B (60 marks)

Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(A) answer book.

7. Let $f(x)=\frac{(x+15)(x+1)^2}{(x-6)^2}$ $(x\neq 6)$.

(a) Find $f'(x)$ and $f''(x)$.

(3 marks)

(b) Solve each of the following inequalities:

(i) $f'(x)>0$,

(ii) $f''(x)>0$.

(2 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y=f(x)$.

(4 marks)

(d) Find the asymptote(s) of the graph of $y=f(x)$.

(3 marks)

(e) Sketch the graph of $y=f(x)$.

(3 marks)

8. Let $f:\mathbf{R}\to\mathbf{R}$ and $g:\mathbf{R}\to\mathbf{R}$ be continuous functions such that

$$f(x)=e^{8x}+3e^{8x}\int_0^x e^{-8t}g(t)\,dt \quad \text{and} \quad g(x)=e^{2x}-3e^{2x}\int_0^x e^{-2t}f(t)\,dt$$

for all $x\in\mathbf{R}$.

(a) Find $f(0)$ and $g(0)$.

(1 mark)

(b) Prove that $f'(x)=8f(x)+3g(x)$ for all $x\in\mathbf{R}$.

Hence find $f'(0)$.

(3 marks)

(c) Prove that $f''(x)-10f'(x)+25f(x)=0$ for all $x\in\mathbf{R}$.

(4 marks)

(d) By considering the second derivative of $e^{-5x}f(x)$, find $f(x)$.

(5 marks)

(e) Find $g(x)$.

(2 marks)

Go on to the next page

9. (a) For each non-negative integer n, define $I_n = \int_0^1 \frac{x^{4n}}{x^4+4}\,dx$.

(i) Verify that $\frac{8}{x^4+4} = \frac{x+2}{x^2+2x+2} - \frac{x-2}{x^2-2x+2}$.

Hence evaluate I_0 .

(ii) Prove that $I_{n+1} + 4I_n = \frac{1}{4n+1}$.

Hence prove that $I_{n+1} = (-4)^{n+1} I_0 + (-4)^n \sum_{k=0}^{n} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.

(iii) Prove that $I_{n+1} \le I_n$.

Hence, or otherwise, prove that $\lim_{n\to\infty} \left(\frac{-1}{4}\right)^n I_{n+1} = 0$.

(12 marks)

(b) Using (a), evaluate $\sum_{k=0}^{\infty} \frac{1}{4k+1}\left(\frac{-1}{4}\right)^k$.

(3 marks)

10. (a) Using integration by substitution, find $\int \sqrt{a^2-x^2}\,dx$, where a is a positive constant.

(3 marks)

(b) Let a and b be positive constants. Consider the curve $E:\ y = \frac{b}{a}\sqrt{a^2-x^2}$, where $0 < x < a$. Prove that the straight line $y = mx + c$ is a tangent to E if and only if $m < 0$ and $c = \sqrt{a^2m^2+b^2}$.

(4 marks)

(c) Consider the curve $E_1:\ y = \sqrt{27-3x^2}$, where $0 < x < 3$, and the curve $E_2:\ y = \sqrt{9-\frac{x^2}{3}}$, where $0 < x < 3\sqrt{3}$.

(i) Find the point of intersection of E_1 and E_2 .

(ii) Let L be the common tangent to E_1 and E_2 .

(1) Find the equation of L .

(2) Find the area of the region bounded by E_1 , E_2 and L .

(8 marks)

11. (a) Suppose that $0 \le a < b$. Using Mean Value Theorem, or otherwise, prove that

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2} .$$

(4 marks)

(b) (i) Prove that $\tan\frac{3\pi}{8} = 1+\sqrt{2}$.

(ii) Using (b)(i), prove that $\tan\frac{\pi}{24} = \sqrt{6}+\sqrt{2}-\sqrt{3}-2$.

(7 marks)

(c) Using (a) and (b)(ii), prove that $3(\sqrt{6}-\sqrt{2}) < \pi < 24(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)$.

(4 marks)

END OF PAPER