香港考試及評核局
HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

2006年香港高級程度會考
HONG KONG ADVANCED LEVEL EXAMINATION 2006

純粹數學 高級程度 試卷一
PURE MATHEMATICS A-LEVEL PAPER 1

本評卷參考乃香港考試及評核局專爲今年本科考試而編寫，供閱卷員參考之用。閱卷員在完成閱卷工作後，若將本評卷參考提供其任教會考班的本科同事參閱，本局不表反對，但須切記，在任何情況下均不得容許本評卷參考落入學生手中。學生若索閱或求取此等文件，閱卷員/教師應嚴詞拒絕，因學生極可能將評卷參考視爲標準答案，以致但知硬背死記，活剝生吞。這種落伍的學習態度，既不符現代教育原則，亦有違考試着重理解能力與運用技巧之旨。因此，本局籲請各閱卷員/教師通力合作，堅守上述原則。

This marking scheme has been prepared by the Hong Kong Examinations and Assessment Authority for markers' reference. The Authority has no objection to markers sharing it, after the completion of marking, with colleagues who are teaching the subject. However, under no circumstances should it be given to students because they are likely to regard it as a set of model answers. Markers/teachers should therefore firmly resist students' requests for access to this document. Our examinations emphasise the testing of understanding, the practical application of knowledge and the use of processing skills. Hence the use of model answers, or anything else which encourages rote memorisation, should be considered outmoded and pedagogically unsound. The Authority is counting on the co-operation of markers/teachers in this regard.

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Advanced Level Pure Mathematics

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. Marks may be deducted for poor presentation (*pp*). The symbol (*pp-1*) should be used to denote 1 mark deducted for *pp*. At most deducted 1 mark from Section A and 1 mark from Section B for *pp*. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

7. In the marking scheme, 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

Solution	Marks
1. Note that $\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)-\left(\frac{3-\sqrt{17}}{2}\right)\right]=\frac{\sqrt{17}}{\sqrt{17}}=1=a_1$. So, the statement is true for $n=1$. Note also that $\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^2-\left(\frac{3-\sqrt{17}}{2}\right)^2\right]=\frac{3\sqrt{17}}{\sqrt{17}}=3=a_2$. So, the statement is true for $n=2$.	1 pp−1 for writing '$n=1$ and $n=2$ are true'
Assume that $\begin{cases} a_k=\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^k-\left(\frac{3-\sqrt{17}}{2}\right)^k\right] \\ a_{k+1}=\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^{k+1}-\left(\frac{3-\sqrt{17}}{2}\right)^{k+1}\right] \end{cases}$ for some positive integer k. Then,	1M pp−1 for writing '$n=k$ and $n=k+1$ are true'
a_{k+2} $=3a_{k+1}+2a_k$	
$=\frac{3}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^{k+1}-\left(\frac{3-\sqrt{17}}{2}\right)^{k+1}\right]+\frac{2}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^k-\left(\frac{3-\sqrt{17}}{2}\right)^k\right]$	1M for using induction assumption
$=\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^k\left(\frac{9+3\sqrt{17}}{2}+2\right)-\left(\frac{3-\sqrt{17}}{2}\right)^k\left(\frac{9-3\sqrt{17}}{2}+2\right)\right]$	1
$=\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^k\left(\frac{13+3\sqrt{17}}{2}\right)-\left(\frac{3-\sqrt{17}}{2}\right)^k\left(\frac{13-3\sqrt{17}}{2}\right)\right]$	
$=\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^k\left(\frac{3+\sqrt{17}}{2}\right)^2-\left(\frac{3-\sqrt{17}}{2}\right)^k\left(\frac{3-\sqrt{17}}{2}\right)^2\right]$	1M for using $\left(\frac{3\pm\sqrt{17}}{2}\right)^2=\frac{13\pm3\sqrt{17}}{2}$
$=\frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^{k+2}-\left(\frac{3-\sqrt{17}}{2}\right)^{k+2}\right]$	1
Therefore, the statement is true for $n=k+2$ if it is true for $n=k$ and $n=k+1$. Thus, by mathematical induction, the statement is true for any positive integer n.	
	----------(6)

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Solution	Marks
2. (a) $(1-x^2)^n$	
$=(1+x)^n(1-x)^n$	
$=\sum_{r=0}^{n} C_r^n x^r \sum_{m=0}^{n} (-1)^m C_m^n x^m$	1A for either correct
$=\sum_{r=0}^{n}\sum_{m=0}^{n} (-1)^m C_r^n C_m^n x^{r+m}$	
The coefficient of x^n in the expansion of $(1-x^2)^n$	
$=\sum_{k=0}^{n} (-1)^k C_k^n C_{n-k}^n$	1M
$=\sum_{k=0}^{n} (-1)^k C_k^n C_k^n$	
$=\sum_{k=0}^{n} (-1)^k (C_k^n)^2$	1
(b) (i) $\sum_{k=0}^{2005} (-1)^k (C_k^{2005})^2$	
= the coefficient of x^{2005} in $(1-x^2)^{2005}$	
= the coefficient of x^{2005} in $\sum_{k=0}^{2005} (-1)^k C_k^{2005} x^{2k}$	1M for using (a)
$=0$	1A
$\sum_{k=0}^{2005} (-1)^k (C_k^{2005})^2$	
$=\left(C_0^{2005}\right)^2-\left(C_1^{2005}\right)^2+\cdots+\left(C_{2004}^{2005}\right)^2-\left(C_{2005}^{2005}\right)^2$	
$=\left(C_0^{2005}\right)^2-\left(C_0^{2005}\right)^2+\left(C_1^{2005}\right)^2-\left(C_1^{2005}\right)^2+\cdots+\left(C_{1002}^{2005}\right)^2-\left(C_{1002}^{2005}\right)^2$	1M for using $C_{n-k}^n=C_k^n$
$=0$	1A
(ii) $\sum_{k=0}^{2006} (-1)^k (C_k^{2006})^2$	
= the coefficient of x^{2006} in $(1-x^2)^{2006}$	
= the coefficient of x^{2006} in $\sum_{k=0}^{2006} (-1)^k C_k^{2006} x^{2k}$	1M for using (a)
$=-C_{1003}^{2006}$	1A
$=\dfrac{-2006!}{(1003!)^2}$	
	----------(7)

Solution	Marks
3. (a) (i) Note that $p(n)=\frac{n}{n+1}$ for all $n=0,1,2,3,4$.	
$q(n)=(n+1)p(n)-n$	
$=(n+1)\frac{n}{n+1}-n$	
$=n-n$	
$=0$	1A for all correct
Thus, we have $q(0)=q(1)=q(2)=q(3)=q(4)=0$.	
(ii) Since $q(0)=q(1)=q(2)=q(3)=q(4)=0$ and $q(x)$ is a polynomial of degree 5 with real coefficients, we have	
$q(x)=kx(x-1)(x-2)(x-3)(x-4)$ for some constant k.	1M for $x(x-1)(x-2)(x-3)(x-4)$ + 1A
$(x+1)p(x)-x=kx(x-1)(x-2)(x-3)(x-4)$	
Putting $x=-1$, we have	1M for finding k
$1=-120k$	
$k=\frac{-1}{120}$	1A
Thus, we have $q(x)=\frac{-x}{120}(x-1)(x-2)(x-3)(x-4)$.	
(b) $p(5)$	
$=\frac{1}{6}(5+q(5))$	
$=\frac{1}{6}\left(5+\frac{-5}{120}(4)(3)(2)(1)\right)$	
$=\frac{2}{3}$	1A
	----------(6)

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Solution	Marks
4. (a) Let $\frac{9x+36}{x(x+2)(x+3)}=\frac{C_1}{x}+\frac{C_2}{x+2}+\frac{C_3}{x+3}$.	1M can be absorbed
$9x+36 \equiv C_1(x+2)(x+3)+C_2x(x+3)+C_3x(x+2)$	
Putting $x=0, -2, -3$, we have $C_1=6$, $C_2=-9$, $C_3=3$.	1A for all correct
Thus, we have $\frac{9x+36}{x(x+2)(x+3)}=\frac{6}{x}-\frac{9}{x+2}+\frac{3}{x+3}$.	
(b) $\sum_{k=1}^{n}\frac{9k+36}{k(k+2)(k+3)}$	
$=\sum_{k=1}^{n}(\frac{6}{k}-\frac{9}{k+2}+\frac{3}{k+3})$	
$=6\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right)-3\sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)$	1M
$=6\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)-3\left(\frac{1}{3}-\frac{1}{n+3}\right)$	1A for either sum correct
$=8-\frac{6}{n+1}-\frac{6}{n+2}+\frac{3}{n+3}$	1A
(c) $\sum_{k=1}^{n}\frac{9k+36}{k(k+2)(k+3)}$	
$=8-\frac{6}{n+1}-\frac{6}{n+2}+\frac{3}{n+3}$	
$=8-\frac{6(n+2)(n+3)+6(n+1)(n+3)-3(n+1)(n+2)}{(n+1)(n+2)(n+3)}$	
$=8-\frac{9n^2+45n+48}{(n+1)(n+2)(n+3)}$	1M
<8 for any positive integer n	
Thus, there is no positive integer N such that $\sum_{k=1}^{N}\frac{9k+36}{k(k+2)(k+3)}\geq 8$.	1A must show reasons
$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{9k+36}{k(k+2)(k+3)}=\lim_{n\to\infty}(8-\frac{6}{n+1}-\frac{6}{n+2}+\frac{3}{n+3})=8$	1M
Note that $\sum_{k=1}^{n}\frac{9k+36}{k(k+2)(k+3)}$ is strictly increasing.	
Thus, there is no positive integer N such that $\sum_{k=1}^{N}\frac{9k+36}{k(k+2)(k+3)}\geq 8$.	1A must show reasons
	----------(7)

Solution	Marks
5. (a) $x_{n+1} - x_n$	
$= \sum_{k=1}^{n+1} \frac{1}{n+k+1} - \sum_{k=1}^{n} \frac{1}{n+k}$	
$= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$	1A (either one)
$= \frac{2n+1+2n+2-2(2n+1)}{(2n+2)(2n+1)}$	
$= \frac{1}{(2n+1)(2n+2)}$	
> 0 for all positive integers n	
Thus, the sequence $\{x_n\}$ is strictly increasing.	1
$y_{n+1} - y_n$	
$= \sum_{k=1}^{n+2} \frac{1}{n+k+1} - \sum_{k=1}^{n+1} \frac{1}{n+k}$	
$= \frac{1}{2n+3} + \frac{1}{2n+2} - \frac{1}{n+1}$	
$= \frac{2n+2+2n+3-4n-6}{(2n+3)(2n+2)}$	
$= \frac{-1}{(2n+3)(2n+2)}$	
< 0 for all positive integers n	
Thus, the sequence $\{y_n\}$ is strictly decreasing.	1
(b) Note that $y_n = x_n + \frac{1}{2n+1} > x_n$.	
By (a), we have $\frac{1}{2} = x_1 \le x_n < y_n \le y_1 = \frac{5}{6}$.	
By (a), $\{x_n\}$ is strictly increasing and bounded above by $\frac{5}{6}$	
while $\{y_n\}$ is strictly decreasing and bounded below by $\frac{1}{2}$.	
Therefore, both the sequences $\{x_n\}$ and $\{y_n\}$ are convergent.	1 + 1
Since $y_n = x_n + \frac{1}{2n+1}$ and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$, we have	
$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} \frac{1}{2n+1}$	1M
$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$	1
Thus, the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same limit.	

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Solution	Marks
x_n	
$= \sum_{k=1}^{n} \frac{1}{n+k}$	
$= \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}$	
$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}$	
$= \int_0^1 \frac{dx}{1+x}$	1A
$= \left[\ln(1+x) \right]_0^1$	
$= \ln 2$	
So, we have $\lim_{n\to\infty} x_n = \ln 2$.	1A
$\because \quad y_n = x_n + \frac{1}{2n+1}$, $\lim_{n\to\infty} x_n = \ln 2$ and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$	1M
$\therefore \quad \lim_{n\to\infty} y_n = \ln 2$	1A
Therefore, we have $\lim_{n\to\infty} x_n = \ln 2 = \lim_{n\to\infty} y_n$.	
Thus, the sequences $\{ x_n \}$ and $\{ y_n \}$ converge to the same limit.	
	----------(7)

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Solution	Marks
6. (a) (i) By Cauchy-Schwarz's inequality, we have	
$\left((a)(1)+(b)(1)+(c)(1)\right)^2 \le (a^2+b^2+c^2)(1^2+1^2+1^2)$	
$(a+b+c)^2 \le 3(a^2+b^2+c^2)$	
$(a+b+c)^2 \le 3^2$	
$a+b+c \le 3$	1
(ii) By Cauchy-Schwarz's inequality, we have	
$\left[\left(a^{\frac{3}{2}}\right)^2+\left(b^{\frac{3}{2}}\right)^2+\left(c^{\frac{3}{2}}\right)^2\right]\left[\left(a^{\frac{1}{2}}\right)^2+\left(b^{\frac{1}{2}}\right)^2+\left(c^{\frac{1}{2}}\right)^2\right] \ge \left(a^{\frac{3}{2}}a^{\frac{1}{2}}+b^{\frac{3}{2}}b^{\frac{1}{2}}+c^{\frac{3}{2}}c^{\frac{1}{2}}\right)^2$	1M
$(a^3+b^3+c^3)(a+b+c) \ge (a^2+b^2+c^2)^2$	
$a^3+b^3+c^3 \ge \dfrac{3^2}{a+b+c}$ (since $a+b+c>0$)	
$a^3+b^3+c^3 \ge \dfrac{3^2}{3}$ (by (a)(i))	
$a^3+b^3+c^3 \ge 3$	1
(b) If $P(k)$ is true, then $a^k+b^k+c^k \ge 3$.	
(i) By Cauchy-Schwarz's inequality, we have	
$\left((a^k)^2+(b^k)^2+(c^k)^2\right)\left(1^2+1^2+1^2\right) \ge \left(a^k+b^k+c^k\right)^2$	1M
$3(a^{2k}+b^{2k}+c^{2k}) \ge 3^2$	
$a^{2k}+b^{2k}+c^{2k} \ge 3$	1
Thus, $P(2k)$ is true.	
(ii) By Cauchy-Schwarz's inequality, we have	
$\left(\left(a^{\frac{2k-1}{2}}\right)^2+\left(b^{\frac{2k-1}{2}}\right)^2+\left(c^{\frac{2k-1}{2}}\right)^2\right)\left(\left(a^{\frac{1}{2}}\right)^2+\left(b^{\frac{1}{2}}\right)^2+\left(c^{\frac{1}{2}}\right)^2\right) \ge \left(a^k+b^k+c^k\right)^2$	1M
$(a^{2k-1}+b^{2k-1}+c^{2k-1})(a+b+c) \ge (a^k+b^k+c^k)^2$	
$(a^{2k-1}+b^{2k-1}+c^{2k-1})(a+b+c) \ge 3^2$	
$a^{2k-1}+b^{2k-1}+c^{2k-1} \ge \dfrac{3^2}{a+b+c}$ (since $a+b+c>0$)	
$a^{2k-1}+b^{2k-1}+c^{2k-1} \ge \dfrac{3^2}{3}$ (by (a)(i))	
$a^{2k-1}+b^{2k-1}+c^{2k-1} \ge 3$	1
Thus, $P(2k-1)$ is true.	
	----------(7)

Solution | **Marks**

7. (a) (E) has a unique solution

$\Leftrightarrow \quad \Delta \neq 0$

$$\Leftrightarrow \quad \Delta = \begin{vmatrix} 1 & a & 1 \\ 1 & 2-a & 3b-1 \\ 2 & a+1 & b+1 \end{vmatrix} \neq 0$$

1M

$\Leftrightarrow \quad (2-a)(b+1)+(a+1)+2a(3b-1)-2(2-a)-(a+1)(3b-1)-a(b+1) \neq 0$

$\Leftrightarrow \quad ab-b \neq 0$

$\Leftrightarrow \quad (a-1)b \neq 0$ — 1A

$\Leftrightarrow \quad a \neq 1$ and $b \neq 0$ — 1

The augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} 1 & a & 1 & 4 \\ 1 & 2-a & 3b-1 & 3 \\ 2 & a+1 & b+1 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & 1 & 4 \\ 0 & 2-2a & 3b-2 & -1 \\ 0 & 1-a & b-1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & 4 \\ 0 & 2a-2 & 2-3b & 1 \\ 0 & 1-a & b-1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & 1 & 4 \\ 0 & 2a-2 & 2-3b & 1 \\ 0 & 0 & b & 1 \end{array}\right)$$

1A

(E) has unique solution

$\Leftrightarrow \quad b \neq 0$ and $2a-2 \neq 0$ — 1M

$\Leftrightarrow \quad a \neq 1$ and $b \neq 0$ — 1

In this case,

$$x = \frac{\begin{vmatrix} 4 & a & 1 \\ 3 & 2-a & 3b-1 \\ 7 & a+1 & b+1 \end{vmatrix}}{(a-1)b} = \frac{2ab-4b+1}{(a-1)b}$$

1M for Cramer's Rule

$$y = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 1 & 3 & 3b-1 \\ 2 & 7 & b+1 \end{vmatrix}}{(a-1)b} = \frac{2b-1}{(a-1)b}$$

$$z = \frac{\begin{vmatrix} 1 & a & 4 \\ 1 & 2-a & 3 \\ 2 & a+1 & 7 \end{vmatrix}}{(a-1)b} = \frac{1}{b}$$

1A + 1A (1A for any one, 1A for all)

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Solution	Marks
In this case, the augmented matrix of (E)	
$\sim\left(\begin{array}{ccc\|c} 1 & a & 1 & 4 \\ 0 & 2a-2 & 2-3b & 1 \\ 0 & 0 & 1 & \frac{1}{b} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & a & 1 & 4 \\ 0 & 1 & 0 & \frac{2b-1}{b(a-1)} \\ 0 & 0 & 1 & \frac{1}{b} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{2ab-4b+1}{(a-1)b} \\ 0 & 1 & 0 & \frac{2b-1}{(a-1)b} \\ 0 & 0 & 1 & \frac{1}{b} \end{array}\right)$	1M
$\therefore \quad x=\frac{2ab-4b+1}{(a-1)b}$, $y=\frac{2b-1}{(a-1)b}$, $z=\frac{1}{b}$	1A + 1A (1A for any one, 1A for all)
	----------(6)
(b) (i) When $a=1$, the augmented matrix of (E)	
$\sim\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 2-3b & 1 \\ 0 & 0 & b & 1 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 2-3b & 1 \\ 0 & 0 & 4b-2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4b-2 & 0 \end{array}\right)$	
(E) is consistent when $4b-2=0$.	1M
So, (E) is consistent when $b=\frac{1}{2}$.	1A
Therefore, the solution set $\{(2-t,\ t,\ 2)\ :\ t \in \mathbf{R}\}$.	1A or equivalent
(ii) Putting $x=2-t$, $y=t$ and $z=2$ in $x^2-2y^2-z=14$, we have	1M
$(2-t)^2-2t^2-2=14$	
$4-4t+t^2-2t^2-2=14$	
$t^2+4t+12=0$	1A or equivalent
Note that $4^2-4(1)(12)=-32<0$.	1M for considering discriminant
Therefore, there is no real solution of $\begin{cases} x+y+z=4 \\ 2x+2y+z=6 \\ 4x+4y+3z=14 \end{cases}$	
which satisfies $x^2-2y^2-z=14$.	1A f.t.
	----------(7)
(c) When $b=0$, the augmented matrix of (E)	
$\sim\left(\begin{array}{ccc\|c} 1 & a & 1 & 4 \\ 0 & 2a-2 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$	1M
Therefore, (E) is inconsistent for $b=0$.	1A f.t.
	----------(2)

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Solution	Marks
8. (a) $M^2 = \begin{pmatrix} m & -m \\ m & m \end{pmatrix}\begin{pmatrix} m & -m \\ m & m \end{pmatrix} = \begin{pmatrix} 0 & -2m^2 \\ 2m^2 & 0 \end{pmatrix}$	1A
	----------(1)
(b) (i) $MX = \begin{pmatrix} m & -m \\ m & m \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} am-cm & bm-dm \\ am+cm & bm+dm \end{pmatrix}$	1A for either correct
$XM = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m & -m \\ m & m \end{pmatrix} = \begin{pmatrix} am+bm & bm-am \\ cm+dm & dm-cm \end{pmatrix}$	
For $MX = XM$, with the help of $m > 0$, we have $\begin{pmatrix} a-c & b-d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} a+b & b-a \\ c+d & d-c \end{pmatrix}$.	
So, we have $\begin{cases} a-c=a+b \\ b-d=b-a \\ a+c=c+d \\ b+d=d-c \end{cases}$.	1M for comparing elements
Thus, we have $c=-b$ and $d=a$.	1
(ii) Now, $\det X = ad-bc = a^2+b^2$.	1M for considering $\det X$
Note that $X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a non-zero real matrix.	
So, at least one of a and b is non-zero real.	
Therefore, $\det X > 0$.	
Hence, $\det X \neq 0$.	
Thus, X is a non-singular matrix.	1
(iii) (1) $X^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	1M
$X - 6X^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} - \frac{6}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
So, we have $a - \frac{6a}{a^2+b^2} = 1$ and $b + \frac{6b}{a^2+b^2} = 0$.	
Solving, we have $\begin{cases} a=3 \\ b=0 \end{cases}$ or $\begin{cases} a=-2 \\ b=0 \end{cases}$.	
Thus, we have $X = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ or $X = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.	1A + 1A

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Solution	Marks
$X-6X^{-1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$	
$X^2-6I=X$	
$X^2-X-6I=\begin{pmatrix}0&0\\0&0\end{pmatrix}$	
$\begin{pmatrix}a^2-b^2&2ab\\-2ab&a^2-b^2\end{pmatrix}-\begin{pmatrix}a&b\\-b&a\end{pmatrix}-6\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$	1M
$\begin{pmatrix}a^2-b^2-a-6&2ab-b\\b-2ab&a^2-b^2-a-6\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$	
So, we have $a^2-b^2-a-6=0$ and $2ab-b=0$.	
Solving, we have $\begin{cases}a=3\\b=0\end{cases}$ or $\begin{cases}a=-2\\b=0\end{cases}$.	
Thus, we have $X=\begin{pmatrix}3&0\\0&3\end{pmatrix}$ or $X=\begin{pmatrix}-2&0\\0&-2\end{pmatrix}$.	1A + 1A
(2) $\left[\begin{pmatrix}m&-m\\m&m\end{pmatrix}-\begin{pmatrix}3k&0\\0&3k\end{pmatrix}\right]^2=-\begin{pmatrix}0&-2m^2\\2m^2&0\end{pmatrix}$	1M for using (b)(iii)(1) and (a)
$\begin{pmatrix}m-3k&-m\\m&m-3k\end{pmatrix}\begin{pmatrix}m-3k&-m\\m&m-3k\end{pmatrix}=\begin{pmatrix}0&2m^2\\-2m^2&0\end{pmatrix}$	
$\begin{pmatrix}(m-3k)^2-m^2&-2m(m-3k)\\2m(m-3k)&(m-3k)^2-m^2\end{pmatrix}=\begin{pmatrix}0&2m^2\\-2m^2&0\end{pmatrix}$	
So, we have $(m-3k)^2-m^2=0$ and $-2m(m-3k)=2m^2$.	
Solving, we have $k=\frac{2m}{3}$.	1A
	----------(10)
(c) By (b)(iii)(2), we have $\left[\begin{pmatrix}m&-m\\m&m\end{pmatrix}-\begin{pmatrix}2m&0\\0&2m\end{pmatrix}\right]^2=-M^2$.	
So, we have $\begin{pmatrix}-m&-m\\m&-m\end{pmatrix}^2=-M^2$.	1M
Therefore, we have $\begin{pmatrix}-m&-m\\m&-m\end{pmatrix}^4=(-M^2)^2=(-1)^2M^4=M^4$.	1A
Note that $\begin{pmatrix}m&m\\-m&m\end{pmatrix}^4=(-1)^4\begin{pmatrix}-m&-m\\m&-m\end{pmatrix}^4=\begin{pmatrix}-m&-m\\m&-m\end{pmatrix}^4=M^4$.	1M for considering $-\begin{pmatrix}-m&-m\\m&-m\end{pmatrix}$ + 1A
Thus, two required real matrices are $\begin{pmatrix}-m&-m\\m&-m\end{pmatrix}$ and $\begin{pmatrix}m&m\\-m&m\end{pmatrix}$.	
	----------(4)

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Solution	Marks
9. (a) (i) $\begin{cases} \alpha+\beta+\gamma=-b \\ \alpha\beta+\beta\gamma+\gamma\alpha=c \\ \alpha\beta\gamma=-d \end{cases}$	
S_1 $=\alpha+\beta+\gamma$	
$=-b$	1A
S_2 $=\alpha^2+\beta^2+\gamma^2$	
$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	1M for using $(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$=b^2-2c$	1A
S_3 $=\alpha^3+\beta^3+\gamma^3$	
$=(-b\alpha^2-c\alpha-d)+(-b\beta^2-c\beta-d)+(-b\gamma^2-c\gamma-d)$	1M
$=-b(\alpha^2+\beta^2+\gamma^2)-c(\alpha+\beta+\gamma)-3d$ $=-bS_2-cS_1-3d$ $=-b(b^2-2c)-c(-b)-3d$	
$=-b^3+3bc-3d$	1A
S_3 $=\alpha^3+\beta^3+\gamma^3$	
$=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)+3\alpha\beta\gamma$	1M
$=-b(b^2-2c-c)-3d$	
$=-b^3+3bc-3d$	1A
(ii) Note that $\begin{cases} \alpha^3+b\alpha^2+c\alpha+d=0 \\ \beta^3+b\beta^2+c\beta+d=0 \\ \gamma^3+b\gamma^2+c\gamma+d=0 \end{cases}$.	
So, we have $\begin{cases} \alpha^{k+3}+b\alpha^{k+2}+c\alpha^{k+1}+d\alpha^k=0 \\ \beta^{k+3}+b\beta^{k+2}+c\beta^{k+1}+d\beta^k=0 \\ \gamma^{k+3}+b\gamma^{k+2}+c\gamma^{k+1}+d\gamma^k=0 \end{cases}$.	1M
Therefore, we have $\alpha^{k+3}+\beta^{k+3}+\gamma^{k+3}+b(\alpha^{k+2}+\beta^{k+2}+\gamma^{k+2})+c(\alpha^{k+1}+\beta^{k+1}+\gamma^{k+1})+d(\alpha^k+\beta^k+\gamma^k)=0$	
$S_{k+3}+bS_{k+2}+cS_{k+1}+dS_k=0$	1

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Solution	Marks
(iii) By (a)(ii), we have $S_{k+3}+bS_{k+2}+cS_{k+1}+dS_k=0$.	
So, we have $S_{k+3}+bS_{k+2}=-c(S_{k+1}+bS_k)$ for all positive integers k .	1M
$S_{2n+1}+bS_{2n}$	
$=-c(S_{2n-1}+bS_{2n-2})$	
$=(-c)^2(S_{2n-3}+bS_{2n-4})$	1M (for either)
$=\cdots$	
$=(-c)^{n-1}(S_3+bS_2)$	
$=(-c)^{n-1}(-b^3+b(b^2-2c))$ (by (a)(i))	
$=(-c)^{n-1}(-2bc)$	
$=(-1)^n 2bc^n$ (which is also true for $n=1$)	1
$S_{2n}+bS_{2n-1}$	
$=-c(S_{2n-2}+bS_{2n-3})$	
$=(-c)^2(S_{2n-4}+bS_{2n-5})$	
$=\cdots$	
$=(-c)^{n-1}(S_2+bS_1)$	
$=(-c)^{n-1}(b^2-2c+b(-b))$ (by (a)(i))	
$=(-1)^n 2c^n$ (which is also true for $n=1$)	1
By (a)(i), we have $S_3+bS_2=-b^3+b(b^2-2c)=-2bc$	
and $S_2+bS_1=b^2-2c+b(-b)=-2c$.	
So, the statement is true for $n=1$.	1M for attempting to use induction
Assume that $S_{2k+1}+bS_{2k}=(-1)^k 2bc^k$ and $S_{2k}+bS_{2k-1}=(-1)^k 2c^k$ for some positive integer k .	
Then, we have	
$S_{2k+3}+bS_{2k+2}$	
$=-c(S_{2k+1}+bS_{2k})$ (by (a)(ii))	1M (for either)
$=-c\left((-1)^k 2bc^k\right)$ (by induction assumption)	
$=(-1)^{k+1}2bc^{k+1}$	1
Also,	
$S_{2k+2}+bS_{2k+1}$	
$=-c(S_{2k}+bS_{2k-1})$ (by (a)(ii))	
$=-c\left((-1)^k 2c^k\right)$ (by induction assumption)	
$=(-1)^{k+1}2c^{k+1}$	1
Thus, by mathematical induction, the statement is true for any positive integer n .	
	----------(11)
(b) Let three required numbers be α,β and γ.	
Suppose that the roots of $x^3+bx^2+cx+d=0$ are α,β and γ .	
Then, we have $S_1=3$, $S_2=3203$ and $S_3=9603$.	
Therefore, we have $b=-3$, $c=-1597$ and $d=1599$.	1A for all correct
So, the roots of $x^3-3x^2-1597x+1599=0$ are α,β and γ .	
Note that $x^3-3x^2-1597x+1599=(x-1)(x+39)(x-41)$.	1M for factorization + 1A
Thus, three required numbers are 1 , −39 and 41 .	1A for all correct
	----------(4)

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Solution	Marks
10. (a) Let $f(x) = x\ln x - x$ for $x > 0$.	
Then, we have	
$f'(x)$	
$= \ln x + 1 - 1$	
$= \ln x$	1A
$\begin{cases} < 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ > 0 & \text{if } x > 1 \end{cases}$	1M for testing + 1A
So, $f(x)$ attains its least value when $x = 1$. Therefore,	
$f(x) \ge f(1)$ for all $x > 0$	
$x\ln x - x \ge -1$ for all $x > 0$	
$x\ln x - x + 1 \ge 0$ for all $x > 0$	1
Let $f(x) = x\ln x - x$ for $x > 0$.	
Then, we have	
$f'(x)$	
$= \ln x + 1 - 1$	
$= \ln x$	1A
$f''(x) = \frac{1}{x}$	
$f'(x) = 0 \Leftrightarrow x = 1$	
$f''(1) = 1 > 0$	1M for testing + 1A
Note that $f(x)$ has only one local minimum.	
So, $f(x)$ attains its least value when $x = 1$. Therefore,	
$f(x) \ge f(1)$ for all $x > 0$	
$x\ln x - x \ge -1$ for all $x > 0$	
$x\ln x - x + 1 \ge 0$ for all $x > 0$	1
	----------(4)
(b) Define $g(x) = \frac{a^x - 1}{x}$ for all $x > 0$.	
Then, we have	
$g'(x)$	
$= \frac{(a^x \ln a)x - (a^x - 1)}{x^2}$	1A
$= \frac{a^x(x\ln a) - a^x + 1}{x^2}$	
$= \frac{a^x \ln a^x - a^x + 1}{x^2}$	
Note that $a^x \ln a^x - a^x + 1 \ge 0$ for all $x > 0$. (by (a))	1M for using (a)
So, we have $g'(x) \ge 0$ for all $x > 0$.	
Thus, g is increasing.	1
	----------(3)

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Solution	Marks
(c) (i) By (b), we have $\dfrac{a_k{}^p-1}{p} \ge \dfrac{a_k{}^q-1}{q}$ for all $k=1,2,\dots,n$.	1M for using (b) repeatedly
So, we have $\displaystyle\sum_{k=1}^{n}\frac{a_k{}^p-1}{p} \ge \sum_{k=1}^{n}\frac{a_k{}^q-1}{q}$.	1M for summation
Therefore, we have $\dfrac{1}{p}\left(\left(\displaystyle\sum_{k=1}^{n}a_k{}^p\right)-n\right) \ge \dfrac{1}{q}\left(\left(\displaystyle\sum_{k=1}^{n}a_k{}^q\right)-n\right)$.	
Note that $\dfrac{1}{q}\left(\left(\displaystyle\sum_{k=1}^{n}a_k{}^q\right)-n\right) = 0$ (since $\displaystyle\sum_{k=1}^{n}a_k{}^q=n$) .	
Thus, we have $\displaystyle\sum_{k=1}^{n}a_k{}^p \ge n$.	1
(ii) Define $B=\left(\dfrac{1}{n}\displaystyle\sum_{r=1}^{n}b_r{}^q\right)^{\frac{1}{q}}$. Then, we have $B>0$.	
Note that $\displaystyle\sum_{k=1}^{n}\left(\frac{b_k}{B}\right)^q = \frac{1}{B^q}\sum_{k=1}^{n}b_k{}^q = \frac{n}{\sum_{r=1}^{n}b_r{}^q}\left(\sum_{k=1}^{n}b_k{}^q\right) = n$.	1M for checking condition
Putting $a_k=\dfrac{b_k}{B}$ for all $k=1,2,\dots,n$ in (c)(i), we have	1A
$\displaystyle\sum_{k=1}^{n}\left(\frac{b_k}{B}\right)^p \ge n$	
$\displaystyle\frac{1}{n}\sum_{k=1}^{n}b_k{}^p \ge B^p$	
$\displaystyle\frac{1}{n}\sum_{k=1}^{n}b_k{}^p \ge \left(\frac{1}{n}\sum_{r=1}^{n}b_r{}^q\right)^{\frac{p}{q}}$	
$\displaystyle\left(\frac{1}{n}\sum_{k=1}^{n}b_k{}^p\right)^{\frac{1}{p}} \ge \left(\frac{1}{n}\sum_{r=1}^{n}b_r{}^q\right)^{\frac{1}{q}}$	
$\displaystyle\left(\frac{1}{n}\sum_{k=1}^{n}b_k{}^p\right)^{\frac{1}{p}} \ge \left(\frac{1}{n}\sum_{k=1}^{n}b_k{}^q\right)^{\frac{1}{q}}$	1
Since $p>q>0$, we have $\dfrac{1}{q}>\dfrac{1}{p}>0$.	1M for checking condition
Hence we have	
$\displaystyle\left(\frac{1}{n}\sum_{k=1}^{n}b_k{}^{\frac{1}{q}}\right)^{\frac{1}{\frac{1}{q}}} \ge \left(\frac{1}{n}\sum_{k=1}^{n}b_k{}^{\frac{1}{p}}\right)^{\frac{1}{\frac{1}{p}}}$	
$\displaystyle\left(\frac{1}{n}\sum_{k=1}^{n}b_k{}^{\frac{1}{p}}\right)^{p} \le \left(\frac{1}{n}\sum_{k=1}^{n}b_k{}^{\frac{1}{q}}\right)^{q}$	1
	----------(8)

Solution	Marks
11. (a) $z^2 = \cos\theta + i\sin\theta$	
$z = \cos\frac{\theta+2k\pi}{2} + i\sin\frac{\theta+2k\pi}{2}$, $k = 0, 1$	1A + 1A or equivalent
$z = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$ or $z = \cos\left(\frac{\theta}{2}+\pi\right) + i\sin\left(\frac{\theta}{2}+\pi\right)$	
$z = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$ or $z = -\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}$	
	----------(2)
(b) (i) $(u+1)^2 = \cos\theta + i\sin\theta$	
$u = \cos\frac{\theta}{2} - 1 + i\sin\frac{\theta}{2}$ or $u = -\cos\frac{\theta}{2} - 1 - i\sin\frac{\theta}{2}$ (by (a))	1M for using (a)
Since $\mathrm{Im}(u_1) < 0$, we have	
$u_1 = -\cos\frac{\theta}{2} - 1 - i\sin\frac{\theta}{2}$ and $u_2 = \cos\frac{\theta}{2} - 1 + i\sin\frac{\theta}{2}$.	1A for both correct or equivalent
(ii) $\frac{u_2}{u_1}$	
$= \dfrac{\cos\frac{\theta}{2} - 1 + i\sin\frac{\theta}{2}}{-\cos\frac{\theta}{2} - 1 - i\sin\frac{\theta}{2}}$	
$= \dfrac{-2\sin^2\frac{\theta}{4} + 2i\sin\frac{\theta}{4}\cos\frac{\theta}{4}}{-2\cos^2\frac{\theta}{4} - 2i\sin\frac{\theta}{4}\cos\frac{\theta}{4}}$	1M
$= \dfrac{2i\sin\frac{\theta}{4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)}{-2\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)}$	1M
$= \dfrac{-i\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}}$	
$= -i\tan\frac{\theta}{4}$	1
$\left(\frac{u_2}{u_1}\right)^n$	
$= (-1)^n i^n \tan^n\frac{\theta}{4}$	
Note that $\tan^n\frac{\theta}{4} \neq 0$ since $0 < \theta < \pi$.	
$\left(\frac{u_2}{u_1}\right)^n$ is a real number if and only if i^n is a real number.	1M can be absorbed
Thus, $\left(\frac{u_2}{u_1}\right)^n$ is a real number if and only if n is an even integer.	1A

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Solution	Marks
$\frac{u_2}{u_1}$	
$=\dfrac{\cos\frac{\theta}{2}-1+i\sin\frac{\theta}{2}}{-\cos\frac{\theta}{2}-1-i\sin\frac{\theta}{2}}$	
$=\dfrac{\left(\cos\frac{\theta}{2}-1+i\sin\frac{\theta}{2}\right)\left(-\cos\frac{\theta}{2}-1+i\sin\frac{\theta}{2}\right)}{\left(-\cos\frac{\theta}{2}-1-i\sin\frac{\theta}{2}\right)\left(-\cos\frac{\theta}{2}-1+i\sin\frac{\theta}{2}\right)}$	1M
$=\dfrac{1-\cos^2\frac{\theta}{2}+i\cos\frac{\theta}{2}\sin\frac{\theta}{2}-i\sin\frac{\theta}{2}-i\sin\frac{\theta}{2}\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}+1\right)^2+\sin^2\frac{\theta}{2}}$	
$=\dfrac{-2i\sin\frac{\theta}{2}}{2+2\cos\frac{\theta}{2}}$	
$=\dfrac{-i\sin\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}$	
$=\dfrac{-2i\sin\frac{\theta}{4}\cos\frac{\theta}{4}}{2\cos^2\frac{\theta}{4}}$	1M
$=\dfrac{-i\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}}$	
$=-i\tan\frac{\theta}{4}$	1
$\left(\frac{u_2}{u_1}\right)^n$	
$=(-1)^n i^n \tan^n\frac{\theta}{4}$	
Note that $\tan^n\frac{\theta}{4}\neq 0$ since $0<\theta<\pi$.	
$\left(\frac{u_2}{u_1}\right)^n$ is a real number if and only if i^n is a real number.	1M can be absorbed
Thus, $\left(\frac{u_2}{u_1}\right)^n$ is a real number if and only if n is an even integer.	1A

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Solution	Marks
u_1	
$= -\left(\cos\frac{\theta}{2}+1\right) - i\sin\frac{\theta}{2}$	
$= -\left(2\cos^2\frac{\theta}{4} + 2i\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)$	1M (either one)
$= -2\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)$	1M (either one)
u_2	
$= \cos\frac{\theta}{2} - 1 + i\sin\frac{\theta}{2}$	
$= -2\sin^2\frac{\theta}{4} + 2i\sin\frac{\theta}{4}\cos\frac{\theta}{4}$	
$= -2\sin\frac{\theta}{4}\left(\sin\frac{\theta}{4} - i\cos\frac{\theta}{4}\right)$	
$= -2\sin\frac{\theta}{4}\left(\cos\left(\frac{\theta}{4}-\frac{\pi}{2}\right) + i\sin\left(\frac{\theta}{4}-\frac{\pi}{2}\right)\right)$	
$\frac{u_2}{u_1}$	
$= \dfrac{-2\sin\frac{\theta}{4}\left(\cos(\frac{\theta}{4}-\frac{\pi}{2}) + i\sin(\frac{\theta}{4}-\frac{\pi}{2})\right)}{-2\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)}$	
$= \tan\frac{\theta}{4}\left(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})\right)$	
$= -i\tan\frac{\theta}{4}$	1
$\left(\frac{u_2}{u_1}\right)^n$	
$= (-1)^n i^n \tan^n\frac{\theta}{4}$	
Note that $\tan^n\frac{\theta}{4} \neq 0$ since $0 < \theta < \pi$.	
$\left(\frac{u_2}{u_1}\right)^n$ is a real number if and only if i^n is a real number.	1M can be absorbed
Thus, $\left(\frac{u_2}{u_1}\right)^n$ is a real number if and only if n is an even integer.	1A

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Solution	Marks
(iii) $u_1 = -\cos\frac{\theta}{2} - 1 - i\sin\frac{\theta}{2}$	
$= -\left(\cos\frac{\theta}{2} + 1\right) - i\sin\frac{\theta}{2}$	
$= -\left(2\cos^2\frac{\theta}{4} + 2i\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)$	
$= -2\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)$	
$u_1^{12} = \left(-2\cos\frac{\theta}{4}(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4})\right)^{12}$	
$= 2^{12}\cos^{12}\frac{\theta}{4}\left(\cos\frac{12\theta}{4} + i\sin\frac{12\theta}{4}\right)$	
$= 2^{12}\cos^{12}\frac{\theta}{4}(\cos 3\theta + i\sin 3\theta)$	1
Note that $u_1^{12} - u_2^{12} = u_1^{12}\left(1 - \left(\frac{u_2}{u_1}\right)^{12}\right)$.	1M
By (b)(ii), since 12 is an even integer, $\left(\frac{u_2}{u_1}\right)^{12}$ is a real number.	
Hence, $\left(\frac{u_2}{u_1}\right)^{12} = \left(-i\tan\frac{\theta}{4}\right)^{12}$	
$= \tan^{12}\frac{\theta}{4}$	
$\neq 1$ (since $0 < \tan\frac{\theta}{4} < 1$ as $0 < \frac{\theta}{4} < \frac{\pi}{4}$)	1M (either one)
So, $u_1^{12} - u_2^{12}$ is a real number if and only if u_1^{12} is a real number.	1A (either one)
Note also that $\cos\frac{\theta}{4} \neq 0$ as $0 < \theta < \pi$.	
$u_1^{12} - u_2^{12}$ is a real number	
$\Leftrightarrow$ $\sin 3\theta = 0$	1A
$\Leftrightarrow$ $\theta = \frac{\pi}{3}$ or $\theta = \frac{2\pi}{3}$ (since $0 < \theta < \pi$)	1A for both correct

Solution	Marks
$u_1 = -\cos\frac{\theta}{2} - 1 - i\sin\frac{\theta}{2}$	
$= -\left(\cos\frac{\theta}{2} + 1\right) - i\sin\frac{\theta}{2}$	
$= -\left(2\cos^2\frac{\theta}{4} + 2i\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)$	
$= -2\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)$	
$u_1^{12} = \left(-2\cos\frac{\theta}{4}(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4})\right)^{12}$	
$= 2^{12}\cos^{12}\frac{\theta}{4}\left(\cos\frac{12\theta}{4} + i\sin\frac{12\theta}{4}\right)$	
$= 2^{12}\cos^{12}\frac{\theta}{4}(\cos 3\theta + i\sin 3\theta)$	1
$u_2^{12} = \left(-i\tan\frac{\theta}{4}\right)^{12} u_1^{12}$ (by (b)(ii))	1M for using (b)(ii)
$= \tan^{12}\frac{\theta}{4}\left(2^{12}\cos^{12}\frac{\theta}{4}(\cos 3\theta + i\sin 3\theta)\right)$	
$= 2^{12}\sin^{12}\frac{\theta}{4}(\cos 3\theta + i\sin 3\theta)$	
So, we have $u_1^{12} - u_2^{12} = 2^{12}\left(\cos^{12}\frac{\theta}{4} - \sin^{12}\frac{\theta}{4}\right)(\cos 3\theta + i\sin 3\theta)$.	1A
$u_1^{12} - u_2^{12}$ is a real number	
$\Leftrightarrow$ $\sin 3\theta\left(\cos^{12}\frac{\theta}{4} - \sin^{12}\frac{\theta}{4}\right) = 0$	
$\Leftrightarrow$ $\sin 3\theta = 0$ or $\cos^{12}\frac{\theta}{4} = \sin^{12}\frac{\theta}{4}$	
$\Leftrightarrow$ $\sin 3\theta = 0$ or $\tan\frac{\theta}{4} = 1$	
Note that $\tan\frac{\theta}{4} \neq 1$ since $0 < \theta < \pi$.	1M
$u_1^{12} - u_2^{12}$ is a real number	
$\Leftrightarrow$ $\sin 3\theta = 0$	1A
$\Leftrightarrow$ $\theta = \frac{\pi}{3}$ or $\theta = \frac{2\pi}{3}$ (since $0 < \theta < \pi$) .	1A for both correct

Solution	Marks
$u_1=-2\cos\frac{\theta}{4}\left(\cos\frac{\theta}{4}+i\sin\frac{\theta}{4}\right)$	
$u_1^{12}=\left(-2\cos\frac{\theta}{4}(\cos\frac{\theta}{4}+i\sin\frac{\theta}{4})\right)^{12}$	
$=2^{12}\cos^{12}\frac{\theta}{4}\left(\cos\frac{12\theta}{4}+i\sin\frac{12\theta}{4}\right)$	
$=2^{12}\cos^{12}\frac{\theta}{4}(\cos 3\theta+i\sin 3\theta)$	1
$u_2^{12}=\left(-2\sin\frac{\theta}{4}\left(\cos\left(\frac{\theta}{4}-\frac{\pi}{2}\right)+i\sin\left(\frac{\theta}{4}-\frac{\pi}{2}\right)\right)\right)^{12}$	
$=2^{12}\sin^{12}\frac{\theta}{4}(\cos(3\theta-6\pi)+i\sin(3\theta-6\pi))$	1M
$=2^{12}\sin^{12}\frac{\theta}{4}(\cos 3\theta+i\sin 3\theta)$	
So, we have $u_1^{12}-u_2^{12}=2^{12}\left(\cos^{12}\frac{\theta}{4}-\sin^{12}\frac{\theta}{4}\right)(\cos 3\theta+i\sin 3\theta)$.	1A
$u_1^{12}-u_2^{12}$ is a real number	
$\Leftrightarrow$ $\sin 3\theta\left(\cos^{12}\frac{\theta}{4}-\sin^{12}\frac{\theta}{4}\right)=0$	
$\Leftrightarrow$ $\sin 3\theta=0$ or $\cos^{12}\frac{\theta}{4}=\sin^{12}\frac{\theta}{4}$	
$\Leftrightarrow$ $\sin 3\theta=0$ or $\tan\frac{\theta}{4}=1$	
Note that $\tan\frac{\theta}{4}\neq 1$ since $0<\theta<\pi$.	1M
$u_1^{12}-u_2^{12}$ is a real number	
$\Leftrightarrow$ $\sin 3\theta=0$	1A
$\Leftrightarrow$ $\theta=\frac{\pi}{3}$ or $\theta=\frac{2\pi}{3}$ (since $0<\theta<\pi$) .	1A for both correct
	----------(13)

香港考試及評核局
HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

2006年香港高級程度會考
HONG KONG ADVANCED LEVEL EXAMINATION 2006

純粹數學　高級程度　試卷二
PURE MATHEMATICS　A-LEVEL　PAPER 2

本評卷參考乃香港考試及評核局專爲今年本科考試而編寫，供閱卷員參考之用。閱卷員在完成閱卷工作後，若將本評卷參考提供其任教會考班的本科同事參閱，本局不表反對，但須切記，在任何情況下均不得容許本評卷參考落入學生手中。學生若索閱或求取此等文件，閱卷員／教師應嚴詞拒絕，因學生極可能將評卷參考視爲標準答案，以致但知硬背死記，活剝生吞。這種落伍的學習態度，既不符現代教育原則，亦有違考試着重理解能力與運用技巧之旨。因此，本局籲請各閱卷員/教師通力合作，堅守上述原則。

This marking scheme has been prepared by the Hong Kong Examinations and Assessment Authority for markers' reference. The Authority has no objection to markers sharing it, after the completion of marking, with colleagues who are teaching the subject. However, under no circumstances should it be given to students because they are likely to regard it as a set of model answers. Markers/teachers should therefore firmly resist students' requests for access to this document. Our examinations emphasise the testing of understanding, the practical application of knowledge and the use of processing skills. Hence the use of model answers, or anything else which encourages rote memorisation, should be considered outmoded and pedagogically unsound. The Authority is counting on the co-operation of markers/teachers in this regard.

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Advanced Level Pure Mathematics

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. Marks may be deducted for poor presentation (*pp*). The symbol (*pp-1*) should be used to denote 1 mark deducted for *pp*. At most deducted 1 mark from Section A and 1 mark from Section B for *pp*. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

7. In the marking scheme, 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

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Solution	Marks
1. (a) Note that $\frac{d}{dx}\int_1^{3x+1}\sqrt{t^5+t^3+1}\ dt = 3\sqrt{(3x+1)^5+(3x+1)^3+1}$.	1A
$\lim_{x\to 0}\frac{\int_1^{3x+1}\sqrt{t^5+t^3+1}\ dt}{\ln(x+1)}$	
$=\lim_{x\to 0}\frac{3\sqrt{(3x+1)^5+(3x+1)^3+1}}{\frac{1}{x+1}}$	1M
$=3\sqrt{3}$	1A
(b) Note that $-1\le \sin\frac{1}{x}\le 1$ for all $x\neq 0$.	1M
So, we have $-\lvert\sin x\rvert\le \sin x\sin\frac{1}{x}\le\lvert\sin x\rvert$ for all $x\neq 0$.	
Since $\lim_{x\to 0}(-\lvert\sin x\rvert)=0=\lim_{x\to 0}\lvert\sin x\rvert$, we have	1M
$\lim_{x\to 0}\sin x\ \sin\frac{1}{x}=0$ (by Sandwich Theorem).	1A f.t.
Note that $\left\lvert\sin\frac{1}{x}\right\rvert\le 1$ for all $x\neq 0$.	1M
Note also that $\lim_{x\to 0}\sin x=0$.	1M
Thus, we have $\lim_{x\to 0}\sin x\ \sin\frac{1}{x}=0$.	1A f.t.
	------------(6)

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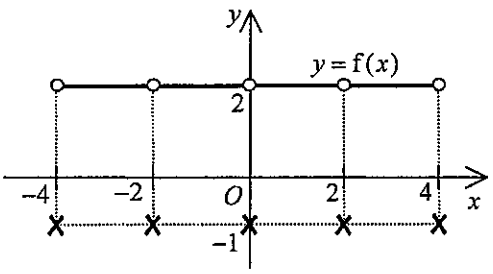
Solution	Marks

2. (a) (i)



1M for the shape of the graph
1A for all being correct

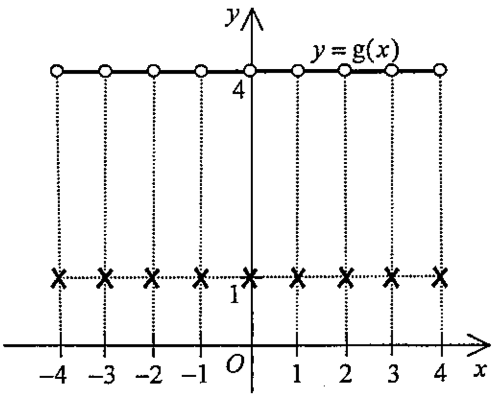
(ii) Case 1: x is an even number.
$x+2$ is also an even number.
So, $f(x+2)=-1=f(x)$.

Case 2: x is not an even number.
$x+2$ is also not an even number.
So, $f(x+2)=2=f(x)$.

Hence, $f(x+2)=f(x)$ for all $x \in \mathbf{R}$. — 1A
Thus, f is a periodic function with period 2. — 1A f.t.

(b) Note that $g(x)=\begin{cases} 1 & \text{when } x \text{ is an integer,} \\ 4 & \text{when } x \text{ is not an integer.} \end{cases}$

(i)



1M for the shape of the graph
1A for all being correct

(ii) $\because$ $g(3)=1=g(4)$ but $3 \neq 4$
$\therefore$ g is not an injective function. — 1A f.t.

----------(7)

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Solution	Marks
3. (a) $I_{m+2,n+2}$	
$=\int_0^{\frac{\pi}{4}}\frac{\sin^{m+2}\theta}{\cos^{n+2}\theta}d\theta$	
$=\int_0^{\frac{\pi}{4}}\frac{\sin^{m+1}\theta}{n+1}d\left(\frac{1}{\cos^{n+1}\theta}\right)$	1M can be absorbed
$=\frac{1}{n+1}\left[\frac{\sin^{m+1}\theta}{\cos^{n+1}\theta}\right]_0^{\frac{\pi}{4}}-\frac{m+1}{n+1}\int_0^{\frac{\pi}{4}}\frac{\sin^m\theta\cos\theta}{\cos^{n+1}\theta}d\theta$	1A
$=\frac{1}{n+1}\frac{\left(\frac{1}{\sqrt{2}}\right)^{m+1}}{\left(\frac{1}{\sqrt{2}}\right)^{n+1}}-\frac{m+1}{n+1}\int_0^{\frac{\pi}{4}}\frac{\sin^m\theta}{\cos^n\theta}d\theta$	
$=\frac{1}{n+1}\left(\frac{1}{\sqrt{2}}\right)^{m-n}-\frac{m+1}{n+1}I_{m,n}$	1
(b) Let $u=\cos\theta$. Then, we have $\frac{du}{d\theta}=-\sin\theta$.	
$I_{3,1}$	
$=\int_0^{\frac{\pi}{4}}\frac{\sin^3\theta}{\cos\theta}d\theta$	
$=-\int_1^{\frac{1}{\sqrt{2}}}\frac{1-u^2}{u}du$	1A
$=\int_{\frac{1}{\sqrt{2}}}^1\left(\frac{1}{u}-u\right)du$	
$=\left[\ln u-\frac{u^2}{2}\right]_{\frac{1}{\sqrt{2}}}^1$	
$=\frac{1}{2}\ln 2-\frac{1}{4}$	1A
(c) $I_{7,5}$	
$=\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2-\frac{6}{4}I_{5,3}$ (by (a))	
$=\frac{1}{8}-\frac{3}{2}\left(\frac{1}{4}-2I_{3,1}\right)$ (by (a))	1M for using (a) repeatedly
$=\frac{-1}{4}+3I_{3,1}$	
$=\frac{-1}{4}+3\left(\frac{1}{2}\ln 2-\frac{1}{4}\right)$ (by (b))	
$=\frac{3}{2}\ln 2-1$	1A
	----------(7)

Solution	Marks
4. (a) Let $t=\sqrt{1+x^2}$. Then, we have $\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{x}{\sqrt{1+x^2}}$.	1A
$\displaystyle\int\frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x$	
$\displaystyle=\int(t^2-1)\,\mathrm{d}t$	1A pp−1 for omitting $\mathrm{d}t$
$\displaystyle=\frac{1}{3}t^3-t+C$	
$\displaystyle=\frac{1}{3}(1+x^2)^{\frac{3}{2}}-(1+x^2)^{\frac{1}{2}}+C$, where C is a constant	1A pp−1 for omitting C
(b) $\displaystyle\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{k^3}{\sqrt{n^2+k^2}}$	
$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)^3}{\sqrt{1+\left(\frac{k}{n}\right)^2}}$	1M can be absorbed
$\displaystyle=\int_0^1\frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x$	1A
$\displaystyle=\left[\frac{1}{3}(1+x^2)^{\frac{3}{2}}-(1+x^2)^{\frac{1}{2}}\right]_0^1$ (by (a))	1M
$\displaystyle=\frac{2\sqrt{2}}{3}-\sqrt{2}-\frac{1}{3}+1$	
$\displaystyle=\frac{2-\sqrt{2}}{3}$	1A
	----------(7)

Solution	Marks
5. (a) $\int \ln y \, dy$	
$= y\ln y - \int dy$	1M for integration by parts
$= y\ln y - y + C$, where C is a constant	1A pp−1 for omitting C
$\frac{d}{dy}(y\ln y - y)$	1M for considering $\frac{d}{dy}(y\ln y - y)$
$= \ln y + \frac{y}{y} - 1$	
$= \ln y$	
Thus, we have	
$\int \ln y \, dy$	
$= y\ln y - y + C$, where C is a constant	1A pp−1 for omitting C
(b) The required volume	
$= \pi \int_1^2 x^2 \, dy$	1M pp−1 for omitting dy
$= \pi \int_1^2 \frac{\ln y}{\ln 2} \, dy$	1A
$= \frac{\pi}{\ln 2}[y\ln y - y]_1^2$	1M for using the result of (a)
$= \frac{\pi}{\ln 2}(2\ln 2 - 1)$	1A
$= \pi\left(2 - \frac{1}{\ln 2}\right)$	
	----------(6)
The required volume	
$= 2\pi \int_0^1 x(2-y) \, dx$	1M pp−1 for omitting dx
$= 2\pi \int_0^1 x(2-2^{x^2}) \, dx$	1A
$= 2\pi\left(\int_0^1 2x \, dx - \int_0^1 x2^{x^2} \, dx\right)$	
$= 2\pi\left(\int_0^1 2x \, dx - \frac{1}{2}\int_0^1 2^{x^2} \, d(x^2)\right)$	
$= 2\pi\left(\left[x^2\right]_0^1 - \frac{1}{2\ln 2}\left[2^{x^2}\right]_0^1\right)$	1M
$= 2\pi\left(1 - \frac{1}{2\ln 2}\right)$	1A
$= \pi\left(2 - \frac{1}{\ln 2}\right)$	

Solution	Marks
6. (a) (i) Note that $\frac{(a\cos\theta)^2}{a^2}+\frac{(b\sin\theta)^2}{b^2}=\cos^2\theta+\sin^2\theta=1$.	
Thus, P lies on E.	1
(ii) Differentiating both sides of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$.	
So, we have $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-b^2x}{a^2y}$ if $y\neq 0$.	
The slope of the tangent to E at P $=\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\rvert_P$ $=\frac{-b^2a\cos\theta}{a^2b\sin\theta}$ $=\frac{-b}{a}\cot\theta$	1A (for either correct)
The slope of the straight line passing through P and Q $=\frac{(a+b)\sin\theta-b\sin\theta}{(a+b)\cos\theta-a\cos\theta}$ $=\frac{a}{b}\tan\theta$	
The product of the slopes $=\left(\frac{-b}{a}\cot\theta\right)\left(\frac{a}{b}\tan\theta\right)$ $=-1$	
Thus, the straight line passing through P and Q is the normal to E at P.	1
(b) Putting $y=\frac{\sin\theta}{\cos\theta}x-\frac{c}{\cos\theta}$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have	
$\frac{x^2}{a^2}+\frac{\left(\frac{\sin\theta}{\cos\theta}x-\frac{c}{\cos\theta}\right)^2}{b^2}=1$	
$(a^2\sin^2\theta+b^2\cos^2\theta)x^2-2a^2c\sin\theta\, x+a^2(c^2-b^2\cos^2\theta)=0$	1A
Since the straight line is a tangent to E, we have $(-2a^2c\sin\theta)^2-4a^2(a^2\sin^2\theta+b^2\cos^2\theta)(c^2-b^2\cos^2\theta)=0$	1M
$c^2=a^2\sin^2\theta+b^2\cos^2\theta$	
The distance between P and Q $=\sqrt{(a\sin\theta)^2+(b\cos\theta)^2}$	1M for using distance formula
$=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ $=\sqrt{c^2}$	
$=\lvert c\rvert$	1A do not accept omitting the absolute value

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Solution	Marks
Suppose that the straight line is a tangent to E at the point $(a\cos\phi, b\sin\phi)$.	
The equation of the tangent at $(a\cos\phi, b\sin\phi)$ is $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$.	1A
Since the equation of the straight line is $x\sin\theta - y\cos\theta = c$ and $0<\theta<\frac{\pi}{2}$, it follows that $c\neq 0$.	
So, we have $\frac{\cos\phi}{a\sin\theta}=\frac{\sin\phi}{-b\cos\theta}=\frac{1}{c}$.	1M
Therefore, we have $\cos\phi=\frac{a}{c}\sin\theta$ and $\sin\phi=\frac{-b}{c}\cos\theta$.	
Hence, we have $\left(\frac{a}{c}\sin\theta\right)^2+\left(\frac{-b}{c}\cos\theta\right)^2=1$.	
So, we have $c^2=a^2\sin^2\theta+b^2\cos^2\theta$.	
The distance between P and Q	
$=\sqrt{(a\sin\theta)^2+(b\cos\theta)^2}$	1M for using distance formula
$=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$	
$=\sqrt{c^2}$	
$=\|c\|$	1A do not accept omitting the absolute value

Solution	Marks
Suppose that the straight line is a tangent to E at the point $(a\cos\phi, b\sin\phi)$.	
The equation of the tangent at $(a\cos\phi, b\sin\phi)$ is $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$.	1A
Since the equation of the straight line is $x\sin\theta - y\cos\theta = c$ and $0<\theta<\frac{\pi}{2}$, it follows that $c\neq 0$.	
So, we have $\frac{\cos\phi}{a\sin\theta}=\frac{\sin\phi}{-b\cos\theta}=\frac{1}{c}$.	1M
Therefore, we have $\cos\phi=\frac{a}{c}\sin\theta$ and $\sin\phi=\frac{-b}{c}\cos\theta$.	
So, the straight line is a tangent to E at the point $\left(\frac{a^2\sin\theta}{c}, \frac{-b^2\cos\theta}{c}\right)$.	
Hence, we have $\frac{a^2\sin^2\theta}{c}+\frac{b^2\cos^2\theta}{c}=c$.	
So, we have $c^2=a^2\sin^2\theta+b^2\cos^2\theta$.	
The distance between P and Q	
$=\sqrt{(a\sin\theta)^2+(b\cos\theta)^2}$	1M for using distance formula
$=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$	
$=\sqrt{c^2}$	
$=\|c\|$	1A do not accept omitting the absolute value
	----------(7)

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Solution	Marks
7. (a) $f'(x)=\dfrac{x(x+12)}{(x+6)^2}$	1A or equivalent
$f''(x)=\dfrac{72}{(x+6)^3}$	1A or equivalent
	------------(2)
(b) Note that $f'(x)=0 \Leftrightarrow x=0$ or $x=-12$.	

x	$(-\infty,-12)$	-12	$(-12,-6)$	$(-6,0)$	0	$(0,\infty)$
$f'(x)$	+	0	−	−	0	+
$f''(x)$	−	−	−	+	+	+
$f(x)$	↗	−25	↘	↘	−1	↗

Solution	Marks
(i) $f'(x)>0 \Leftrightarrow x<-12$ or $x>0$.	1A
(ii) $f'(x)<0 \Leftrightarrow -12<x<-6$ or $-6<x<0$.	1A accept $-12<x<0$
(iii) $f''(x)>0 \Leftrightarrow x>-6$. (iv) $f''(x)<0 \Leftrightarrow x<-6$.	} 1A for both correct
	------------(3)
(c) From the table in (b), the relative maximum point is $(-12,-25)$	1A
and the relative minimum point is $(0,-1)$.	1A
	------------(2)

(d) $\because$ $\lim\limits_{x\to-6^-} f(x)=\lim\limits_{x\to-6^-}\dfrac{x^2-x-6}{x+6}=-\infty$ and

$\lim\limits_{x\to-6^+} f(x)=\lim\limits_{x\to-6^+}\dfrac{x^2-x-6}{x+6}=+\infty$

$\therefore$ the vertical asymptote is $x=-6$. — 1A

$\because$ $\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\pm\infty}\dfrac{x^2-x-6}{x(x+6)}=\lim\limits_{x\to\pm\infty}\dfrac{1-\frac{1}{x}-\frac{6}{x^2}}{1+\frac{6}{x}}=1$ and — 1M

$\lim\limits_{x\to\pm\infty}\left(f(x)-x\right)=\lim\limits_{x\to\pm\infty}\left(\dfrac{x^2-x-6}{x+6}-x\right)=\lim\limits_{x\to\pm\infty}\dfrac{-7-\frac{6}{x}}{1+\frac{6}{x}}=-7$

$\therefore$ the oblique asymptote is $y=x-7$. — 1A

Solution	Marks
$\because$ $\lim\limits_{x\to-6^-} f(x)=-\infty$ and $\lim\limits_{x\to-6^+} f(x)=+\infty$	
$\therefore$ the vertical asymptote is $x=-6$.	1A
$\because$ $f(x)=x-7+\dfrac{36}{x+6}$	1M for division
$\therefore$ $\lim\limits_{x\to\pm\infty}\left(f(x)-(x-7)\right)=\lim\limits_{x\to\pm\infty}\dfrac{36}{x+6}=0$	
Thus, the oblique asymptote is $y=x-7$.	1A
	----------(3)

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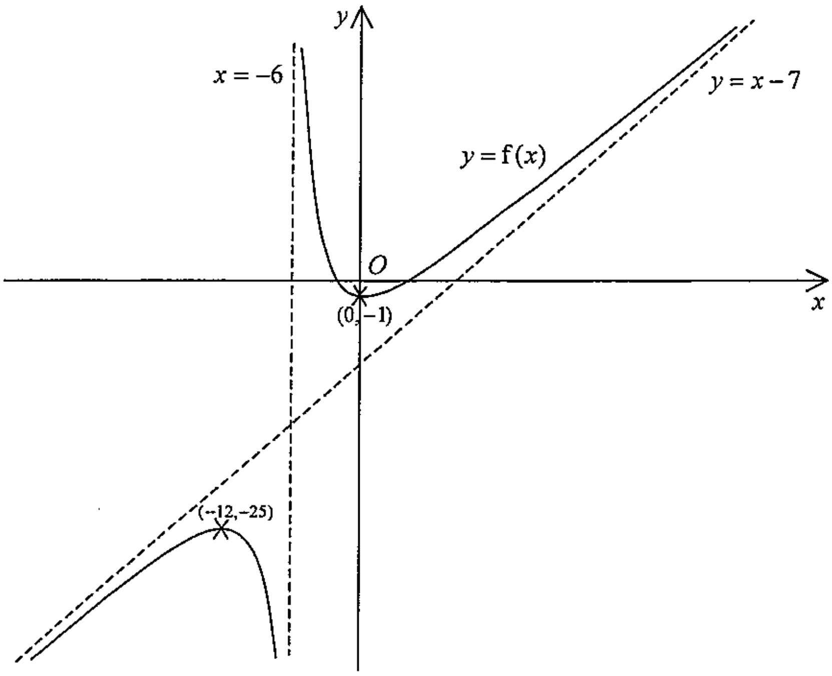
Solution	Marks
(e) 	1A for the relative extreme points and the asymptotes 1A for all being correct ------- (2)
(f) The required area $= \int_{-2}^{3} -f(x)\,dx$	1A accept $\int_{-2}^{3} \lvert f(x)\rvert\,dx$ and $\left\lvert \int_{-2}^{3} f(x)\,dx \right\rvert$
$= \int_{-2}^{3} \frac{-x^2+x+6}{x+6}\,dx$	pp−1 for omitting dx
$= \int_{-2}^{3} \left(-x+7-\frac{36}{x+6}\right)dx$	
$= \left[\frac{-x^2}{2}+7x-36\ln\lvert x+6\rvert\right]_{-2}^{3}$	1M
$= \frac{65}{2}+36\ln\frac{4}{9}$	1A
$= \frac{65}{2}-72\ln\frac{3}{2}$	------- (3)

Solution	Marks
8. (a) (i) $\int_0^{\pi} \cos mx \cos nx \,\mathrm{d}x$	
$= \frac{1}{2}\int_0^{\pi} \left(\cos(m+n)x + \cos(m-n)x\right)\mathrm{d}x$	1A pp−1 for omitting $\mathrm{d}x$
$= \frac{1}{2(m+n)}\left[\sin(m+n)x\right]_0^{\pi} + \frac{1}{2(m-n)}\left[\sin(m-n)x\right]_0^{\pi}$	
$= 0.$	1A
(ii) $\int_0^{\pi} \cos nx \,\mathrm{d}x$	
$= \frac{1}{n}\left[\sin nx\right]_0^{\pi}$	
$= 0$	1A
$\int_0^{\pi} \cos^2 nx \,\mathrm{d}x$	
$= \frac{1}{2}\int_0^{\pi} \left(1+\cos 2nx\right)\mathrm{d}x$	1A
$= \frac{1}{2}\left[x + \frac{1}{2n}\sin 2nx\right]_0^{\pi}$	
$= \frac{\pi}{2}$	1A
	------------(5)
(b) $\int_0^{\pi} x^2 \cos nx \,\mathrm{d}x$	
$= \frac{1}{n}\int_0^{\pi} x^2 \,\mathrm{d}\sin nx$	
$= \frac{1}{n}\left[x^2 \sin nx\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin nx \,\mathrm{d}x$	1M for integration by parts
$= \frac{-2}{n}\int_0^{\pi} x \sin nx \,\mathrm{d}x$	
$= \frac{2}{n^2}\int_0^{\pi} x \,\mathrm{d}\cos nx$	
$= \frac{2}{n^2}\left[x\cos nx\right]_0^{\pi} - \frac{2}{n^2}\int_0^{\pi} \cos nx \,\mathrm{d}x$	1A
$= \frac{2\pi \cos n\pi}{n^2} - \frac{2}{n^2}(0)$	
$= \frac{(-1)^n 2\pi}{n^2}$	1
	------------(3)

Solution	Marks
(c) (i) $\int_0^{\pi}\left(f(x)-x^2\right)dx=0$	
$\int_0^{\pi}(a_0+\sum_{m=1}^{N}a_m\cos mx-x^2)\,dx=0$	
$a_0\int_0^{\pi}dx+\sum_{m=1}^{N}a_m\int_0^{\pi}\cos mx\,dx=\int_0^{\pi}x^2\,dx$	1M for $\int_0^{\pi}\sum_{m=1}^{N}g_m(x)\,dx=\sum_{m=1}^{N}\int_0^{\pi}g_m(x)\,dx$
$\pi a_0=\left[\frac{x^3}{3}\right]_0^{\pi}$ (by (a)(ii))	
$a_0=\frac{\pi^2}{3}$	1A
(ii) For each $n=1,2,\ldots,N$, we have	
$\int_0^{\pi}\left(f(x)-x^2\right)\cos nx\,dx=0$	
$\int_0^{\pi}(a_0+\sum_{m=1}^{N}a_m\cos mx-x^2)\cos nx\,dx=0$	
$a_0\int_0^{\pi}\cos nx\,dx+\sum_{m=1}^{N}a_m\int_0^{\pi}\cos mx\cos nx\,dx=\int_0^{\pi}x^2\cos nx\,dx$	------------ for either one ------------
$\frac{\pi a_n}{2}=\frac{(-1)^n 2\pi}{n^2}$ (by (a)(ii), (a)(i) and (b))	1M for $\sum_{m=1}^{N}a_m\int_0^{\pi}\cos mx\cos nx\,dx=\frac{\pi a_n}{2}$
$a_n=\frac{(-1)^n 4}{n^2}$	1
Thus, we have $a_n=\frac{(-1)^n 4}{n^2}$ for all $n=1,2,\ldots,N$.	
(iii) I_k	
$=\int_0^{\pi}\left(f(x)-x^2\right)\cos(N+k)x\,dx$	
$=\int_0^{\pi}f(x)\cos(N+k)x\,dx-\int_0^{\pi}x^2\cos(N+k)x\,dx$	
$=\int_0^{\pi}(a_0+\sum_{m=1}^{N}a_m\cos mx)\cos(N+k)x\,dx-\frac{(-1)^{N+k}2\pi}{(N+k)^2}$ (by (b))	
$=a_0\int_0^{\pi}\cos(N+k)x\,dx+\sum_{m=1}^{N}a_m\int_0^{\pi}\cos mx\cos(N+k)x\,dx+\frac{(-1)^{N+k+1}2\pi}{(N+k)^2}$	
$=\frac{(-1)^{N+k+1}2\pi}{(N+k)^2}$ for all $k\in\mathbf{N}$ (by (a)(ii) and (a)(i))	1A accept $\frac{-(-1)^{N+k}2\pi}{(N+k)^2}$
So, we have $\frac{-2\pi}{(N+k)^2}\le I_k\le\frac{2\pi}{(N+k)^2}$ for all $k\in\mathbf{N}$.	1M for using Sandwich Theorem
Note that $\lim_{k\to\infty}\frac{-2\pi}{(N+k)^2}=0=\lim_{k\to\infty}\frac{2\pi}{(N+k)^2}$.	
Thus, we have $\lim_{k\to\infty}I_k=0$ (by Sandwich Theorem).	1A f.t.
	------------(7)

Solution	Marks
9. (a) $\lim_{x\to 0^+}\dfrac{x^{-n}}{e^{\frac{1}{x}}}$	
$=\lim_{x\to 0^+}\dfrac{-nx^{-n-1}}{\frac{-1}{x^2}e^{\frac{1}{x}}}$	1M
$=\lim_{x\to 0^+}\dfrac{nx^{-n+1}}{e^{\frac{1}{x}}}$	
$=\cdots$	
$=\lim_{x\to 0^+}\dfrac{n!}{e^{\frac{1}{x}}}$	1A
$=0$	1
Note that $\lim_{x\to 0^+}\dfrac{x^{-1}}{e^{\frac{1}{x}}}=\lim_{x\to 0^+}\dfrac{\frac{-1}{x^2}}{\frac{-1}{x^2}e^{\frac{1}{x}}}=\lim_{x\to 0^+}\dfrac{1}{e^{\frac{1}{x}}}=0$.	1
So, the statement is true for $n=1$.	
Assume that $\lim_{x\to 0^+}\dfrac{x^{-k}}{e^{\frac{1}{x}}}=0$ for some positive integer k .	
Then, we have	
$\lim_{x\to 0^+}\dfrac{x^{-(k+1)}}{e^{\frac{1}{x}}}$	
$=\lim_{x\to 0^+}\dfrac{-(k+1)x^{-k-2}}{\frac{-1}{x^2}e^{\frac{1}{x}}}$	1M
$=(k+1)\lim_{x\to 0^+}\dfrac{x^{-k}}{e^{\frac{1}{x}}}$	
$=(k+1)(0)$ (by induction assumption)	
$=0$	1
Thus, by mathematical induction, the statement is true any positive integer n .	
	----------(3)
(b) (i) $f'(x)=\begin{cases}0 & \text{when } x<0,\\ \dfrac{1}{x^2}e^{\frac{-1}{x}} & \text{when } x>0.\end{cases}$	1A for both cases correct

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Solution	Marks
(ii) $\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{0-0}{x}=\lim_{x\to0^-}0=0$ $\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{e^{\frac{-1}{x}}}{x}=\lim_{x\to0^+}\frac{x^{-1}}{e^{\frac{1}{x}}}=0$ (by (a))	1M for attempting to find $f_-'(0)$ or $f_+'(0)$
So, we have $\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=0=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}$.	
Thus, we have $f'(0)=0$.	1
By (b)(i), $f'(x)=0$ when $x<0$, and $f'(x)=e^{\frac{-1}{x}}\frac{1}{x^2}$ when $x>0$.	
Therefore, we have $\lim_{x\to0^+}f'(x)=\lim_{x\to0^+}e^{\frac{-1}{x}}\frac{1}{x^2}=\lim_{x\to0^+}\frac{x^{-2}}{e^{\frac{1}{x}}}=0$ (by (a)).	1M for considering $\lim_{x\to0^+}f'(x)$ -- either one
Moreover, we have $\lim_{x\to0^-}f'(x)=\lim_{x\to0^-}0=0$.	-- either one
Note that $f'(0)=0$. Hence, we have $\lim_{x\to0^+}f'(x)=f'(0)=\lim_{x\to0^-}f'(x)$.	
Thus, $f'(x)$ is continuous at $x=0$.	1
(iii) For any $x>0$, we have $f(x)=e^{\frac{-1}{x}}$. By (b)(i), we have $f'(x)=e^{\frac{-1}{x}}\frac{1}{x^2}=e^{\frac{-1}{x}}p_1\left(\frac{1}{x}\right)$, where $p_1(t)=t^2$ which is a polynomial in t .	
So, the statement is true for $n=1$.	1
Assume that $f^{(k)}(x)=e^{\frac{-1}{x}}p_k\left(\frac{1}{x}\right)$ for some positive integer k, where $p_k(t)$ is a polynomial in t. Then, we have $f^{(k+1)}(x)$ $=\frac{d}{dx}\left(e^{\frac{-1}{x}}p_k\left(\frac{1}{x}\right)\right)$ (by induction assumption)	
$=\frac{-1}{x^2}e^{\frac{-1}{x}}p_k'\left(\frac{1}{x}\right)+\frac{1}{x^2}e^{\frac{-1}{x}}p_k\left(\frac{1}{x}\right)$	1M
$=e^{\frac{-1}{x}}\left(\frac{-1}{x^2}p_k'\left(\frac{1}{x}\right)+\frac{1}{x^2}p_k\left(\frac{1}{x}\right)\right)$	
Putting $p_{k+1}(t)=-t^2p_k'(t)+t^2p_k(t)$, we have $p_{k+1}\left(\frac{1}{x}\right)=\frac{-1}{x^2}p_k'\left(\frac{1}{x}\right)+\frac{1}{x^2}p_k\left(\frac{1}{x}\right)$ and $p_{k+1}(t)$ is a polynomial in t . Hence, we have $f^{(k+1)}(x)$	1M for $p_{k+1}(t)$
$=e^{\frac{-1}{x}}p_{k+1}\left(\frac{1}{x}\right)$, where $p_{k+1}(t)$ is a polynomial in t .	1
Thus, by mathematical induction, the statement is true for any positive integer n.	

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Solution	Marks
(iv) By (b)(ii), we have $f'(0)=0$.	
So, the statement is true for $n=1$.	
Assume that $f^{(k)}(0)=0$ for some positive integer k.	
Then, we have	
$\lim_{x\to 0^-}\frac{f^{(k)}(x)-f^{(k)}(0)}{x-0}$	
$=\lim_{x\to 0^-}\frac{0-0}{x}$ (by (b)(i) and the induction assumption)	1M (either one)
$=\lim_{x\to 0^-} 0$	
$=0$	1
Also, we have	
$\lim_{x\to 0^+}\frac{f^{(k)}(x)-f^{(k)}(0)}{x-0}$	
$=\lim_{x\to 0^+}\frac{e^{\frac{-1}{x}}\,p_k\left(\frac{1}{x}\right)-0}{x}$ (by (b)(iii) and the induction assumption)	(either one)
$=\lim_{x\to 0^+}\frac{x^{-1}\,p_k(x^{-1})}{e^{\frac{1}{x}}}$	
$=0$ (by (a))	1
So, we have $\lim_{x\to 0^-}\frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} = 0 = \lim_{x\to 0^+}\frac{f^{(k)}(x)-f^{(k)}(0)}{x-0}$.	
Hence, we have $f^{(k+1)}(0)=0$ if $f^{(k)}(0)=0$.	
Thus, by mathematical induction, $f^{(n)}(0)=0$ for any positive integer n .	
	----------(12)

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Solution	Marks
10. (a) Case 1. $t=0$	
The equation of the normal to P at $(0,0)$ is $y=0$.	
Case 2. $t \neq 0$	
Differentiate both sides of $y^2=4ax$ w.r.t. x, we have	
$2y\frac{\mathrm{d}y}{\mathrm{d}x}=4a$	
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2a}{y}$	
$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{(at^2,\,2at)}=\frac{2a}{2at}=\frac{1}{t}$	1A
The equation of the normal to P at $(at^2,2at)$ is	
$\frac{y-2at}{x-at^2}=-t$	1M
$y-2at=-tx+at^3$	
$tx+y-2at-at^3=0$	1A
By combining the above two cases, the required equation is $tx+y-2at-at^3=0$.	
	------------(3)
(b) (i) By (a), the equation of the normal to P at $(at_1{}^2,2at_1)$ is	
$t_1x+y-2at_1-at_1{}^3=0$.	
Also, the equation of the normal to P at $(at_2{}^2,2at_2)$ is	
$t_2x+y-2at_2-at_2{}^3=0$.	
Since (h,k) lies on $t_1x+y-2at_1-at_1{}^3=0$ and $t_2x+y-2at_2-at_2{}^3=0$,	
we have $t_1h+k-2at_1-at_1{}^3=0$ and $t_2h+k-2at_2-at_1{}^3=0$.	1M for either one
So, we have $at_1{}^3+(2a-h)t_1-k=0$ and $at_2{}^3+(2a-h)t_2-k=0$.	
Thus, t_1 and t_2 are roots of the equation $at^3+(2a-h)t-k=0$.	1
Since $t_1 \neq t_2$, suppose that the roots of the equation $at^3+(2a-h)t-k=0$ are t_1 , t_2 and s .	
Then, we have $t_1+t_2+s=\frac{0}{a}=0$.	1M
So, we have $s=-(t_1+t_2)$.	
Since $t_3=-(t_1+t_2)$, we have $s=t_3$.	
Thus, the roots of the equation $at^3+(2a-h)t-k=0$ are t_1 , t_2 and t_3 .	1

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Solution	Marks
(ii) By (b)(i), we have $at_3^{\ 3}+(2a-h)t_3-k=0$.	
By (a), the equation of the normal to P at $(at_3^{\ 2},2at_3)$ is	
$t_3x+y-2at_3-at_3^{\ 3}=0$.	1M for using (a)
Note that $t_3h+k-2at_3-at_3^{\ 3}=-(at_3^{\ 3}+(2a-h)t_3-k)=0$.	
Thus, the normal to P at $(at_3^{\ 2},2at_3)$ passes through the point (h,k) .	1A f.t.
By (a), the equation of the normal to P at $(at_3^{\ 2},2at_3)$ is $t_3x+y-2at_3-at_3^{\ 3}=0$.	1M for using (a)
Note that $t_3h+k-2at_3-at_3^{\ 3}$ $=-(t_1+t_2)h+k+2a(t_1+t_2)+a(t_1+t_2)^3$ $=-(t_1+t_2)h+k+2a(t_1+t_2)+a(t_1^{\ 3}+3t_1^{\ 2}t_2+3t_1t_2^{\ 2}+t_2^{\ 3})$ $=(at_1^{\ 3}+(2a-h)t_1)+(at_2^{\ 3}+(2a-h)t_2)+k+3at_1t_2(t_1+t_2)$ $=k+k+k-3at_1t_2t_3$ $=3k-3a\left(\frac{k}{a}\right)$ (by (b)(i)) $=3k-3k$ $=0$	
Thus, the normal to P at $(at_3^{\ 2},2at_3)$ passes through the point (h,k) .	1A f.t.
(iii) By (b)(i), we have $t_1t_2+t_2t_3+t_3t_1=\frac{2a-h}{a}$ and $t_1t_2t_3=\frac{k}{a}$.	1A + 1A
	------------(8)
(c) Let $A(at_1^{\ 2},2at_1)$ and $B(at_2^{\ 2},2at_2)$ be two points on P at which the normals to P are perpendicular to each other.	
Suppose that the two normals intersect at the point (h,k) .	
Let $t_3=-(t_1+t_2)$.	
The slopes of the normals to P at A and B are $-t_1$ and $-t_2$ respectively.	
Therefore, we have $(-t_1)(-t_2)=-1$.	1M
So, we have $t_1t_2=-1$.	
By (b)(iii), we have $t_1t_2t_3=\frac{k}{a}$.	
Hence, we have $t_3=\frac{-k}{a}$.	1A (for either correct)
Again, by (b)(iii), we have $t_1t_2+t_2t_3+t_3t_1=\frac{2a-h}{a}$.	
Hence, we have $-1-t_3^{\ 2}=\frac{2a-h}{a}$.	(for either correct)
So, we have $-1-\left(\frac{-k}{a}\right)^2=\frac{2a-h}{a}$.	1M for eliminating t_3
Therefore, we have $k^2=a(h-3a)$.	1A
Thus, the required equation is $y^2=a(x-3a)$.	

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Solution	Marks
Let $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ be two points on P at which the normals to P are perpendicular to each other.	
Suppose that the two normals intersect at the point (x, y) .	
By the proof in (b)(i), we have $y = -t_1 x + 2at_1 + at_1^3$ and $y = -t_2 x + 2at_2 + at_3^3$.	
The slopes of the normals to P at A and B are $-t_1$ and $-t_2$ respectively.	
Therefore, we have $(-t_1)(-t_2) = -1$.	1M
So, we have $t_1 t_2 = -1$.	
Now, x	
$= 2a + a(t_1^2 + t_1 t_2 + t_2^2)$	
$= a(t_1^2 + t_2^2) + a$	
$= a(t_1 + t_2)^2 + 3a$	1A (either one)
Also, y	
$= -t_1(2a + a(t_1^2 + t_1 t_2 + t_2^2)) + 2at_1 + at_1^3$	
$= -at_1(t_1^2 + t_1 t_2 + t_2^2) + at_1^3$	
$= -at_1(t_1 t_2 + t_2^2)$	
$= a(t_1 + t_2)$	(either one)
So, we have	
$x = a\left(\dfrac{y}{a}\right)^2 + 3a$	1M for eliminating $t_1 + t_2$
$ax = y^2 + 3a^2$	
$y^2 = a(x - 3a)$	1A
Thus, the required equation is $y^2 = a(x - 3a)$.	
	------------(4)

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Solution	Marks
11. (a) (i) $h(a)=f(a)-f(a)-0=0$ $h(b)=f(b)-f(a)-(f(b)-f(a))=0$	1A for both correct
(ii) By Mean Value Theorem, there exists $\beta\in(a,b)$ such that	
$\frac{h(b)-h(a)}{b-a}=h'(\beta)$.	1M
By (a)(i), we have $h'(\beta)=0$.	1A
Now, we have $h'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x)$.	1A
So, we have $f'(\beta)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(\beta)=0$.	
Note that $g'(\beta)\neq 0$.	
Thus, we have $\frac{f'(\beta)}{g'(\beta)}=\frac{f(b)-f(a)}{g(b)-g(a)}$.	1
	------------(5)
(b) Note that F and G are differentiable in **R**, $G(x)\neq G(c)$ and $G'(t)\neq 0$ for all $t\in I$.	
By (a)(ii), there exists $\gamma\in I$ such that $\frac{F'(\gamma)}{G'(\gamma)}=\frac{F(c)-F(x)}{G(c)-G(x)}$.	1M withhold 1M for no checking
However, $F(x)=u(x)-u(x)-0=0$ and $G(x)=\frac{(x-x)^2}{2}=0$.	
So, there exists $\gamma\in I$ such that $\frac{F'(\gamma)}{G'(\gamma)}=\frac{F(c)}{G(c)}$.	1
Also, $F'(t)=-u'(t)-u''(t)(x-t)+u'(t)=-u''(t)(x-t)$. Moreover, $G'(t)=-(x-t)$.	1A for both correct
Therefore, we have $F'(\gamma)=-u''(\gamma)(x-\gamma)$ and $G'(\gamma)=-(x-\gamma)$.	
So, we have $\frac{-u''(\gamma)(x-\gamma)}{-(x-\gamma)}=\frac{u(x)-u(c)-u'(c)(x-c)}{\frac{(x-c)^2}{2}}$.	1M
Thus, we have $u(x)=u(c)+u'(c)(x-c)+\frac{u''(\gamma)}{2}(x-c)^2$.	1
	------------(5)
(c) (i) $v(0)$ $=\lim_{x\to 0}v(x)$ (since v is continuous at 0) $=\lim_{x\to 0}\left(x\left(\frac{v(x)}{x}\right)\right)$ $=\left(\lim_{x\to 0}x\right)\left(\lim_{x\to 0}\frac{v(x)}{x}\right)$ $=(0)(2006)$	
$=0$	1
$v'(0)$ $=\lim_{x\to 0}\frac{v(x)-v(0)}{x-0}$ $=\lim_{x\to 0}\frac{v(x)}{x}$	
$=2006$	1A

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Solution	Marks
(ii) Case 1. $x=0$	
$LS = v(0) = 0$ (by (c)(i))	
$RS = (2006)(0) + 0^2 = 0$	
So, the inequality holds for $x=0$.	1
Case 2. $x \neq 0$	
Putting $u = v$ and $c=0$ in the latter result of (b),	
there exists $d \in \mathbf{R}$ such that $v(x) = v(0) + v'(0)x + \frac{v''(d)}{2}x^2$.	1M
So, we have $v(x) = 2006x + \frac{v''(d)}{2}x^2$ (by (c)(i)) .	
Note that $v''(d) \geq 2$ and $x^2 \geq 0$.	
Thus, we have $v(x) \geq 2006x + x^2$.	1
By combining the above two cases, $v(x) \geq 2006x + x^2$ for all $x \in \mathbf{R}$.	
Case 1. $x \geq 0$	
$v''(t) \geq 2$ for all $t \in \mathbf{R}$	
$\int_0^s v''(t)\,dt \geq \int_0^s 2\,dt$ for all $s \geq 0$	1M (for either)
$v'(s) - v'(0) \geq 2s$ for all $s \geq 0$	
$v'(s) \geq 2s + 2006$ for all $s \geq 0$ (by (c)(i))	
$\int_0^x v'(s)\,ds \geq \int_0^x (2s+2006)\,ds$	
$v(x) - v(0) \geq x^2 + 2006x$	
$v(x) \geq 2006x + x^2$ (by (c)(i))	1
Case 2. $x < 0$	
$v''(t) \geq 2$ for all $t \in \mathbf{R}$	
$\int_s^0 v''(t)\,dt \geq \int_s^0 2\,dt$ for all $s < 0$	
$v'(0) - v'(s) \geq -2s$ for all $s < 0$	
$2006 - v'(s) \geq -2s$ for all $s < 0$ (by (c)(i))	
$-v'(s) \geq -(2s+2006)$ for all $s < 0$	
$-\int_x^0 v'(s)\,ds \geq -\int_x^0 (2s+2006)\,ds$	
$-v(0) + v(x) \geq x^2 + 2006x$	
$v(x) \geq 2006x + x^2$ (by (c)(i))	1
By combining the above two cases, $v(x) \geq 2006x + x^2$ for all $x \in \mathbf{R}$.	
	-----------(5)