SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. A sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 3$ and $a_{n+2} = 3a_{n+1} + a_n$ for all $n = 1, 2, 3 \ldots$. Using mathematical induction, prove that

$$a_n = \frac{1}{\sqrt{17}}\left[\left(\frac{3+\sqrt{17}}{2}\right)^n - \left(\frac{3-\sqrt{17}}{2}\right)^n\right]$$ for any positive integer n. (6 marks)

2. Let C_k^n be the coefficient of x^k in the expansion of $(1+x)^n$.

   (a) Using the identity $(1-x^2)^n = (1+x)^n(1-x)^n$, prove that the coefficient of x^n in the expansion of $(1-x^2)^n$ is $\sum_{k=0}^{n}(-1)^k\left(C_k^n\right)^2$.

   (b) Evaluate

   (i) $\sum_{k=0}^{2005}(-1)^k\left(C_k^{2005}\right)^2$,

   (ii) $\sum_{k=0}^{2006}(-1)^k\left(C_k^{2006}\right)^2$. (7 marks)

3. Let $p(x)$ be a polynomial of degree 4 with real coefficients satisfying $p(0)=0$, $p(1)=\frac{1}{2}$, $p(2)=\frac{2}{3}$, $p(3)=\frac{3}{4}$ and $p(4)=\frac{4}{5}$.

   (a) Let $q(x)=(x+1)p(x)-x$.

   (i) Evaluate $q(0)$, $q(1)$, $q(2)$, $q(3)$ and $q(4)$.

   (ii) Express $q(x)$ as a product of linear polynomials.

   (b) Evaluate $p(5)$. (6 marks)

4.

   (a) Resolve $\frac{9x+36}{x(x+2)(x+3)}$ into partial fractions.

   (b) Express $\sum_{k=1}^{n}\frac{9k+36}{k(k+2)(k+3)}$ in the form $A+\frac{B}{n+1}+\frac{C}{n+2}+\frac{D}{n+3}$, where A, B, C and D are constants.

   (c) Is there a positive integer N such that $\sum_{k=1}^{N}\frac{9k+36}{k(k+2)(k+3)} \geq 8$? Explain your answer. (7 marks)

5. For every positive integer n, define $x_n = \sum_{k=1}^{n}\frac{1}{n+k}$ and $y_n = \sum_{k=1}^{n+1}\frac{1}{n+k}$.

   (a) Prove that the sequence $\{x_n\}$ is strictly increasing and that the sequence $\{y_n\}$ is strictly decreasing.

   (b) Prove that the sequence $\{x_n\}$ and $\{y_n\}$ converge to the same limit. (7 marks)

6. Let a, b and c be positive real numbers such that $a^2+b^2+c^2=3$.

   (a) Using Cauchy-Schwarz's inequality, prove that

   (i) $a+b+c \leq 3$.

   (ii) $a^3+b^3+c^3 \geq 3$.

   (b) For every $n = 2, 3, 4\ldots$, let $P(n)$ be the statement $a^n+b^n+c^n \geq 3$. Prove that for any integer $k \geq 2$,

   (i) If $P(k)$ is true, then $P(2k)$ is true;

   (ii) If $P(k)$ is true, then $P(2k-1)$ is true. (7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write you answer in the AL(A) answer book.

7. Consider the system of linear equations in x, y, z

$$(E):\begin{cases} x+ay+z=4 \\ x+(2-a)y+(3b-1)z=3 \\ 2x+(a+1)y+(b+1)z=7 \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

   (a) Prove that (E) has a unique solution if and only if $a \neq 1$ and $b \neq 0$. Solve (E) in this case. (6 marks)

   (b)

   (i) For a=1, find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b.

   (ii) Is there a real solution (x, y, z) of

$$\begin{cases} x+y+z=4 \\ 2x+2y+z=6 \\ 4x+4y+3z+14 \end{cases}$$

   satisfying $x^2-2y^2-z=14$? Explain you answer. (7 marks)

   (c) Is (E) consistent for b=0? Explain your answer. (2 marks)

8. Let $M=\begin{pmatrix} m & -m \\ m & m \end{pmatrix}$, where $m>0$.

(a) Evaluate M^2. (1 mark)

(b) Let $X=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-zero real matrix such that $MX=XM$.

(i) Prove that $c=-b$ and $d=a$.

(ii) Prove that X is a non-singular matrix.

(iii) Suppose that $X-6X^{-1}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(1) Find X.

(2) If $a>0$ and $(M-kX)^2=-M^2$, express k in terms of m.

(10 marks)

(c) Using the result of (b)(iii)(2), find two real matrices P and Q, other than M and $-M$, such that $P^4=Q^4=M^4$. (4 marks)

9.

(a) Let $b, c, d \in \mathbf{R}$ and α, β and γ be the roots of the equation $x^3+bx^2+cx+d=0$.

for every positive integer k, define $S_k=\alpha^k+\beta^k+\gamma^k$.

(i) Using relations between coefficients and roots, express S_1, S_2 and S_3 in terms of b, c and d.

(ii) Prove that $S_{k+1}+bS_{k+2}+cS_{k+1}+dS_k=0$ for any positive integer k.

(iii) Suppose $d=bc$. Using the results of (a)(i) and (a)(ii) prove that

$S_{2n1}+bS_{2n}=(-1)^n 2bc^n$ and $S_{2n}+bS_{2n-1}=(-1)^n 2c^n$ for any positive integer n.

(11 marks)

(b) Find three numbers such that their sum is 3, the sum of their square is 3203 and the sum of their cubes is 9603. (4 marks)

10.

(a) By differentiation $f(x)=x\ln x-x$ prove that $x\ln x-1+1\geq 0$ for all $x>0$. (4 marks)

(b) Let a be a positive real number. Define $g(x)=\dfrac{a^x-1}{x}$ for all $x>0$. Prove that g is increasing. (3 marks)

(c) Let p and q be real numbers such that $p>q>0$.

(i) Suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $\sum_{k=1}^{n} a_k{}^q=n$.

Using (b), prove that $\sum_{k=1}^{n} a_k{}^p \geq n$.

(ii) Suppose that $b_1, b_2, \ldots, b_n$ are positive real numbers.

Using (c)(i), prove that $\left(\dfrac{1}{n}\sum_{k=1}^{n} b_k{}^p\right)^{\frac{1}{p}} \geq \left(\dfrac{1}{n}\sum_{k=1}^{n} b_k{}^q\right)^{\frac{1}{q}}$.

Hence prove that $\left(\dfrac{1}{n}\sum_{k=1}^{n} b_k{}^{\frac{1}{p}}\right)^p \geq \left(\dfrac{1}{n}\sum_{k=1}^{n} b_k{}^{\frac{1}{q}}\right)^q$. (8 marks)

11. Let $0<\theta<\pi$.

(a) Solve the equation $z^2=\cos\theta+i\sin\theta$. (2 marks)

(b) Let u_1 and u_2 be the roots of the equation $(u+1)^2=\cos\theta+i\sin\theta$, where $\mathrm{Im}(u_1)<0$.

(i) Find u_1 and u_2.

(ii) Prove that $\dfrac{u_2}{u_1}=-i\tan\dfrac{\theta}{4}$.

Hence, find all the integers n for which $\left(\dfrac{u_2}{u_1}\right)^n$ is a real number.

(iii) Prove that $u_1{}^{12}=2^{12}\cos^{12}\dfrac{\theta}{4}(\cos 3\theta+i\sin 3\theta)$.

Hence, find all the values of θ for which $u_1{}^{12}-u_2{}^{12}$ is a real number. (13 marks)

END OF PAPER

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answer in the AL(E) answer book.

1. Evaluate

(a) $\lim_{x\to 0}\dfrac{\int^{3x+1}\sqrt{t^5+t^3+1}\,dt}{\ln(x+1)}$,

(b) $\lim_{x\to 0}\sin x\sin\dfrac{1}{x}$. (6 marks)

2. Let $f:\mathbf{R}\to\mathbf{R}$ be defined by

$$f(x)=\begin{cases}-1 \text{ when } x \text{ is an even number,}\\ 2 \text{ when } x \text{ is not an even number.}\end{cases}$$

(a)
  (i) Sketch the graph of $y=f(x)$ for $-4\le x\le 4$.
  (ii) Is f a periodic function? Explain your answer.

(b) Let $g:\mathbf{R}\to\mathbf{R}$ be defined by $g(x)=f(x+1)+f(x)$.
  (i) Sketch the graph of $y=g(x)$ for $-4\le x\le 4$.
  (ii) Is g an injective function? Explain your answer.

(7 marks)

3. For any positive integers m and n, define $I_{m,n}=\int_0^{\frac{\pi}{4}}\dfrac{\sin^m\theta}{\cos^n\theta}d\theta$.

(a) Prove that $I_{m+2,n+2}=\dfrac{1}{n+1}\left(\dfrac{1}{\sqrt{2}}\right)^{m-n}-\dfrac{m+1}{n+1}I_{m,n}$.

(b) Using the substitution $u=\cos\theta$, evaluate $I_{3,1}$.

(c) Using the results of (a) and (b), evaluate $I_{7,5}$. (7 marks)

4.

(a) Using the substitution $t=\sqrt{1+x^2}$, find $\int\dfrac{x^3}{\sqrt{1+x^2}}dx$.

(b) Evaluate $\lim_{n\to\infty}\dfrac{1}{n^3}\sum_{k=1}^{n}\dfrac{k^3}{\sqrt{n^2+k^2}}$. (7 marks)

5.

(a) Find $\int\ln y\,dy$.

(b) Find the volume of the solid of revolution generated by revolving the region bounded by the curve $y=2^{x^2}$ and the straight line $y=2$ about the y-axis.

(6 marks)

6. Let the equation of ellipse E be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where a and b are two distinct positive constants. The coordinates of the points P and Q are ($a\cos\theta$, $b\sin\theta$) and ($(a+b)\cos\theta$, $(a+b)\sin\theta$) respectively, where $0<\theta<\dfrac{\pi}{2}$.

(a) Prove that
  (i) P lies on E,
  (ii) The straight line passing through P and Q is the normal to E at P.

(b) Let c be a constant such that the straight line $x\sin\theta-y\cos\theta=c$ is a tangent to E. Express the distance between P and Q in terms of c.

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answer in the AL (A) answer book.

7. Let $f(x)=\dfrac{x^2-x-6}{x+6}$ $(x\ne 6)$.

(a) Find $f'(x)$ and $f''(x)$. (2 marks)

(b) Solve each of the following inequalities:
  (i) $f'(x)>0$
  (ii) $f'(x)<0$
  (iii) $f''(x)>0$
  (iv) $f''(x)<0$. (3 marks)

(c) Find the relative extreme point(s) of the graph of $y=f(x)$. (2 marks)

(d) Find the asymptote(s) of the graph of $y=f(x)$. (3 marks)

(e) Sketch the graph of $y=f(x)$. (2 marks)

(f) Find the area of the region bounded by the graph of $y=f(x)$ and the x-axis. (3 marks)

8.

(a)

(i) For any two distinct positive integers m and n, evaluate

$\int_0^\pi \cos mx \cos nx dx$.

(ii) For any positive integer n, evaluate $\int_0^\pi \cos nx dx$ and $\int_0^\pi \cos^2 nx dx$.

(5 marks)

(b) Using integration by parts, prove that $\int_0^\pi x^2 \cos nx dx = \frac{(-1)^n 2\pi}{n^2}$ for any positive integer n. (3 marks)

(c) Let N be a positive integer and $f(x) = a_0 + \sum_{m=1}^{N} a_m \cos mx$, where a_0, a_1, … a_N are constants. It is given that $\int_0^\pi (f(x) - x^2)dx = 0$ and

$\int_0^\pi (f(x) - x^2)\cos nx dx = 0$ for all $n = 1, 2, \ldots, N$.

(i) Find a_0.

(ii) Prove that $a_n = \frac{(-1)^n 4}{n^2}$ for all $n = 1, 2, \ldots, N$.

(iii) For any positive integer k, let $I_k = \int_0^\pi (f(x) - x^2)\cos(N+k)x dx$.

Evaluate $\lim_{k\to\infty} I_k$. (7 marks)

9.

(a) Prove that $\lim_{x\to 0^+} \frac{x^{-n}}{e^{\frac{1}{x}}} = 0$ for any positive integer n. (3 marks)

(b) Let $f(x) = \begin{cases} 0 \text{ when } x \le 0. \\ e^{\frac{-1}{x}} \text{ when } x > 0. \end{cases}$

(i) Find $f'(x)$ for all $x \ne 0$.

(ii) Prove that $f'(0) = 0$.

Hence prove that $f'(x)$ is continuous at $x = 0$.

(iii) For any $x>0$, prove that $f^{(n)}(x) = e^{\frac{-1}{x}} p_n\left(\frac{1}{x}\right)$ for any positive integer n, where $p_n(t)$ is a polynomial in t.

(iv) Prove that $f^{(n)}(0) = 0$ for any positive integer n. (12 marks)

10. Let the equation of the parabola P be $y^2 = 4ax$, where a is a non-zero constant.

(a) Find the equation of the normal to P at the point (at^2, $2at$). (3 marks)

(b) The normals to P at two distinct points (at_1^2, $2at_1$) and (at_2^2, $2at_2$) intersect at the point (h, k). Let $t_3 = -(t_1 + t_2)$.

(i) Prove that the roots of the equation $at^3 + (2a - h)t - k = 0$ are t_1, t_2 and t_3.

(ii) Does the normal to P at the point (at_3^2, $2at_3$) pass through the point (h,k)? Explain your answer.

(iii) Express $t_1t_2 + t_2t_3 + t_3t_1$ and $t_1t_2t_3$ in terms of a, h and k. (8 marks)

(c) Let A and B be two points on P at which the normals to P are perpendicular to each other. Using the results of (b)(iii), or otherwise, find the equation of the locus of the point of intersection of the two normals as A and B vary.

(4 marks)

11.

(a) Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be continuous on [a, b] and differentiable in (a, b), where $a<b$. Suppose that $g(a) \ne g(b)$ and $g'(x) \ne 0$ for all $x \in (a, b)$. Define $h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$ for all $x \in \mathbf{R}$.

(i) Find $h(a)$ and $h(b)$.

(ii) Using Meaning Value Theorem, prove that there exists $\beta \in (a, b)$ such that $\frac{f'(\beta)}{g'(\beta)} = \frac{f(b) - f(a)}{g(b) - g(a)}$. (5 marks)

(b) Let $u: \mathbf{R} \to \mathbf{R}$ be twice differentiable.

For each $x \in \mathbf{R}$, $F: \mathbf{R} \to \mathbf{R}$ and $G: \mathbf{R} \to \mathbf{R}$ be defined by

$F(t) = u(x) - u(t) - u'(t)(x - t)$ and $G(t) = \frac{(x-t)^2}{2}$. For each $c \ne x$, prove that there exist $\gamma \in I$ such that $\frac{F'(\gamma)}{G'(\gamma)} = \frac{F(c)}{G(c)}$ and

$u(x) = u(c) + u'(c)(x - c) + \frac{u''(\gamma)}{2}(x - c)^2$, where I is the open interval with end points c and x. (5 marks)

(c) Let $v: \mathbf{R} \to \mathbf{R}$ be twice differentiable. It is given that $\lim_{x\to 0} \frac{v(x)}{x} = 2006$.

(i) Prove that $v(0) = 0$. Hence find $v'(0)$.

(ii) Suppose that $v''(x) \ge 2$ for all $x \in \mathbf{R}$. Prove that $v(x) \ge 2006x + x^2$ for all $x \in \mathbf{R}$. (5 marks)