Solution	Marks
1. (a) $2S_{n+1}+4S_n$	
$=2\left((1+\sqrt{5})^{n+1}+(1-\sqrt{5})^{n+1}\right)+4\left((1+\sqrt{5})^n+(1-\sqrt{5})^n\right)$	
$=(1+\sqrt{5})^n\left(2(1+\sqrt{5})+4\right)+(1-\sqrt{5})^n\left(2(1-\sqrt{5})+4\right)$	
$=(1+\sqrt{5})^n(6+2\sqrt{5})+(1-\sqrt{5})^n(6-2\sqrt{5})$	
$=(1+\sqrt{5})^n(1+\sqrt{5})^2+(1-\sqrt{5})^n(1-\sqrt{5})^2$	1M for using $(1\pm\sqrt{5})^2=6\pm2\sqrt{5}$
$=(1+\sqrt{5})^{n+2}+(1-\sqrt{5})^{n+2}$	
$=S_{n+2}$	1
(b) Note that $S_1=2$ which is divisible by 2^1 and $S_2=12=(3)(2^2)$ which is divisible by 2^2.	1 pp−1 for writing '$n=1$ and $n=2$ are true'
Assume that $S_k=2^k p$ and $S_{k+1}=2^{k+1}q$ for some positive integer k, where p and q are integers.	1M
Then, we have	
S_{k+2}	
$=2S_{k+1}+4S_k$ (by (a))	
$=2(2^{k+1}q)+4(2^k p)$ (by induction assumption)	1M for using induction assumption
$=2^{k+2}(p+q)$	
Therefore, S_{k+2} is divisible by 2^{k+2} if S_{k+1} is divisible by 2^{k+1} and S_k is divisible by 2^k.	
Thus, by mathematical induction, S_n is divisible by 2^n for every positive integer n.	1
	----------(6)

Solution	Marks
2. (a) $T_{r+1} \le T_r$	
$\Leftrightarrow C_r^{kn} x^r \le C_{r-1}^{kn} x^{r-1}$	
$\Leftrightarrow \frac{(kn)!}{r!(kn-r)!} x^r \le \frac{(kn)!}{(r-1)!(kn-r+1)!} x^{r-1}$	1A for both C_r^{kn} and C_{r-1}^{kn} correct
$\Leftrightarrow \frac{kn-r+1}{r} x \le 1$ (since $x > 0$)	
$T_{r+1} \le T_r$	
$\Leftrightarrow C_r^{kn} x^r \le C_{r-1}^{kn} x^{r-1}$	
$\Leftrightarrow \frac{(kn)!}{r!(kn-r)!} x^r - \frac{(kn)!}{(r-1)!(kn-r+1)!} x^{r-1} \le 0$	1A for both C_r^{kn} and C_{r-1}^{kn} correct
$\Leftrightarrow \frac{(kn)! x^{r-1}}{(r-1)!(kn-r)!}\left(\frac{x}{r} - \frac{1}{kn-r+1}\right) \le 0$	
Since $x = \frac{2}{k}$, we have	
$T_{r+1} \le T_r$	
$\Leftrightarrow \left(\frac{kn-r+1}{r}\right)\left(\frac{2}{k}\right) \le 1$	1M
$\Leftrightarrow 2kn - 2r + 2 \le rk$	
$\Leftrightarrow r \ge \frac{2(kn+1)}{k+2}$	1A
(b) Putting $n = 17$ and $k = 3$ in (a), we have	
$T_{r+1} \le T_r \Leftrightarrow r \ge \frac{2(51+1)}{3+2}$.	1M for putting $n = 17$ and $k = 3$
Therefore, we have $T_{r+1} \le T_r \Leftrightarrow r \ge 20.8$.	1A
So, we have $T_1 \le T_2 \le \cdots \le T_{21}$ and $T_{21} \ge T_{22} \ge \cdots \ge T_{52}$.	
Thus, the greatest term is $T_{21} = C_{20}^{51}\left(\frac{2}{3}\right)^{20}$.	1A
	----------(6)

Solution	Marks
3. (a) Let $f(x) = x - \ln(x+1)$ for all $x > -1$. Then, we have	
$f'(x)$	
$= 1 - \frac{1}{x+1}$	1A
$= \frac{x}{x+1}$	
$\begin{cases} < 0 & \text{if } -1 < x < 0 \\ = 0 & \text{if } x = 0 \\ > 0 & \text{if } x > 0 \end{cases}$	1M for testing +1A
Therefore, f attains its absolute minimum at 0.	
Thus, we have	
$f(x) \geq f(0)$ for all $x > -1$	
$x - \ln(x+1) \geq 0$ for all $x > -1$	
$x \geq \ln(x+1)$ for all $x > -1$	1
(b) Let n be a positive integer.	
Putting $x = \frac{1}{n}$ in (a), we have $\frac{1}{n} \geq \ln\left(\frac{1}{n} + 1\right)$.	1M
Hence, we have $\frac{1}{n} \geq \ln \frac{n+1}{n}$.	
So, we have $\sum_{n=1}^{k} \frac{1}{n} \geq \sum_{n=1}^{k} \ln \frac{n+1}{n}$ for all positive integers k.	
Therefore, we have $\sum_{n=1}^{k} \frac{1}{n} \geq \ln \prod_{n=1}^{k} \frac{n+1}{n}$ for all positive integers k.	
Hence, we have $\sum_{n=1}^{k} \frac{1}{n} \geq \ln(k+1)$ for all positive integers k.	1A
Since $\ln(k+1) \to \infty$ as $k \to \infty$, we have $\sum_{n=1}^{k} \frac{1}{n} \to \infty$ as $k \to \infty$.	
Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.	1
	----------(7)

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Solution	Marks
4. (a) Let $r(x)=Ax+B$, where A and B are constants.	
Note that $f(x)=(x-2)(x+3)q_1(x)+Ax+B$ for some polynomial $q_1(x)$.	1M for using Division Algorithm
Since $f(2)=4$ and $f(-3)=-6$,	
we have $2A+B=4$ and $-3A+B=-6$.	1M
Solving, we have $A=2$ and $B=0$.	1A for both correct
Thus, we have $r(x)=2x$.	
(b) By (a), we have $g(x)=(x-2)(x+3)q_1(x)$.	
Note that $g(x)=(x^2+1)q_2(x)$ for some polynomial $q_2(x)$.	1A
$\because$ $(x-2)(x+3)$ and x^2+1 are relatively prime and degree of $g(x)$ is 4 .	
$\therefore$ $g(x)=k(x-2)(x+3)(x^2+1)$ for some non-zero constant k	1A
$\because$ $g(1)=-16$	
$\therefore$ $-8k=-16$	1M for finding the leading coefficient
Solving, we have $k=2$.	1A
Thus, we have $g(x)$ $=2(x-2)(x+3)(x^2+1)$ $=2(x^4+x^3-5x^2+x-6)$ $=2x^4+2x^3-10x^2+2x-12$	
	-----------(7)

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Solution	Marks
5. (a) (i) $y=\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}=\begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix}$.	1A
(ii) Note that $A=\begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}$.	
Thus, T_1 is the rotation in the Cartesian plane anticlockwise about the	1A for rotation
origin by $\frac{\pi}{6}$.	1A for the details
(iii) By (a)(ii), we have $A^{12}=I$.	1M
Thus, we have A^{2005} $=A(A^{12})^{167}$ $=A(I^{167})$ $=A$	1A
Note that $A^{2005}=\begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}^{2005}=\begin{pmatrix}\cos\frac{2005\pi}{6} & -\sin\frac{2005\pi}{6}\\ \sin\frac{2005\pi}{6} & \cos\frac{2005\pi}{6}\end{pmatrix}$.	1M
So, we have $A^{2005}=A$.	1A
(b) Since the angle between $\begin{pmatrix}0\\2\end{pmatrix}$ and $\begin{pmatrix}-\sqrt{2}\\ \sqrt{2}\end{pmatrix}$ is $\frac{\pi}{4}$ which is not a multiple of $\frac{\pi}{6}$,	1M
there is no positive integer m such that T_m transforms $\begin{pmatrix}0\\2\end{pmatrix}$ to $\begin{pmatrix}-\sqrt{2}\\ \sqrt{2}\end{pmatrix}$.	1A must show reasons
By (a)(ii), we have $A^m=\begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}^m=\begin{pmatrix}\cos\frac{m\pi}{6} & -\sin\frac{m\pi}{6}\\ \sin\frac{m\pi}{6} & \cos\frac{m\pi}{6}\end{pmatrix}$.	
Suppose that $\begin{pmatrix}-\sqrt{2}\\ \sqrt{2}\end{pmatrix}=\begin{pmatrix}\cos\frac{m\pi}{6} & -\sin\frac{m\pi}{6}\\ \sin\frac{m\pi}{6} & \cos\frac{m\pi}{6}\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix}$ for some $m\in\mathbb{N}$.	
Then, we have $-2\sin\frac{m\pi}{6}=-\sqrt{2}$ and $2\cos\frac{m\pi}{6}=\sqrt{2}$.	1M
So, we have $\frac{m\pi}{6}=2k\pi+\frac{\pi}{4}$ for some integer k .	
Therefore, we have $2m=24k+3$ which is impossible.	
Thus, there is no positive integer m such that T_m transforms $\begin{pmatrix}0\\2\end{pmatrix}$ to $\begin{pmatrix}-\sqrt{2}\\ \sqrt{2}\end{pmatrix}$.	1A must show reasons
	-----------(7)

Solution	Marks
6. (a) $z^2+2z\cos 6\theta+1=0$	
$z=\dfrac{-2\cos 6\theta\pm\sqrt{(2\cos 6\theta)^2-4}}{2}$	1A
$z=-\cos 6\theta\pm\sqrt{\cos^2 6\theta-1}$	
$z=-\cos 6\theta+i\sin 6\theta$ or $z=-\cos 6\theta-i\sin 6\theta$	1A or equivalent
(b) For $x^6+2x^3\cos 6\theta+1=0$, we have $x^3=-\cos 6\theta\pm i\sin 6\theta$.	1M for putting $z=x^3$ in (a)
Therefore, we have $x^3=-(\cos(\pm 6\theta)+i\sin(\pm 6\theta))$.	
So, all the roots of $x^6+2x^3\cos 6\theta+1=0$ are	
$x=-\left(\cos\dfrac{2k\pi\pm 6\theta}{3}+i\sin\dfrac{2k\pi\pm 6\theta}{3}\right)$, where $k=0,1,2$.	1A or equivalent
Let $x_k=-\left(\cos\dfrac{2k\pi+6\theta}{3}+i\sin\dfrac{2k\pi+6\theta}{3}\right)$, where $k=0,1,2$.	
Note that $\overline{x_0}=-\left(\cos\dfrac{6\theta}{3}-i\sin\dfrac{6\theta}{3}\right)=-\left(\cos\dfrac{6\pi-6\theta}{3}+i\sin\dfrac{6\pi-6\theta}{3}\right)$.	
Also, for each $k=1,2$, we have	
$\overline{x_k}=-\left(\cos\dfrac{2k\pi+6\theta}{3}-i\sin\dfrac{2k\pi+6\theta}{3}\right)$	
$=-\left(\cos(2\pi-\dfrac{2k\pi+6\theta}{3})+i\sin(2\pi-\dfrac{2k\pi+6\theta}{3})\right)$	
$=-\left(\cos\dfrac{2(3-k)\pi-6\theta}{3}+i\sin\dfrac{2(3-k)\pi-6\theta}{3}\right)$	
Therefore, all the roots of $x^6+2x^3\cos 6\theta+1=0$ are x_0, $\overline{x_0}$, x_1, $\overline{x_1}$, x_2 and $\overline{x_2}$.	
Thus,	
$x^6+2x^3\cos 6\theta+1$	
$=\prod_{k=0}^{2}\left((x-x_k)(x-\overline{x_k})\right)$	1M for writing in factor form
$=\prod_{k=0}^{2}\left(x^2-(x_k+\overline{x_k})x+\lvert x_k\rvert^2\right)$	1A
$=\prod_{k=0}^{2}\left(x^2+2x\cos\dfrac{2k\pi+6\theta}{3}+1\right)$	1A or equivalent
$=\left(x^2+2x\cos 2\theta+1\right)\left(x^2+2x\cos\dfrac{2\pi+6\theta}{3}+1\right)\left(x^2+2x\cos\dfrac{4\pi+6\theta}{3}+1\right)$	
	---------(7)

Solution	Marks
7. (a) (i) (E) has a unique solution	
$\Leftrightarrow \quad \Delta \neq 0$	
$\Leftrightarrow \quad \Delta = \begin{vmatrix} 1 & a & 1 \\ 2 & a+3 & a-1 \\ 3 & a^2 & 4a+1 \end{vmatrix} \neq 0$	1M
$\Leftrightarrow \quad -a^3+2a^2+5a-6 \neq 0$	1A
$\Leftrightarrow \quad -(a+2)(a-1)(a-3) \neq 0$	
$\Leftrightarrow \quad a \neq -2$, $a \neq 1$ and $a \neq 3$	1A
$\Leftrightarrow \quad a < -2$, $-2 < a < 1$, $1 < a < 3$ or $a > 3$	
The augmented matrix of (E) is	
$\left(\begin{array}{ccc\|c} 1 & a & 1 & b \\ 2 & a+3 & a-1 & 0 \\ 3 & a^2 & 4a+1 & -b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & a & 1 & b \\ 0 & 3-a & a-3 & -2b \\ 0 & a^2-3a & 4a-2 & -4b \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & a & 1 & b \\ 0 & 3-a & a-3 & -2b \\ 0 & 0 & (a+2)(a-1) & -2b(a+2) \end{array}\right)$	1A
(E) has a unique solution	
$\Leftrightarrow \quad (a+2)(a-1) \neq 0$ and $3-a \neq 0$	1M
$\Leftrightarrow \quad a \neq -2$, $a \neq 1$ and $a \neq 3$	1A
$\Leftrightarrow \quad a < -2$, $-2 < a < 1$, $1 < a < 3$ or $a > 3$	
When (E) has a unique solution,	
$x = \dfrac{\begin{vmatrix} b & a & 1 \\ 0 & a+3 & a-1 \\ -b & a^2 & 4a+1 \end{vmatrix}}{\Delta} = \dfrac{-b(a+2)(a^2-6a-3)}{\Delta}$	1M for Cramer's rule
$= \dfrac{b(a^2-6a-3)}{(a-1)(a-3)}$	
$y = \dfrac{\begin{vmatrix} 1 & b & 1 \\ 2 & 0 & a-1 \\ 3 & -b & 4a+1 \end{vmatrix}}{\Delta} = \dfrac{-4b(a+2)}{\Delta}$	
$= \dfrac{4b}{(a-1)(a-3)}$	
$z = \dfrac{\begin{vmatrix} 1 & a & b \\ 2 & a+3 & 0 \\ 3 & a^2 & -b \end{vmatrix}}{\Delta} = \dfrac{2b(a+2)(a-3)}{\Delta}$	
$= \dfrac{-2b}{a-1}$	1A+1A (1A for any one, 1A for all)

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Solution	Marks

When (E) has a unique solution, the augmented matrix of (E) becomes

$$\left(\begin{array}{ccc|c} 1 & a & 1 & b \\ 0 & 3-a & a-3 & -2b \\ 0 & 0 & 1 & \frac{-2b}{a-1} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & 1 & 0 \\ 0 & 1 & -1 & \frac{2b}{a-3} \\ 0 & 0 & 1 & \frac{-2b}{a-1} \end{array}\right)$$ 1M

$$\sim \left(\begin{array}{ccc|c} 1 & a & 0 & \frac{b(a+1)}{a-1} \\ 0 & 1 & 0 & \frac{4b}{(a-1)(a-3)} \\ 0 & 0 & 1 & \frac{-2b}{a-1} \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{b(a^2-6a-3)}{(a-1)(a-3)} \\ 0 & 1 & 0 & \frac{4b}{(a-1)(a-3)} \\ 0 & 0 & 1 & \frac{-2b}{a-1} \end{array}\right)$$

$\therefore \quad x = \dfrac{b(a^2-6a-3)}{(a-1)(a-3)}$, $y = \dfrac{4b}{(a-1)(a-3)}$, $z = \dfrac{-2b}{a-1}$. 1A+1A (1A for any one, 1A for all)

(ii) (1) When $a = 1$, the augmented matrix of (E) becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 2 & -2 & -2b \\ 0 & 0 & 0 & -6b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & -b \\ 0 & 0 & 0 & b \end{array}\right)$$

(E) is consistent when $b = 0$. 1A

Therefore, the solution set is $\{(-2t, t, t) : t \in \mathbf{R}\}$. 1A or equivalent

(2) When $a = -2$, the augmented matrix of (E) becomes

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & 5 & -5 & -2b \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & \frac{3b}{5} \\ 0 & 1 & -1 & \frac{-2b}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(E) is consistent for all real values of b . 1A

Therefore, the solution set is $\left\{\left(\dfrac{5t+3b}{5}, t, \dfrac{5t+2b}{5}\right) : t \in \mathbf{R}\right\}$. 1A or equivalent

---------(10)

Solution	Marks
8. (a) (i) Note that $\det M = 0 \Rightarrow qr = ps$.	
M^2	
$=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}$	
$=\begin{pmatrix} p^2+qr & pq+qs \\ pr+rs & qr+s^2 \end{pmatrix}$	1M
$=\begin{pmatrix} p^2+ps & pq+qs \\ pr+rs & ps+s^2 \end{pmatrix}$	
$=\begin{pmatrix} (p+s)p & (p+s)q \\ (p+s)r & (p+s)s \end{pmatrix}$	
$=(p+s)\begin{pmatrix} p & q \\ r & s \end{pmatrix}$	
$=(p+s)M$	1
Assume that $M^{k+1}=(p+s)^k M$ for some positive integer k . Then,	
M^{k+2}	
$=M^{k+1}M$	
$=(p+s)^k M^2$ (by induction assumption)	1M
$=(p+s)^k(p+s)M$ (by the case of $n=1$)	
$=(p+s)^{k+1}M$	1
Thus, by mathematical induction, $M^{n+1}=(p+s)^n M$ for any positive integer n .	
(ii) (1) $(p+s)^2-(4)(1)(ps-qr)$	1M for considering the discriminant
$=p^2+2ps+s^2-4ps+4qr$	
$=p^2-2ps+s^2+4qr$	
$=(p-s)^2+4qr$	
>0 (since $qr>0$)	1
Thus, α and β are two distinct real numbers.	
(2) $M^2-(\alpha+\beta)M+\alpha\beta I$	
$=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}-(p+s)\begin{pmatrix} p & q \\ r & s \end{pmatrix}+(ps-qr)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
$=\begin{pmatrix} p^2+qr-p^2-ps+ps-qr & pq+qs-pq-qs \\ pr+rs-pr-rs & qr+s^2-ps-s^2+ps-qr \end{pmatrix}$	
$=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1

Solution	Marks
(b) By (a)(ii)(2), putting $x=\frac{5t+3b}{5}$, $y=t$ and $z=\frac{5t+2b}{5}$ in	
$x^2+y^2+z^2=b+3$, we have $\left(\frac{5t+3b}{5}\right)^2+t^2+\left(\frac{5t+2b}{5}\right)^2=b+3$.	1M
So, we have $75t^2+50bt+13b^2-25b-75=0$.	1A or equivalent
$\because$ t is a real number.	
$\therefore$ we have $(50b)^2-4(75)(13b^2-25b-75)\geq 0$.	1M for using discriminant ≥ 0
Hence, we have $14b^2-75b-225\leq 0$.	1A
Therefore, we have $(2b-15)(7b+15)\leq 0$.	
Thus, we have $-\frac{15}{7}\leq b\leq\frac{15}{2}$.	1A
	----------(5)

Solution	Marks
(3) $AB=(M-\alpha I)(M-\beta I)=M^2-(\alpha+\beta)M+\alpha\beta I=0$ (by (2)) $BA=(M-\beta I)(M-\alpha I)=M^2-(\alpha+\beta)M+\alpha\beta I=0$ (by (2))	
Thus, we have $AB=BA=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Also,	1
$\det A=\det(M-\alpha I)=(\alpha-p)(\alpha-s)-qr=\alpha^2-(p+s)\alpha+\det M$ $\det B=\det(M-\beta I)=(\beta-p)(\beta-s)-qr=\beta^2-(p+s)\beta+\det M$ Note that the roots of $x^2-(p+s)x+\det M=0$ are α and β.	
Therefore, we have $\det A=\det B=0$.	1
Moreover, for $M=\lambda A+\mu B$, it is equivalent to have $M=(\lambda+\mu)M-(\alpha\lambda+\beta\mu)I$. It is sufficient to choose λ and μ such that $\lambda+\mu=1$ and $\alpha\lambda+\beta\mu=0$.	1M
By (1), we have $\lambda=\dfrac{\beta}{\beta-\alpha}$ and $\mu=\dfrac{\alpha}{\alpha-\beta}$.	1A for both correct
	---------(11)
(b) Putting $p=1$, $q=2$, $r=4$ and $s=3$ in (a), we have $\det M=-5$ and $qr=8>0$. Choose $\alpha=5$ and $\beta=-1$.	1A accept $\alpha=-1$ and $\beta=5$
Also, we have $A=M-\alpha I=\begin{pmatrix}-4&2\\4&-2\end{pmatrix}$ and $B=M-\beta I=\begin{pmatrix}2&2\\4&4\end{pmatrix}$.	
Moreover, we have $\lambda=\dfrac{-1}{-1-5}=\dfrac{1}{6}$ and $\mu=\dfrac{5}{5-(-1)}=\dfrac{5}{6}$.	
Thus, we have $\begin{pmatrix}1&2\\4&3\end{pmatrix}^n$	
$=\left(\dfrac{1}{6}\begin{pmatrix}-4&2\\4&-2\end{pmatrix}+\dfrac{5}{6}\begin{pmatrix}2&2\\4&4\end{pmatrix}\right)^n$ (by (a)(ii)(3))	1M
$=\left(\begin{pmatrix}\frac{-2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{-1}{3}\end{pmatrix}+\begin{pmatrix}\frac{5}{3}&\frac{5}{3}\\\frac{10}{3}&\frac{10}{3}\end{pmatrix}\right)^n$	
$=\begin{pmatrix}\frac{-2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{-1}{3}\end{pmatrix}^n+\begin{pmatrix}\frac{5}{3}&\frac{5}{3}\\\frac{10}{3}&\frac{10}{3}\end{pmatrix}^n$ (by (a)(ii)(3))	1M
$=(-1)^{n-1}\begin{pmatrix}\frac{-2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{-1}{3}\end{pmatrix}+5^{n-1}\begin{pmatrix}\frac{5}{3}&\frac{5}{3}\\\frac{10}{3}&\frac{10}{3}\end{pmatrix}$ (by (a)(i))	
$=\begin{pmatrix}\dfrac{5^n+2(-1)^n}{3}&\dfrac{5^n+(-1)^{n-1}}{3}\\\dfrac{2(5^n)+2(-1)^{n-1}}{3}&\dfrac{2(5^n)+(-1)^n}{3}\end{pmatrix}$ (which is also true for $n=1$)	1A
	---------(4)

Solution	Marks
9. (a) Note that $a_n > 0$ and $b_n > 0$ for all $n = 1, 2, 3, \ldots$.	
(i) $a_{n+1} - b_{n+1}$	
$= \frac{a_n^2 + b_n^2}{a_n + b_n} - \frac{2a_n b_n}{a_n + b_n}$	1M
$= \frac{(a_n - b_n)^2}{a_n + b_n}$	
≥ 0 for all $n = 1, 2, 3, \ldots$	1
Thus, $a_n \geq b_n$ for all $n = 1, 2, 3, \ldots$.	
(ii) $a_{n+1} - a_n$	
$= \frac{a_n^2 + b_n^2}{a_n + b_n} - a_n$	
$= \frac{-b_n(a_n - b_n)}{a_n + b_n}$	
≤ 0 for all $n = 1, 2, 3, \ldots$ (by (a)(i))	
Therefore, $a_{n+1} \leq a_n$ for all $n = 1, 2, 3, \ldots$.	
Thus, $\{ a_n \}$ is monotonic decreasing. Also,	1
$b_{n+1} - b_n$	
$= \frac{2a_n b_n}{a_n + b_n} - b_n$	
$= \frac{b_n(a_n - b_n)}{a_n + b_n}$	
≥ 0 for all $n = 1, 2, 3, \ldots$ (by (a)(i))	
Therefore, $b_{n+1} \geq b_n$ for all $n = 1, 2, 3, \ldots$.	
Thus, $\{ b_n \}$ is monotonic increasing.	1
(iii) Note that $a_1 \geq a_n \geq b_n \geq b_1$ for all $n = 1, 2, 3, \ldots$ (by (a)(i) and (a)(ii)).	
$\{ a_n \}$ is monotonic decreasing and bounded below by b_1 .	1A
$\{ b_n \}$ is monotonic increasing and bounded above by a_1 .	1A
Thus, $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist.	
(iv) Let $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} b_n = l_2$. Then, we have $l_1 = \frac{l_1^2 + l_2^2}{l_1 + l_2}$.	1M accept $l_1 = \frac{2l_1 l_2}{l_1 + l_2}$
So, we have $l_1^2 + l_1 l_2 = l_1^2 + l_2^2$. Hence, we have $(l_1 - l_2) l_2 = 0$.	
Note that $l_2 = \lim_{n\to\infty} b_n \geq b_1$ and $b_1 > 0 \Rightarrow l_2 > 0$.	1M
Therefore, we have $l_1 = l_2$.	1
Thus, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.	
(v) Note that $a_n + b_n = \frac{(a_{n-1} + b_{n-1})^2}{a_{n-1} + b_{n-1}} = a_{n-1} + b_{n-1} = \cdots = a_1 + b_1$.	1A
Therefore, we have $\lim_{n\to\infty} (a_n + b_n) = a_1 + b_1$.	1A
By (a)(iv), $\lim_{n\to\infty} a_n = \frac{1}{2}\lim_{n\to\infty} a_n + \frac{1}{2}\lim_{n\to\infty} b_n = \frac{1}{2}\lim_{n\to\infty}(a_n + b_n) = \frac{a_1 + b_1}{2}$.	1A
	---------(12)

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Solution	Marks
10. (a) Note that $x^4+2ax^2+4bx+c\equiv(x^2+2tx+r)(x^2-2tx+s)$.	
(i) By comparing coefficients of x of both sides, we have $2ts-2tr=4b$.	1M
So, we have $t(s-r)=2b$. Since $b\neq 0$, we have $t\neq 0$.	1
(ii) By comparing coefficients of x^2 of both sides, we have $s+r-4t^2=2a$.	either one
By (a)(i), we have $s-r=\frac{2b}{t}$. Thus, we have $\begin{cases} r=a+2t^2-\frac{b}{t} \\ s=a+2t^2+\frac{b}{t}\end{cases}$.	1A 1A
(iii) By comparing the constant terms of both sides, we have $rs=c$.	
By (a)(ii), we have $(a+2t^2-\frac{b}{t})(a+2t^2+\frac{b}{t})=c$.	1M
So, we have $(a+2t^2)^2-\frac{b^2}{t^2}=c$.	
Therefore, we have $t^2(2t^2+a)^2-b^2-ct^2=0$.	
Thus, we have $4t^6+4at^4+(a^2-c)t^2-b^2=0$.	1
	--------(6)
(b) (i) Putting $y=x+h$ in $y^4+4y^3-2y^2+52y+9=0$,	
we have $(x+h)^4+4(x+h)^3-2(x+h)^2+52(x+h)+9=0$.	
So, we have $x^4+(4h+4)x^3+\mathrm{p}(h)x^2+\mathrm{q}(h)x+\mathrm{r}(h)=0$,	
where $\mathrm{p}(h)=6h^2+12h-2$, $\mathrm{q}(h)=4h^3+12h^2-4h+52$	
and $\mathrm{r}(h)=h^4+4h^3-2h^2+52h+9$.	
For the coefficient of x^3 to be zero, we put $4h+4=0$.	1M
So, we have $h=-1$.	1A
Note that $\mathrm{p}(-1)=-8$, $\mathrm{q}(-1)=64$ and $\mathrm{r}(-1)=48$.	
Thus, when $y=x-1$, (*) can be written as $x^4-8x^2+64x-48=0$.	1
(ii) Putting $a=-4$, $b=16$ and $c=-48$ in (a), we have	
$x^4-8x^2+64x-48\equiv(x^2+2tx+r)(x^2-2tx+s)$, where	
$4t^6-16t^4+64t^2-256=0$.	1A
So, we have $t^6-4t^4+16t^2-64=0$.	
Note that $2^6-4(2^4)+16(2^2)-64=64-64+64-64=0$.	1M for finding t by substitution
So, we have $(t-2)(t+2)(t^4+16)=0$.	
A real value of t is 2 .	1A accept $t=-2$
By (a)(ii), we have $r=-4$ and $s=12$. So, we have	
$x^4-8x^2+64x-48=0$	
$(x^2+4x-4)(x^2-4x+12)=0$	1M
$x=\frac{-4\pm\sqrt{16+16}}{2}$ or $x=\frac{4\pm\sqrt{16-48}}{2}$	
$x=-2\pm 2\sqrt{2}$ or $x=2\pm 2\sqrt{2}i$	1A for all the roots being correct
Thus, all the roots of (*) are	
$-3+2\sqrt{2}$, $-3-2\sqrt{2}$, $1+2\sqrt{2}i$ and $1-2\sqrt{2}i$.	1M
	--------(9)

Solution	Marks
(b) If $a_1 \le b_1 < 0$, then $-a_1 \ge -b_1 > 0$.	
Define $A_{n+1} = \dfrac{A_n^{\,2} + B_n^{\,2}}{A_n + B_n}$ and $B_{n+1} = \dfrac{2A_n B_n}{A_n + B_n}$ for all $n = 1, 2, 3, \ldots$, where $A_1 = -a_1$ and $B_1 = -b_1$.	1M
Note that $A_1 B_1 > 0$ and $A_1 \ge B_1 > 0$.	
Then, by (a), $\lim\limits_{n\to\infty} A_n$ and $\lim\limits_{n\to\infty} B_n$ both exist.	1A
Further note that $a_n = -A_n$ and $b_n = -B_n$ for all $n = 1, 2, 3, \ldots$.	
Therefore, we have $\lim\limits_{n\to\infty} a_n = -\lim\limits_{n\to\infty} A_n$ and $\lim\limits_{n\to\infty} b_n = -\lim\limits_{n\to\infty} B_n$.	
Thus, the limits of the sequences $\{a_n\}$ and $\{b_n\}$ both exist.	1A must show reasons
If $a_1 \le b_1 < 0$, then $a_n < 0$ and $b_n < 0$ for all $n = 1, 2, 3, \ldots$.	
$a_{n+1} - b_{n+1} = \dfrac{(a_n - b_n)^2}{a_n + b_n} \le 0$ for all $n = 1, 2, 3, \ldots$	
Therefore, $a_n \le b_n$ for all $n = 1, 2, 3, \ldots$.	
$a_{n+1} - a_n = \dfrac{-b_n(a_n - b_n)}{a_n + b_n} \ge 0$ for all $n = 1, 2, 3, \ldots$	
Therefore, $a_{n+1} \ge a_n$ for all $n = 1, 2, 3, \ldots$.	
$b_{n+1} - b_n = \dfrac{b_n(a_n - b_n)}{a_n + b_n} \le 0$ for all $n = 1, 2, 3, \ldots$	
Therefore, $b_{n+1} \le b_n$ for all $n = 1, 2, 3, \ldots$.	
Note that $a_1 \le a_n \le b_n \le b_1$ for all $n = 1, 2, 3, \ldots$.	1M
$\{a_n\}$ is monotonic increasing and bounded above by b_1 . $\{b_n\}$ is monotonic decreasing and bounded below by a_1 .	1A
Thus, the limits of the sequences $\{a_n\}$ and $\{b_n\}$ both exist.	1A must show reasons
	---------(3)

Solution	Marks
11. (a) Note that $t^n - 1 = (t-1)(t^{n-1} + t^{n-2} + \cdots + t + 1)$.	1M
Case 1: $0 < t < 1$	
Under this case, we have $t - 1 < 0$ and $0 < t^{n-1} + t^{n-2} + \cdots + t + 1 < n$.	
So, we have $(t-1)(t^{n-1} + t^{n-2} + \cdots + t + 1) > n(t-1)$.	
Therefore, we have $t^n - 1 > n(t-1)$.	1
Case 2: $t = 1$	
Under this case, we have $t^n - 1 = 0 = n(t-1)$.	
Case 3: $t > 1$	
Under this case, we have $t - 1 > 0$ and $t^{n-1} + t^{n-2} + \cdots + t + 1 > n$.	
So, we have $(t-1)(t^{n-1} + t^{n-2} + \cdots + t + 1) > n(t-1)$.	
Therefore, we have $t^n - 1 > n(t-1)$.	1
Thus, by combining the three cases, we have $t^n - 1 \ge n(t-1)$ for all $t > 0$.	
Since $t - 1 = t - 1$, the result is true when $n = 1$.	
When $n \ge 2$, let $f(t) = t^n - nt$ for all $t > 0$.	
Then, we have	
$f'(t) = nt^{n-1} - n$	
$= n(t^{n-1} - 1)$	
$\begin{cases} < 0 & \text{if } 0 < t < 1 \\ = 0 & \text{if } t = 1 \\ > 0 & \text{if } t > 1 \end{cases}$	1M+1A
So, $f(t)$ attains its least value when $t = 1$. Therefore,	
$f(t) \ge f(1)$ for all $t > 0$	
$t^n - nt \ge 1 - n$ for all $t > 0$	
$t^n - 1 \ge n(t-1)$ for all $t > 0$	1
	---------(3)
(b) (i) By (a), we have $\left(\dfrac{\sqrt[3]{abc}}{\sqrt{ab}}\right)^3 - 1 \ge 3\left(\dfrac{\sqrt[3]{abc}}{\sqrt{ab}} - 1\right)$.	
Therefore, we have $\dfrac{c}{\sqrt{ab}} - 1 \ge 3\left(\dfrac{\sqrt[3]{abc}}{\sqrt{ab}}\right) - 3$.	
So, we have $c \ge (3\sqrt[3]{abc}) - 2\sqrt{ab}$.	
Note that $c = 3(\dfrac{a+b+c}{3}) - 2(\dfrac{a+b}{2})$.	1M
Hence, we have $3(\dfrac{a+b+c}{3}) - 2(\dfrac{a+b}{2}) \ge 3(\sqrt[3]{abc}) - 2\sqrt{ab}$.	
So, we have $3(\dfrac{a+b+c}{3}) - 3(\sqrt[3]{abc}) \ge 2(\dfrac{a+b}{2}) - 2\sqrt{ab}$.	
Thus, we have $\dfrac{a+b+c}{3} - \sqrt[3]{abc} \ge \dfrac{2}{3}(\dfrac{a+b}{2} - \sqrt{ab})$.	1

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Solution	Marks
(ii) By (a), putting $n=k+1$ and $t=\dfrac{G_{k+1}}{G_k}$,	1M
we have $\left(\dfrac{G_{k+1}}{G_k}\right)^{k+1}-1 \ge (k+1)(\dfrac{G_{k+1}}{G_k}-1)$.	
Therefore, we have $\dfrac{y_{k+1}}{G_k}-1\ge(k+1)(\dfrac{G_{k+1}}{G_k}-1)$.	
Thus, we have $y_{k+1}\ge(k+1)G_{k+1}-kG_k$.	1
(iii) We are going to prove the statement by mathematical induction on n .	
Since $x_1=x_1$, the statement is true for $n=1$.	
Assume that there exists a positive integer k such that	
$\dfrac{x_1+x_2+\cdots+x_k}{k}\ge\sqrt[k]{x_1x_2\cdots x_n}$ for any k positive real numbers $x_1,x_2,\ldots,x_k$.	
Then, for any $k+1$ positive real numbers $y_1,y_2,\ldots,y_{k+1}$, define	
$A_l=\dfrac{y_1+y_2+\cdots+y_l}{l}$ and $G_l=\sqrt[l]{y_1y_2\cdots y_l}$ for $l=k,k+1$.	
Note that $y_{k+1}=(k+1)A_{k+1}-kA_k$.	
By (b)(ii), we have $(k+1)A_{k+1}-kA_k\ge(k+1)G_{k+1}-kG_k$.	1M for using (b)(ii)
So, we have $(k+1)(A_{k+1}-G_{k+1})\ge k(A_k-G_k)$.	1A
Therefore, we have $A_{k+1}-G_{k+1}\ge\dfrac{k}{k+1}(A_k-G_k)$.	
By putting $x_i=y_i$ ($1\le i\le k$) in the induction assumption,	
we have $A_k\ge G_k$.	1M for using induction assumption
Therefore, we have $A_{k+1}\ge G_{k+1}$ if $A_k\ge G_k$.	1
So, the statement is true for $n=k+1$ if the statement is true for $n=k$.	
Thus, by mathematical induction, the statement is true for all positive integers n .	
	----------(8)
(c) Putting $x_i=2i-1$ ($1\le i\le n$) in (b)(iii),	
we have $\dfrac{1+3+5+\cdots+(2n-1)}{n}\ge\sqrt[n]{(1)(3)(5)\cdots(2n-1)}$.	1M
Therefore, we have $\dfrac{(1+2n-1)n}{2n}\ge\sqrt[n]{(1)(3)(5)\cdots(2n-1)}$.	
Thus, we have $n^n\ge(1)(3)(5)\cdots(2n-1)$.	1
Also, putting $x_i=2i$ ($1\le i\le n$) in (b)(iii),	
we have $\dfrac{2+4+6+\cdots+2n}{n}\ge\sqrt[n]{(2)(4)(6)\cdots(2n)}$.	1M
Therefore, we have $\dfrac{(2+2n)n}{2n}\ge\sqrt[n]{(2)(4)(6)\cdots(2n)}$.	
So, we have $(n+1)^n\ge(2)(4)(6)\cdots(2n)$.	
Hence, we have $n^n(n+1)^n\ge(1)(2)(3)(4)\cdots(2n-1)(2n)$.	
Thus, we have $(n^2+n)^n\ge(2n)!$.	1
	----------(4)

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Solution	Marks
12. (a) (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.	
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$	
$=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot[(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\times(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})]$	
$=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot[(b_2c_3-b_3c_2)\mathbf{i}-(b_1c_3-b_3c_1)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}]$	
$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$	1A withhold 1A for no justification
$=a_1\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & c_1\\ b_3 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}$	
$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$	1
Hence, we have	
$\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$	
$=\begin{vmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}$	
$=-\begin{vmatrix}b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$	1M (for either one)
$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$	1 (for both correct)
Also, we have	
$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$	
$=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$	
$=-\begin{vmatrix}a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\\ b_1 & b_2 & b_3\end{vmatrix}$	
$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$	
Thus, we have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.	

Solution	Marks
(ii) Suppose $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\neq 0$. Then, by (a)(i), $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\neq 0$, where $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.	
Let $\mathbf{x}=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$ be a vector in $\mathbf{R}^3$.	
Then, consider the system of linear equations in x , y and z (*): $\begin{cases} a_1x+b_1y+c_1z=x_1 \\ a_2x+b_2y+c_2z=x_2 \\ a_3x+b_3y+c_3z=x_3 \end{cases}$. Since $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\neq 0$,	1M for using $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\neq 0$
there exists a (unique) solution $(\alpha_0,\beta_0,\gamma_0)$ which satisfies (*) .	
So, there exist real numbers $\alpha_0,\beta_0,\gamma_0$ such that $\mathbf{x}=\alpha_0\mathbf{a}+\beta_0\mathbf{b}+\gamma_0\mathbf{c}$.	1A must show reasons
Now, $\mathbf{x}\cdot(\mathbf{b}\times\mathbf{c})=\alpha_0\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+0+0$ ($\because$ $\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})=0$)	1M (either one)
So, we have $\alpha_0=\dfrac{\mathbf{x}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$ ($\because$ $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\neq 0$) .	
Also, $\mathbf{x}\cdot(\mathbf{c}\times\mathbf{a})=\beta_0\,\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})+0+0$ ($\because$ $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{c}\times\mathbf{a})=0$)	either one
So, we have $\beta_0=\dfrac{\mathbf{x}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}$ ($\because$ $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\neq 0$) .	
Moreover, $\mathbf{x}\cdot(\mathbf{a}\times\mathbf{b})=\gamma_0\,\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})+0+0$ ($\because$ $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=0$)	(either one)
So, we have $\gamma_0=\dfrac{\mathbf{x}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}$ ($\because$ $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\neq 0$) .	
Thus, we have $\mathbf{x}=\left(\dfrac{\mathbf{x}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\right)\mathbf{a}+\left(\dfrac{\mathbf{x}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}\right)\mathbf{b}+\left(\dfrac{\mathbf{x}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}\right)\mathbf{c}$.	1
(iii) Suppose $\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}=1$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0$.	
(1) Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0$, we have $\mathbf{b}\times\mathbf{c}=l\mathbf{a}$, $\mathbf{c}\times\mathbf{a}=m\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}=n\mathbf{c}$ for some non-zero real numbers l , m and n .	1M for either one or for considering the volume of the parallelepiped
Note that $\lvert\mathbf{b}\times\mathbf{c}\rvert=\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert=1$ (since $\mathbf{b}\cdot\mathbf{c}=0$) .	
Since $\lvert\mathbf{a}\rvert=1$, we have $\lvert l\rvert=1$.	
Thus, we have $\lvert\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\rvert=\lvert l\rvert\lvert\mathbf{a}\rvert^2=1$.	1
(2) By (a)(iii)(1), we have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\neq 0$, $\mathbf{b}\times\mathbf{c}=l\mathbf{a}$, $\mathbf{c}\times\mathbf{a}=m\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}=n\mathbf{c}$ for some non-zero real numbers l , m and n .	
Let $\mathbf{x}$ be a vector in $\mathbf{R}^3$. Then, by (a)(ii),	
$\mathbf{x}=\left(\dfrac{\mathbf{x}\cdot(l\mathbf{a})}{\mathbf{a}\cdot(l\mathbf{a})}\right)\mathbf{a}+\left(\dfrac{\mathbf{x}\cdot(m\mathbf{b})}{\mathbf{b}\cdot(m\mathbf{b})}\right)\mathbf{b}+\left(\dfrac{\mathbf{x}\cdot(n\mathbf{c})}{\mathbf{c}\cdot(n\mathbf{c})}\right)\mathbf{c}$	1M for using (a)(ii)
$=(\dfrac{l\,\mathbf{x}\cdot\mathbf{a}}{l})\mathbf{a}+(\dfrac{m\,\mathbf{x}\cdot\mathbf{b}}{m})\mathbf{b}+(\dfrac{n\,\mathbf{x}\cdot\mathbf{c}}{n})\mathbf{c}$ ($\because$ $\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{c}=1$)	
$=(\mathbf{x}\cdot\mathbf{a})\mathbf{a}+(\mathbf{x}\cdot\mathbf{b})\mathbf{b}+(\mathbf{x}\cdot\mathbf{c})\mathbf{c}$ ($\because$ l , m and n are non-zero)	1
	---------(12)

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Solution	Marks

(b) Since $\mathbf{u}=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\mathbf{v}=\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{k})$ and $\mathbf{w}=\frac{1}{\sqrt{6}}(\mathbf{i}-2\mathbf{j}+\mathbf{k})$,

we have $\mathbf{u}\cdot\mathbf{u}=\mathbf{v}\cdot\mathbf{v}=\mathbf{w}\cdot\mathbf{w}=1$ and $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{u}=0$.

1M for checking conditions

By (a)(iii)(2), we have $\mathbf{r}=(\mathbf{r}\cdot\mathbf{u})\mathbf{u}+(\mathbf{r}\cdot\mathbf{v})\mathbf{v}+(\mathbf{r}\cdot\mathbf{w})\mathbf{w}$.

1A

Thus, we have

$$\begin{cases}\alpha=\mathbf{r}\cdot\mathbf{u}=\frac{1}{\sqrt{3}}(6\mathbf{i}-\mathbf{j}+10\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=5\sqrt{3}\\ \beta=\mathbf{r}\cdot\mathbf{v}=\frac{1}{\sqrt{2}}(6\mathbf{i}-\mathbf{j}+10\mathbf{k})\cdot(\mathbf{i}-\mathbf{k})=-2\sqrt{2}\\ \gamma=\mathbf{r}\cdot\mathbf{w}=\frac{1}{\sqrt{6}}(6\mathbf{i}-\mathbf{j}+10\mathbf{k})\cdot(\mathbf{i}-2\mathbf{j}+\mathbf{k})=3\sqrt{6}\end{cases}.$$

1A for all correct

It is sufficient to consider the system of linear equations

$$\begin{cases}\frac{\alpha}{\sqrt{3}}+\frac{\beta}{\sqrt{2}}+\frac{\gamma}{\sqrt{6}}=6\\ \frac{\alpha}{\sqrt{3}}-\frac{2\gamma}{\sqrt{6}}=-1\\ \frac{\alpha}{\sqrt{3}}-\frac{\beta}{\sqrt{2}}+\frac{\gamma}{\sqrt{6}}=10\end{cases}.$$

1A

Note that $\begin{vmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{vmatrix}=-1\neq 0$. By Cramer's rule, we have

1M for solving

$$\begin{cases}\alpha=-\begin{vmatrix}6 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ -1 & 0 & \frac{-2}{\sqrt{6}}\\ 10 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{vmatrix}=5\sqrt{3}\\ \beta=-\begin{vmatrix}\frac{1}{\sqrt{3}} & 6 & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -1 & \frac{-2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 10 & \frac{1}{\sqrt{6}}\end{vmatrix}=-2\sqrt{2}\\ \gamma=-\begin{vmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 6\\ \frac{1}{\sqrt{3}} & 0 & -1\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & 10\end{vmatrix}=3\sqrt{6}\end{cases}.$$

1A for all correct

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Solution	Marks
Suppose $\mathbf{u}=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\mathbf{v}=\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{k})$ and $\mathbf{w}=\frac{1}{\sqrt{6}}(\mathbf{i}-2\mathbf{j}+\mathbf{k})$.	
Then, we have $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix}$ (by (a)(i)) $=-1 \neq 0$	1M for checking condition
By (a)(ii), we have $\mathbf{r}=\left(\frac{\mathbf{r}\cdot(\mathbf{v}\times\mathbf{w})}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}\right)\mathbf{u}+\left(\frac{\mathbf{r}\cdot(\mathbf{w}\times\mathbf{u})}{\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})}\right)\mathbf{v}+\left(\frac{\mathbf{r}\cdot(\mathbf{u}\times\mathbf{v})}{\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})}\right)\mathbf{w}$.	1A
Note that $\mathbf{v}\times\mathbf{w}=\frac{-1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\mathbf{w}\times\mathbf{u}=\frac{-1}{\sqrt{2}}(\mathbf{i}-\mathbf{k})$ and $\mathbf{u}\times\mathbf{v}=\frac{-1}{\sqrt{6}}(\mathbf{i}-2\mathbf{j}+\mathbf{k})$.	
Thus, we have $\begin{cases} \alpha=\frac{1}{\sqrt{3}}(6\mathbf{i}-\mathbf{j}+10\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=5\sqrt{3} \\ \beta=\frac{1}{\sqrt{2}}(6\mathbf{i}-\mathbf{j}+10\mathbf{k})\cdot(\mathbf{i}-\mathbf{k})=-2\sqrt{2} \\ \gamma=\frac{1}{\sqrt{6}}(6\mathbf{i}-\mathbf{j}+10\mathbf{k})\cdot(\mathbf{i}-2\mathbf{j}+\mathbf{k})=3\sqrt{6} \end{cases}$.	1A for all correct
	---------(3)

Solution	Marks
1. (a) Let $f(x)=x^x$ for all $x>0$. Then, we have	
$\ln f(x)=x\ln x$.	1M for taking ln on both sides
Differentiate both sides w.r.t. x, we have $\frac{f'(x)}{f(x)}=\ln x+1$.	
Thus, we have $f'(x)=x^x(1+\ln x)$.	1
$\lim_{x\to 1}\frac{x^x-x}{\ln x-x+1}$	
$=\lim_{x\to 1}\frac{x^x(1+\ln x)-1}{\frac{1}{x}-1}$	1M
$=\lim_{x\to 1}\frac{x^x(1+\ln x)^2+(\frac{1}{x})(x^x)}{-x^{-2}}$	1M
$=-2$	1A
(b) Note that $\frac{d}{dx}\int_0^x t\sin(\sin t)\,dt=x\sin(\sin x)$.	1A
$\lim_{x\to 0}\frac{\int_0^x t\sin(\sin t)\,dt}{x^3}$	
$=\lim_{x\to 0}\frac{x\sin(\sin x)}{3x^2}$	either (1M)
$=\frac{1}{3}\lim_{x\to 0}\frac{\sin(\sin x)}{x}$	
$=\frac{1}{3}\lim_{x\to 0}\left(\frac{\sin(\sin x)}{\sin x}\frac{\sin x}{x}\right)$	either (1M)
$=\frac{1}{3}\left(\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)$	
$=\frac{1}{3}$	1A
Note that $\frac{d}{dx}\int_0^x t\sin(\sin t)\,dt=x\sin(\sin x)$.	1A
$\lim_{x\to 0}\frac{\int_0^x t\sin(\sin t)\,dt}{x^3}$	
$=\lim_{x\to 0}\frac{x\sin(\sin x)}{3x^2}$	either (1M)
$=\frac{1}{3}\lim_{x\to 0}\frac{\sin(\sin x)}{x}$	
$=\frac{1}{3}\lim_{x\to 0}\frac{\cos(\sin x)\cos x}{1}$	either (1M)
$=\frac{1}{3}$	1A
	----------(7)

Solution	Marks
2. (a) Since f is continuous at π, we have $\lim\limits_{x \to \pi^-} f(x) = f(\pi)$.	
Therefore, we have $\lim\limits_{x \to \pi^-} (\frac{x^2}{2\pi} - x + a) = a\cos\pi$.	
So, we have $\frac{\pi}{2} - \pi + a = -a$.	
Hence, we have $2a = \frac{\pi}{2}$.	
Thus, we have $a = \frac{\pi}{4}$.	1
(b) $\lim\limits_{x \to \pi^-} \frac{f(x) - f(\pi)}{x - \pi}$	
$= \lim\limits_{x \to \pi^-} \frac{\frac{x^2}{2\pi} - x + \frac{\pi}{4} - \left(\frac{-\pi}{4}\right)}{x - \pi}$	
$= \lim\limits_{x \to \pi^-} \frac{\frac{x}{\pi} - 1}{1}$	1M + 1A (either one)
$= 0$	
$\lim\limits_{x \to \pi^+} \frac{f(x) - f(\pi)}{x - \pi}$	
$= \lim\limits_{x \to \pi^+} \frac{\frac{\pi\cos x}{4} - \left(\frac{-\pi}{4}\right)}{x - \pi}$	
$= \lim\limits_{x \to \pi^+} \frac{\frac{-\pi\sin x}{4}}{1}$	
$= 0$	
Therefore, $\lim\limits_{x \to \pi^-} \frac{f(x) - f(\pi)}{x - \pi}$ [redacted] $= \lim\limits_{x \to \pi^+} \frac{f(x) - f(\pi)}{x - \pi}$.	
Thus, f is differentiable at π [redacted].	1
(c) $f'(x) = \begin{cases} \frac{x}{\pi} - 1 & \text{when } x < \pi, \\ \frac{-\pi\sin x}{4} & \text{when } x > \pi. \end{cases}$	1A for both correct
Note that $\lim\limits_{x \to \pi^-} f'(x) = \lim\limits_{x \to \pi^-} \left(\frac{x}{\pi} - 1\right) = 1 - 1 = 0$	1M for considering $\lim\limits_{x \to \pi^-} f'(x)$ (either one)
and $\lim\limits_{x \to \pi^+} f'(x) = \lim\limits_{x \to \pi^+} \left(\frac{-\pi\sin x}{4}\right) = 0$.	
Further note that $f'(\pi) = 0$ (by (b)).	
Therefore, we have $\lim\limits_{x \to \pi^-} f'(x) = f'(\pi) = \lim\limits_{x \to \pi^+} f'(x)$.	
Thus, f' is continuous at π.	1A must show reasons -----------(7)

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Solution	Marks
3. (a) Let $\dfrac{3-5x}{(1+x)(1-x+2x^2)}=\dfrac{A}{1+x}+\dfrac{B+Cx}{1-x+2x^2}$.	1M
$3-5x \equiv A(1-x+2x^2)+(B+Cx)(1+x)$	
$3-5x \equiv (A+B)+(B+C-A)x+(2A+C)x^2$	
So, we have $A+B=3$, $B+C-A=-5$ and $2A+C=0$.	
Solving, we have $A=2$, $B=1$ and $C=-4$.	1A for all correct
Thus, we have $\dfrac{3-5x}{(1+x)(1-x+2x^2)}=\dfrac{2}{1+x}+\dfrac{1-4x}{1-x+2x^2}$.	
(b) $\displaystyle\int_0^\infty \frac{3-5x}{(1+x)(1-x+2x^2)}\,dx$	
$\displaystyle=\lim_{l\to\infty}\int_0^l \frac{3-5x}{(1+x)(1-x+2x^2)}\,dx$	1M
$\displaystyle=\lim_{l\to\infty}\left(\int_0^l \frac{2}{1+x}\,dx+\int_0^l \frac{1-4x}{1-x+2x^2}\,dx\right)$	
$\displaystyle=\lim_{l\to\infty}\left(2\int_0^l \frac{d(1+x)}{1+x}-\int_0^l \frac{d(1-x+2x^2)}{1-x+2x^2}\right)$	
$\displaystyle=\lim_{l\to\infty}\left(\left[2\ln(1+x)\right]_0^l-\left[\ln(1-x+2x^2)\right]_0^l\right)$	1A for correct integration
$\displaystyle=\lim_{l\to\infty}\left(2\ln(1+l)-\ln(1-l+2l^2)\right)$	
$\displaystyle=\lim_{l\to\infty}\ln\frac{(1+l)^2}{1-l+2l^2}$	1M
$\displaystyle=\ln\lim_{l\to\infty}\frac{(\frac{1}{l}+1)^2}{\frac{1}{l^2}-\frac{1}{l}+2}$	
$=\ln\dfrac{1}{2}$	1A
$=-\ln 2$	
	-----------(6)

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Solution	Marks
4. (a) (i) $\int \sqrt{1+x}\,dx$	
$=\frac{2}{3}(1+x)^{\frac{3}{2}}+C_1$, where C_1 is a constant	1A
(ii) $\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-x}}\,dx$	
$=-2\int_0^{\frac{1}{2}} \arcsin x \,d\sqrt{1-x}$	
$=\left[-2\sqrt{1-x}\arcsin x\right]_0^{\frac{1}{2}}+2\int_0^{\frac{1}{2}}\sqrt{1-x}\frac{dx}{\sqrt{1-x^2}}$	1M
$=\frac{-\sqrt{2}\pi}{6}+2\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1+x}}$	
$=\frac{-\sqrt{2}\pi}{6}+\left[4\sqrt{1+x}\right]_0^{\frac{1}{2}}$	
$=2\sqrt{6}-4-\frac{\sqrt{2}\,\pi}{6}$	1
(b) Consider the curve $C: y=\sqrt{1-x^2}-\arcsin x$, $0\le x\le\frac{1}{2}$.	
$\frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}=\frac{-(1+x)}{\sqrt{1-x^2}}$	1A
$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\left(\frac{-(1+x)}{\sqrt{1-x^2}}\right)^2}=\sqrt{\frac{1-x^2+1+2x+x^2}{1-x^2}}=\sqrt{\frac{2}{1-x}}$	
Note that $y>0$ for all $x\in[0,\frac{1}{2}]$.	
The required surface area	
$=2\pi\int_0^{\frac{1}{2}} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$	
$=2\pi\int_0^{\frac{1}{2}}(\sqrt{1-x^2}-\arcsin x)\sqrt{\frac{2}{1-x}}\,dx$	1M
$=2\sqrt{2}\pi\int_0^{\frac{1}{2}}(\sqrt{1+x}-\frac{\arcsin x}{\sqrt{1-x}})\,dx$	
$=2\sqrt{2}\pi\left(\int_0^{\frac{1}{2}}\sqrt{1+x}\,dx-\int_0^{\frac{1}{2}}\frac{\arcsin x}{\sqrt{1-x}}\,dx\right)$	
$=2\sqrt{2}\pi\left(\left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^{\frac{1}{2}}-\left(2\sqrt{6}-4-\frac{\sqrt{2}\,\pi}{6}\right)\right)$ (by (a))	1M
$=2\sqrt{2}\pi\left(-\frac{3}{2}\sqrt{6}+\frac{10}{3}+\frac{\sqrt{2}\,\pi}{6}\right)$	1A
$=\frac{2\pi}{3}\left(10\sqrt{2}-9\sqrt{3}+\pi\right)$	
	----------(7)

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Solution	Marks
5. (a) Note that $m \neq 0$.	
Putting $y = mx + c$ in $y^2 = 80x$, we have $(mx+c)^2 = 80x$.	
So, we have $m^2x^2 + (2mc - 80)x + c^2 = 0$ which is quadratic since $m \neq 0$.	
The straight line $y = mx + c$ is a tangent to P	
$\Leftrightarrow (2mc-80)^2 - 4m^2c^2 = 0$	1M
$\Leftrightarrow (mc-40)^2 - m^2c^2 = 0$	
$\Leftrightarrow m^2c^2 - 80mc + 1600 - m^2c^2 = 0$	
$\Leftrightarrow mc = 20$	1
Note that $m \neq 0$.	
Putting $x = \dfrac{y-c}{m}$ in $y^2 = 80x$, we have $y^2 = 80\left(\dfrac{y-c}{m}\right)$.	
So, we have $my^2 - 80y + 80c = 0$ which is quadratic since $m \neq 0$.	
The straight line $y = mx + c$ is a tangent to P	
$\Leftrightarrow (-80)^2 - 4(m)(80c) = 0$	1M
$\Leftrightarrow 6400 - 320mc = 0$	
$\Leftrightarrow mc = 20$	1
(b) Suppose that the normal to Γ at $(3\cos\theta, \sin\theta)$ is a tangent to P .	
Then, we have $\theta \neq \pi$.	
The slope of the tangent to Γ at $(3\cos\theta, \sin\theta) = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{\cos\theta}{-3\sin\theta} = \dfrac{-1}{3}\cot\theta$	1A
Therefore, the equation of the normal to Γ at $(3\cos\theta, \sin\theta)$ is:	
$\dfrac{y-\sin\theta}{x-3\cos\theta} = 3\tan\theta$	1M
$y = (3\tan\theta)x - 8\sin\theta$	
Note that $3\tan\theta \neq 0$ since $\theta \neq \pi$.	
By (a), we have $(3\tan\theta)(-8\sin\theta) = 20$.	1M for using (a)
So, we have $\left(\dfrac{\sin\theta}{\cos\theta}\right)(\sin\theta) = \dfrac{-5}{6}$.	
Therefore, we have $6\cos^2\theta - 5\cos\theta - 6 = 0$.	1A
Solving, we have $\cos\theta = \dfrac{-2}{3}$ or $\cos\theta = \dfrac{3}{2}$ (rejected) .	
Therefore, we have $\sin\theta = \pm\dfrac{\sqrt{5}}{3}$ (since $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$) .	
Thus, the coordinates of the two points are $(-2, \dfrac{\sqrt{5}}{3})$ and $(-2, \dfrac{-\sqrt{5}}{3})$.	1A for both correct
	-----------(7)

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Solution	Marks
6. (a) Putting $x=3\lambda-2$, $y=4\lambda+3$ and $z=\lambda+2$ in $x-3=5-y=1-z$,	
we have $3\lambda-2-3=5-4\lambda-3$ and $5-4\lambda-3=1-\lambda-2$.	
Solving the first equation, we have $\lambda=1$ which satisfies the second equation.	
Thus, L_1 and L_2 intersect at a point.	1
Also, the coordinates of the point of intersection are $(1, 7, 3)$.	1A
(b) Let θ be the acute angle between L_1 and L_2. Then, we have	
$\cos\theta=\dfrac{\lvert(3)(1)+(4)(-1)+(1)(-1)\rvert}{\sqrt{3^2+4^2+1^2}\sqrt{1^2+(-1)^2+(-1)^2}}$	1M
$\theta=\cos^{-1}\dfrac{2}{\sqrt{78}}$	1A (accept 76.9° or 1.34)
Thus, the required angle is $\cos^{-1}\dfrac{2}{\sqrt{78}}$.	
(c) Let the equation of the required plane be $kx+ly+mz+n=0$, where k, l and m are not all zero.	
Then, we have $\begin{cases} 3k+4l+m=0 \\ k-l-m=0 \\ k+7l+3m+n=0 \end{cases}$.	1M
Solving, we have $k:l:m:n=3:-4:7:4$.	
Thus, the equation of the required plane is $3x-4y+7z+4=0$.	1A
The equation of the required plane is:	
$\begin{vmatrix} x-1 & y-7 & z-3 \\ 3 & 4 & 1 \\ 1 & -1 & -1 \end{vmatrix}=0$	1M
$\Leftrightarrow\ -3(x-1)+4(y-7)-7(z-3)=0$	
$\Leftrightarrow\ 3x-4y+7z+4=0$	1A
A normal vector of the required plane	
$=(\mathbf{3i}+\mathbf{4j}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}-\mathbf{k})$	1M
$=-3\mathbf{i}+4\mathbf{j}-7\mathbf{k}$	
Let the equation of the required plane be $3x-4y+7z+d=0$.	
$\because$ $(-2, 3, 2)$ lies on the required plane.	
$\therefore$ $-6-12+14+d=0 \Rightarrow d=4$	
Thus, the equation of the required plane is $3x-4y+7z+4=0$.	1A
A normal vector of the required plane	
$=(3\mathbf{i}+4\mathbf{j}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}-\mathbf{k})$	1M
$=-3\mathbf{i}+4\mathbf{j}-7\mathbf{k}$	
Let the equation of the required plane be $\mathbf{r}\cdot(-3\mathbf{i}+4\mathbf{j}-7\mathbf{k})=\rho$.	
$\because$ $(3, 5, 1)$ lies on the required plane.	
$\therefore$ $\rho=(3\mathbf{i}+5\mathbf{j}+\mathbf{k})\cdot(-3\mathbf{i}+4\mathbf{j}-7\mathbf{k}) \Rightarrow \rho=4$	
Thus, the equation of the required plane is $\mathbf{r}\cdot(-3\mathbf{i}+4\mathbf{j}-7\mathbf{k})=4$.	1A
	----------(6)

Solution	Marks
7. (a) (i) For $x>0$, $f(x)=-x+x\sqrt{\frac{x}{x+2}}$. We have	
$f'(x)=-1+\frac{x+3}{x+2}\sqrt{\frac{x}{x+2}}$ and	1A or equivalent
$f''(x)=\frac{3}{x(x+2)^2}\sqrt{\frac{x}{x+2}}$.	1A or equivalent
(ii) For $x<-2$, $f(x)=-x-x\sqrt{\frac{x}{x+2}}$. We have	
$f'(x)=-1-\frac{x+3}{x+2}\sqrt{\frac{x}{x+2}}$ and $f''(x)=\frac{-3}{x(x+2)^2}\sqrt{\frac{x}{x+2}}$.	1M for either one + 1A for both correct
	----------(4)
(b) (i) Case 1: $x>0$ $f'(x)>0$ $\Leftrightarrow -1+\frac{x+3}{x+2}\sqrt{\frac{x}{x+2}}>0$ $\Leftrightarrow \frac{x+3}{x+2}\sqrt{\frac{x}{x+2}}>1$ $\Leftrightarrow \left(\frac{x+3}{x+2}\right)^2\left(\frac{x}{x+2}\right)>1$ $\Leftrightarrow x(x+3)^2-(x+2)^3>0$ $\Leftrightarrow x<\frac{-8}{3}$ Therefore, there is no solution under this case.	
Case 2: $-3<x<-2$ $f'(x)>0$ $\Leftrightarrow -1-\frac{x+3}{x+2}\sqrt{\frac{x}{x+2}}>0$ $\Leftrightarrow \frac{x+3}{x+2}\sqrt{\frac{x}{x+2}}<-1$ $\Leftrightarrow \left(\frac{x+3}{x+2}\right)^2\left(\frac{x}{x+2}\right)>1$ $\Leftrightarrow x(x+3)^2-(x+2)^3<0$ $\Leftrightarrow x>\frac{-8}{3}$ Therefore, the solution under this case is $\frac{-8}{3}<x<-2$.	
Case 3: $x\le -3$ $f'(x)>0$ $\Leftrightarrow -1-\frac{x+3}{x+2}\sqrt{\frac{x}{x+2}}>0$. Note that LS is always negative. Therefore, there is no solution under this case.	
Thus, by combining the three cases, we have $f'(x)>0 \Leftrightarrow \frac{-8}{3}<x<-2$.	1A

Solution	Marks
(ii) Note that $f''(x)=\begin{cases}\dfrac{-3}{x(x+2)^2}\sqrt{\dfrac{x}{x+2}} & \text{if } x<-2\\ \dfrac{3}{x(x+2)^2}\sqrt{\dfrac{x}{x+2}} & \text{if } x>0\end{cases}$ (by (a)(ii) and (a)(i)).	
$f''(x)>0 \Leftrightarrow x<-2$ or $x>0$.	1A + 1A
	----------(3)

(c)

x	$(-\infty, \frac{-8}{3})$	$\frac{-8}{3}$	$(\frac{-8}{3}, -2)$	$(0,\infty)$
$f'(x)$	−	0	+	−
$f(x)$	↘	8	↗	↘

From the above table, the relative minimum point is $(\frac{-8}{3}, 8)$. — 1M for justification + 1A

There is no relative maximum point.

----------(2)

(d) $\because \lim_{x\to-2^-} f(x)=\lim_{x\to-2^-}\left(-x-x\sqrt{\frac{x}{x+2}}\right)=+\infty$

$\therefore$ the vertical asymptote is $x=-2$. — 1A

$\because \lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{-x+x\sqrt{\frac{x}{x+2}}}{x}=\lim_{x\to+\infty}\left(-1+\sqrt{\frac{x}{x+2}}\right)=0$

$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\left(-x+x\sqrt{\frac{x}{x+2}}\right)=\lim_{x\to+\infty}\frac{x^2-\frac{x^3}{x+2}}{-x-x\sqrt{\frac{x}{x+2}}}$ — 1M

$=\lim_{x\to+\infty}\frac{x-\frac{x^2}{x+2}}{-1-\sqrt{\frac{x}{x+2}}}=\lim_{x\to+\infty}\frac{\frac{2x}{x+2}}{-1-\sqrt{\frac{x}{x+2}}}=-1$

either one

$\therefore$ the horizontal asymptote is $y=-1$. — 1A

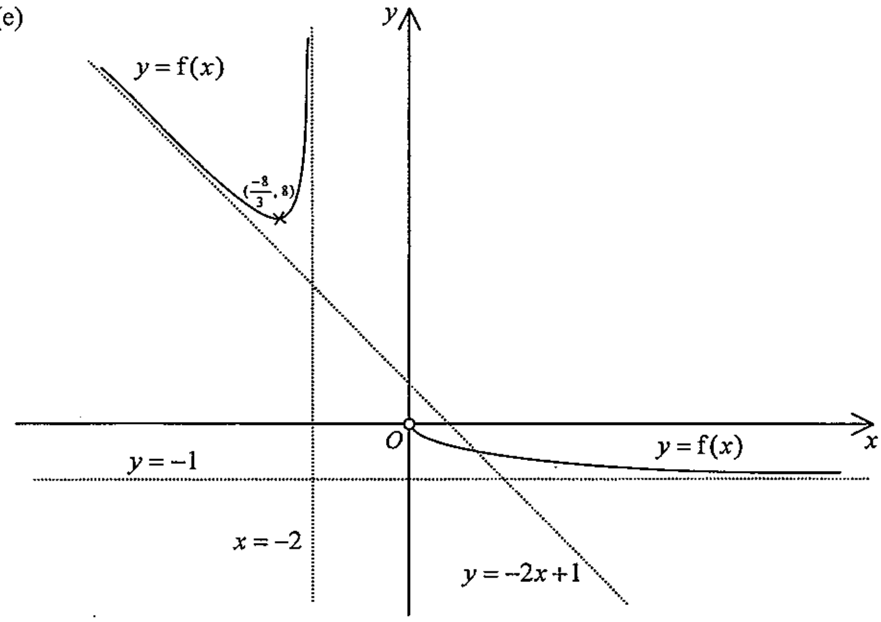
$\because \lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{-x-x\sqrt{\frac{x}{x+2}}}{x}=\lim_{x\to-\infty}\left(-1-\sqrt{\frac{x}{x+2}}\right)=-2$

$\lim_{x\to-\infty}(f(x)+2x)=\lim_{x\to-\infty}\left(x-x\sqrt{\frac{x}{x+2}}\right)=\lim_{x\to-\infty}\frac{x^2-\frac{x^3}{x+2}}{x+x\sqrt{\frac{x}{x+2}}}$

$=\lim_{x\to-\infty}\frac{x-\frac{x^2}{x+2}}{1+\sqrt{\frac{x}{x+2}}}=\lim_{x\to-\infty}\frac{\frac{2x}{x+2}}{1+\sqrt{\frac{x}{x+2}}}=1$

$\therefore$ the oblique asymptote is $y=-2x+1$. — 1A

----------(4)

Solution	Marks
(e) 	1A for the relative minimum point and asymptotes 1A for the shape of the curve ----------(2)

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Solution	Marks
8. (a) By (1), we have $f(0+0)=e^0 f(0)+e^0 f(0)$.	
So, we have $f(0)=f(0)+f(0)$.	
Thus, we have $f(0)=0$.	1A withhold 1A for no justification
	----------(1)
(b) $\lim_{h\to 0} f(h)$	
$=\left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0}\frac{f(h)}{h}\right)$ (since $\lim_{h\to 0}\frac{f(h)}{h}$ exists (by (2)) and $\lim_{h\to 0} h$ exists)	1M withhold 1M for no justification
$=(0)(2005)$ (by (2))	
$=0$	1A
Let $x\in\mathbf{R}$.	
$\lim_{h\to 0} f(x+h)$	
$=\lim_{h\to 0}\left(e^x f(h)+e^h f(x)\right)$ (by (1))	1M
$=e^x\lim_{h\to 0} f(h)+f(x)\lim_{h\to 0} e^h$	
$=e^x(0)+f(x)(1)$	
$=f(x)$	1
Thus, f is a continuous function.	
	----------(4)
(c) (i) Let $x\in\mathbf{R}$.	
$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	1M for considering $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$=\lim_{h\to 0}\frac{e^x f(h)+e^h f(x)-f(x)}{h}$ (by (1))	
$=e^x\lim_{h\to 0}\frac{f(h)}{h}+f(x)\lim_{h\to 0}\frac{e^h-1}{h}$	1M
Note that $\lim_{h\to 0}\frac{e^h-1}{h}=\lim_{h\to 0} e^h=e^0=1$	1A
Therefore, by (2), we have $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=2005\, e^x+f(x)$.	1
Thus, f is differentiable everywhere and $f'(x)=2005e^x+f(x)$ for all $x\in\mathbf{R}$.	
(ii) By (c)(i), we have $f'(x)=2005e^x+f(x)$ for all $x\in\mathbf{R}$.	
So, we have $f''(x)=2005e^x+f'(x)=2005(2)e^x+f(x)$ for all $x\in\mathbf{R}$.	
Therefore, we have $f'''(x)=2005(3)e^x+f(x)$ for all $x\in\mathbf{R}$.	
Inductively, we have $f^{(n)}(x)=(2005n)e^x+f(x)$ for all $x\in\mathbf{R}$.	1A
By (a), we have $f^{(n)}(0)=2005n$ for all positive integers n .	1A
By (c)(i), we have $f'(x)=2005e^x+f(x)$ for all $x\in\mathbf{R}$.	
So, we have $f^{(n)}(x)=2005e^x+f^{(n-1)}(x)$ for all $x\in\mathbf{R}$.	
Therefore, we have $f^{(n)}(0)=2005+f^{(n-1)}(0)$.	1A
By (a), we have $f^{(n)}(0)=2005n$ for all positive integers n .	1A
	----------(6)

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Solution	Marks
(d) $\frac{d}{dx}\left(\frac{f(x)}{e^x}\right)$	
$=\frac{f'(x)e^x - e^x f(x)}{e^{2x}}$	
$=\frac{(2005e^x + f(x))e^x - e^x f(x)}{e^{2x}}$ (by (c)(i))	
$= 2005$	1A
Therefore, we have $\frac{f(x)}{e^x} = 2005x + C$, where C is a constant.	1M for using integration
So, we have $f(x) = 2005xe^x + Ce^x$.	
Since $f(0) = 0$ (by (a)), we have $C = 0$.	1M for finding C
Thus, we have $f(x) = 2005xe^x$ for all $x \in \mathbf{R}$.	1A
$\frac{d}{dx}\left(\frac{f(x)}{e^x}\right)$	
$=\frac{f'(x)e^x - e^x f(x)}{e^{2x}}$	
$=\frac{(2005e^x + f(x))e^x - e^x f(x)}{e^{2x}}$ (by (c)(i))	
$= 2005$	1A
So, we have $\int_0^x \frac{d}{dt}\left(\frac{f(t)}{e^t}\right)dt = \int_0^x 2005\,dt$.	1M for using integration
Therefore, we have $\frac{f(x)}{e^x} - \frac{f(0)}{e^0} = 2005x$.	
Since $f(0) = 0$ (by (a)), we have $\frac{f(x)}{e^x} = 2005x$.	1M for using (a)
Thus, we have $f(x) = 2005xe^x$ for all $x \in \mathbf{R}$.	1A
	-----------(4)

Solution	Marks
9. (a) The equation of L is:	
$\dfrac{2ty+\dfrac{2x}{t}}{2}=4$	1M
$x+t^2y-4t=0$	1A
$H_1 : xy=4$	
$x\dfrac{dy}{dx}+y=0$	
$\dfrac{dy}{dx}=\dfrac{-y}{x}$ for all $x\neq 0$	
$\left.\dfrac{dy}{dx}\right\|_{\left(2t,\frac{2}{t}\right)}=\dfrac{\frac{-2}{t}}{2t}=\dfrac{-1}{t^2}$	1M
The equation of L is:	
$\dfrac{y-\dfrac{2}{t}}{x-2t}=\dfrac{-1}{t^2}$	
$x+t^2y-4t=0$	1A
	----------(2)
(b) (i) Putting $y=\dfrac{4t-x}{t^2}$ in $xy=1$, we have $x\left(\dfrac{4t-x}{t^2}\right)=1$.	1M
So, we have $4tx-x^2=t^2$.	
Therefore, we have $x^2-4tx+t^2=0$.	1A or equivalent
Note that α and β are the roots of the quadratic equation $x^2-4tx+t^2=0$.	
Thus, we have $\alpha+\beta=4t$ and $\alpha\beta=t^2$.	1
(ii) The length of chord AB	
$=\sqrt{(\alpha-\beta)^2+\left(\dfrac{1}{\alpha}-\dfrac{1}{\beta}\right)^2}$	1A
$=\sqrt{(\alpha-\beta)^2\left(1+\dfrac{1}{(\alpha\beta)^2}\right)}$	
$=\sqrt{\left((\alpha+\beta)^2-4\alpha\beta\right)\left(1+\dfrac{1}{(\alpha\beta)^2}\right)}$	1M
$=\sqrt{\left((4t)^2-4t^2\right)\left(1+\dfrac{1}{(t^2)^2}\right)}$ (by (b)(i))	
$=\sqrt{\left(16t^2-4t^2\right)\left(1+\dfrac{1}{t^4}\right)}$	
$=\sqrt{12\left(t^2+\dfrac{1}{t^2}\right)}$	1
	----------(6)

Solution	Marks
(c) (i) Note that the equation of the tangent to H_2 at A is $x+\alpha^2 y-2\alpha=0$ and that the equation of the tangent to H_2 at B is $x+\beta^2 y-2\beta=0$.	1A for either
Let the coordinates of Q be (x,y).	
Then, we have $x+\alpha^2 y-2\alpha=0$ and $x+\beta^2 y-2\beta=0$.	
Solving, we have $x=\frac{2\alpha\beta}{\alpha+\beta}$ and $y=\frac{2}{\alpha+\beta}$.	1M for solving
By (b)(i), we have $x=\frac{t}{2}$ and $y=\frac{1}{2t}$.	1A for both correct
Thus, the coordinates of Q are $(\frac{t}{2},\frac{1}{2t})$.	
(ii) The distance from Q to AB	
$=\frac{\left\lvert \frac{t}{2}+t^2(\frac{1}{2t})-4t\right\rvert}{\sqrt{1+t^4}}$ (by (a))	1M
$=\frac{3\lvert t\rvert}{\sqrt{1+t^4}}$	1A
The area of ΔQAB	
$=\left(\frac{1}{2}\right)\left(\frac{3\lvert t\rvert}{\sqrt{1+t^4}}\right)\sqrt{12(t^2+\frac{1}{t^2})}$ (by (b)(ii))	1M
$=3\sqrt{3}$ which is independent of t	1A
The area of ΔQAB	
$=\left\lvert\frac{1}{2}\begin{vmatrix}\frac{t}{2} & \frac{1}{2t} & 1\\ \alpha & \frac{1}{\alpha} & 1\\ \beta & \frac{1}{\beta} & 1\end{vmatrix}\right\rvert$	1M
$=\frac{1}{2}\left\lvert\frac{t}{2\alpha}+\frac{\alpha}{\beta}+\frac{\beta}{2t}-\frac{\alpha}{2t}-\frac{\beta}{\alpha}-\frac{t}{2\beta}\right\rvert$	1A
$=\frac{1}{2}\lvert\alpha-\beta\rvert\left\lvert\frac{\alpha+\beta}{\alpha\beta}-\frac{t}{2\alpha\beta}-\frac{1}{2t}\right\rvert$	
Note that $\lvert\alpha-\beta\rvert=\sqrt{(\alpha-\beta)^2}=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\sqrt{12}\lvert t\rvert$ (by (b)(i)).	1M
The area of ΔQAB	
$=\frac{1}{2}\sqrt{12}\lvert t\rvert\left\lvert\frac{4}{t}-\frac{1}{2t}-\frac{1}{2t}\right\rvert$ (by (b)(i))	
$=\frac{1}{2}\sqrt{12}\lvert t\rvert\frac{3}{\lvert t\rvert}$	
$=3\sqrt{3}$ which is independent of t	1A
	----------(7)

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Solution	Marks
10. (a) Let $I_n = \int_{-1}^{1} g_n(x)\,dx$. Then, we have	
I_{n+1}	
$= \int_{-1}^{1} g_{n+1}(x)\,dx$	
$= \left[x\,g_{n+1}(x)\right]_{-1}^{1} - \int_{-1}^{1}(n+1)(2x^2)g_n(x)\,dx$	
$= -2(n+1)\int_{-1}^{1} x^2 g_n(x)\,dx$	1A
$= -2(n+1)\int_{-1}^{1}\left((x^2-1)g_n(x)+g_n(x)\right)dx$	1M for using $x^2=(x^2-1)+1$
$= -2(n+1)\int_{-1}^{1} g_{n+1}(x)\,dx - 2(n+1)\int_{-1}^{1} g_n(x)\,dx$	
$= -2(n+1)I_{n+1} - 2(n+1)I_n$	
So, we have $(2n+3)I_{n+1} = -(2n+2)I_n$.	1M
Thus, we have $I_{n+1} = -\dfrac{2n+2}{2n+3}I_n$.	1A
Therefore, we have	
I_{n+1}	
$= -\dfrac{2n+2}{2n+3}I_n$	
$= (-1)^2\dfrac{(2n+2)(2n)}{(2n+3)(2n+1)}I_{n-1}$	
$= \cdots$	
$= (-1)^{n+1}\dfrac{(2n+2)(2n)\cdots(2)}{(2n+3)(2n+1)\cdots(3)}I_0$	
$= (-1)^{n+1}\dfrac{(2n+2)(2n)\cdots(2)}{(2n+3)(2n+1)\cdots(3)}(2)$ (since $I_0 = \int_{-1}^{1} g_0(x)\,dx = \int_{-1}^{1} dx = 2$)	
$= (-1)^{n+1}\dfrac{(2n+2)^2(2n)^2\cdots(2)^2}{(2n+3)!}(2)$	
$= (-1)^{n+1}\dfrac{\left(2^{n+1}(n+1)!\right)^2}{(2n+3)!}(2)$	
$= \dfrac{(-1)^{n+1}2^{2n+3}\left((n+1)!\right)^2}{(2n+3)!}$	1
	----------(5)
(b) Note that $g_{n+1}(x) = (x^2-1)^{n+1} = (x+1)^{n+1}(x-1)^{n+1}$. Therefore, we have	
${g_{n+1}}^{(k)}(x)$	
$= \sum_{r=0}^{k} C_r^k\left((x+1)^{n+1}\right)^{(k-r)}\left((x-1)^{n+1}\right)^{(r)}$	1M
$= \sum_{r=0}^{k} C_r^k\left((n+1)(n)\cdots(n-k+r+2)\right)\left((n+1)(n)\cdots(n-r+2)\right)(x+1)^{n-k+r+1}(x-1)^{n-r+1}$	1A
For each $k=0,1,\ldots,n$, both $(x+1)$ and $(x-1)$ are factors of ${g_{n+1}}^{(k)}(x)$.	
Thus, we have ${g_{n+1}}^{(k)}(-1) = {g_{n+1}}^{(k)}(1) = 0$ for all $k=0,1,\ldots,n$.	1
	----------(3)

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Solution	Marks
(c) (i) $\int_{-1}^{1} \mathrm{p}(x)\,\mathrm{h}_{n+1}(x)\,\mathrm{d}x$	
$=\frac{1}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{p}(x)\,\mathrm{g}_{n+1}{}^{(n+1)}(x)\,\mathrm{d}x$	
$=\frac{1}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{p}(x)\,\mathrm{d}\mathrm{g}_{n+1}{}^{(n)}(x)$	1M for using integration by parts
$=\frac{1}{2^{n+1}(n+1)!}\left[\mathrm{p}(x)\,\mathrm{g}_{n+1}{}^{(n)}(x)\right]_{-1}^{1}-\frac{1}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{g}_{n+1}{}^{(n)}(x)\,\mathrm{p}'(x)\,\mathrm{d}x$	
$=\frac{-1}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{p}'(x)\,\mathrm{g}_{n+1}{}^{(n)}(x)\,\mathrm{d}x$ (by (b))	1A
$=\frac{-1}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{p}'(x)\,\mathrm{d}\mathrm{g}_{n+1}{}^{(n-1)}(x)$	
$=\frac{-1}{2^{n+1}(n+1)!}\left[\mathrm{p}'(x)\,\mathrm{g}_{n+1}{}^{(n-1)}(x)\right]_{-1}^{1}+\frac{(-1)^2}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{g}_{n+1}{}^{(n-1)}(x)\,\mathrm{p}''(x)\,\mathrm{d}x$	
$=\frac{(-1)^2}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{p}''(x)\,\mathrm{g}_{n+1}{}^{(n-1)}(x)\,\mathrm{d}x$ (by (b))	
$=\cdots$	
$=\frac{(-1)^{n+1}}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{p}^{(n+1)}(x)\,\mathrm{g}_{n+1}{}^{(0)}(x)\,\mathrm{d}x$	
$=\frac{(-1)^{n+1}}{2^{n+1}(n+1)!}\int_{-1}^{1}\mathrm{p}^{(n+1)}(x)\,\mathrm{g}_{n+1}(x)\,\mathrm{d}x$	1
(ii) $\int_{-1}^{1} x\,\mathrm{h}_n(x)\,\mathrm{h}_{n+1}(x)\,\mathrm{d}x$	
$=\frac{(-1)^{n+1}}{2^{n+1}(n+1)!}\int_{-1}^{1}(x\,\mathrm{h}_n(x))^{(n+1)}\,\mathrm{g}_{n+1}(x)\,\mathrm{d}x$ (by (c)(i))	
$=\frac{(-1)^{n+1}}{2^{n+1}2^{n}\,(n+1)!\,n!}\int_{-1}^{1}\left(x\,\mathrm{g}_n{}^{(2n+1)}(x)+(n+1)\,\mathrm{g}_n{}^{(2n)}(x)\right)\mathrm{g}_{n+1}(x)\,\mathrm{d}x$	1M can be absorbed
$=\frac{(-1)^{n+1}}{2^{2n+1}(n+1)!\,n!}\int_{-1}^{1}(n+1)(2n)!\,\mathrm{g}_{n+1}(x)\,\mathrm{d}x$ (since $\mathrm{g}_n{}^{(2n+1)}(x)=0$)	1A
$=\frac{(-1)^{n+1}(n+1)(2n)!}{2^{2n+1}(n+1)!\,n!}\int_{-1}^{1}\mathrm{g}_{n+1}(x)\,\mathrm{d}x$	
$=\frac{(-1)^{n+1}(n+1)(2n)!}{2^{2n+1}(n+1)!\,n!}I_{n+1}$	
$=\frac{(-1)^{n+1}(n+1)(2n)!}{2^{2n+1}(n+1)!\,n!}\cdot\frac{(-1)^{n+1}2^{2n+3}((n+1)!)^2}{(2n+3)!}$ (by (a))	1M for using (a)
$=\frac{2(n+1)}{(2n+1)(2n+3)}$	1A
	--------(7)

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Solution	Marks
11. (a) (i) For each $k = 1, 2, \cdots, n+1$, $\cos^k x$ is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.	1M
So, $\cos x + \cos^2 x + \cdots + \cos^{n+1} x$ is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.	1
Thus, f_n is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.	
Since $f_n(x) = \cos x + \cos^2 x + \cdots + \cos^{n+1} x$, we have	
$\frac{d}{dx} f_n(x) = -\sin x - 2\cos x \sin x - \cdots - (n+1)\cos^n x \sin x$	
$= -\sin x\,(1 + 2\cos x + 3\cos^2 x + \cdots + (n+1)\cos^n x)$	
< 0 for all $x \in \left(0, \frac{\pi}{2}\right]$	1M
Note that f_n is continuous at 0 .	
Thus, f_n is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.	1
(ii) By (a)(i), f_n is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.	
It follows that the equation $f_n(x) = 1$ has at most one root in $\left(0, \frac{\pi}{2}\right)$.	
Define $g_n(x) = f_n(x) - 1$ for all $x \in \left[0, \frac{\pi}{2}\right]$. Then,	
$g_n(0) = f_n(0) - 1 = n + 1 - 1 = n > 0$	
$g_n(\frac{\pi}{2}) = f_n(\frac{\pi}{2}) - 1 = 0 - 1 = -1 < 0$	
So, we have $g_n(0)\, g_n(\frac{\pi}{2}) < 0$.	1M
Note that g_n is continuous on $\left[0, \frac{\pi}{2}\right]$.	
Thus, the equation $f_n(x) = 1$ has one and only one root in $\left(0, \frac{\pi}{2}\right)$.	1
	---------(4)
(b) (i) Since $\cos\alpha_1 + \cos^2\alpha_1 = 1$, we have $\cos^2\alpha_1 + \cos\alpha_1 - 1 = 0$.	
Solving, we have $\cos\alpha_1 = \frac{-1 \pm \sqrt{5}}{2}$.	1M
Therefore, we have $\cos\alpha_1 = \frac{\sqrt{5} - 1}{2}$ (since $0 \le \alpha_1 \le \frac{\pi}{2}$).	
$\cos\alpha_1 = \frac{\sqrt{5}-1}{2} = \frac{(\sqrt{5}-1)(\sqrt{5}+1)}{2(\sqrt{5}+1)} = \frac{4}{2(\sqrt{5}+1)} = \frac{2}{\sqrt{5}+1} \le \frac{2}{\sqrt{4}+1} = \frac{2}{3}$	1

Solution	Marks
Assume that $\cos\alpha_1 > \frac{2}{3}$.	
Since $\cos\alpha_1 + \cos^2\alpha_1 = 1$, we have $\frac{2}{3} + \left(\frac{2}{3}\right)^2 < 1$.	1M
Therefore, we have $\frac{10}{9} < 1$ which is impossible.	
Thus, we have $\cos\alpha_1 \le \frac{2}{3}$.	1
(ii) $f_n(\alpha_n) = 1$ $f_n(\alpha_{n+1}) = \cos\alpha_{n+1} + \cos^2\alpha_{n+1} + \cdots + \cos^{n+1}\alpha_{n+1}$ $= f_{n+1}(\alpha_{n+1}) - \cos^{n+2}\alpha_{n+1}$ Note that $f_{n+1}(\alpha_{n+1}) = 1$.	1A either one
So, we have $f_n(\alpha_{n+1}) = 1 - \cos^{n+2}\alpha_{n+1}$ ≤ 1 ($\because$ $\cos^{n+2}\alpha_{n+1} \ge 0$ since $0 \le \alpha_{n+1} \le \frac{\pi}{2}$) So, we have $f_n(\alpha_{n+1}) \le f_n(\alpha_n)$. By (a)(i), f_n is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.	
Therefore, we have $\alpha_{n+1} \ge \alpha_n$.	1M for using (a)(i)
Thus, the sequence $\{\alpha_n\}$ is monotonic increasing.	1A must show reasons
$f_{n+1}(\alpha_{n+1}) = 1$ $f_{n+1}(\alpha_n) = \cos\alpha_n + \cos^2\alpha_n + \cdots + \cos^{n+2}\alpha_n$ $= f_n(\alpha_n) + \cos^{n+2}\alpha_n$ Note that $f_n(\alpha_n) = 1$.	1A either one
So, we have $f_{n+1}(\alpha_n) = 1 + \cos^{n+2}\alpha_n$ ≥ 1 ($\because$ $\cos^{n+2}\alpha_n \ge 0$ since $0 \le \alpha_n \le \frac{\pi}{2}$) So, we have $f_{n+1}(\alpha_{n+1}) \le f_{n+1}(\alpha_n)$. By (a)(i), f_{n+1} is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.	
Therefore, we have $\alpha_{n+1} \ge \alpha_n$.	1M for using (a)(i)
Thus, the sequence $\{\alpha_n\}$ is monotonic increasing.	1A must show reasons

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Solution	Marks
Note that $f_{n+1}(x)-f_n(x)=\cos^{n+2}x\ge 0$ for all $x\in\left[0,\frac{\pi}{2}\right]$.	
Therefore, we have $f_{n+1}(x)\ge f_n(x)$ for all $x\in\left[0,\frac{\pi}{2}\right]$.	
Since $\alpha_n\in\left[0,\frac{\pi}{2}\right]$, we have $f_{n+1}(\alpha_n)\ge f_n(\alpha_n)$.	
Note that $f_n(\alpha_n)=1=f_{n+1}(\alpha_{n+1})$.	1A for either equality
So, we have $f_{n+1}(\alpha_n)\ge f_{n+1}(\alpha_{n+1})$.	
By (a)(i), f_{n+1} is strictly decreasing on $\left[0,\frac{\pi}{2}\right]$.	
Therefore, we have $\alpha_{n+1}\ge\alpha_n$.	1M for using (a)(i)
Thus, the sequence $\{\alpha_n\}$ is monotonic increasing.	1A must show reasons
(iii) By (b)(ii), we have $0\le\alpha_1\le\alpha_n\le\frac{\pi}{2}$.	
So, we have $0\le\cos\alpha_n\le\cos\alpha_1$.	
Therefore, by (b)(i), we have $0\le\cos\alpha_n\le\frac{2}{3}$.	1A
Hence, we have $0\le\cos^n\alpha_n\le\left(\frac{2}{3}\right)^n$ for all positive integers n.	
Since $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0$, we have $\lim_{n\to\infty}\cos^n\alpha_n=0$.	1A must mention $0\le\cos\alpha_n\le C<1$
(iv) $\because$ the sequence $\{\alpha_n\}$ is monotonic increasing (by (b)(ii)) and bounded above by $\frac{\pi}{2}$.	
$\therefore$ the sequence $\{\alpha_n\}$ is convergent.	1
Let $\lim_{n\to\infty}\alpha_n=\alpha$.	
Since $f_n(\alpha_n)=1$, we have $\cos\alpha_n+\cos^2\alpha_n+\cdots+\cos^{n+1}\alpha_n=1$.	
Therefore, we have $\frac{\cos\alpha_n(1-\cos^{n+1}\alpha_n)}{1-\cos\alpha_n}=1$.	1M
Note that $\lim_{n\to\infty}\cos\alpha_n=\cos\alpha$ and $\lim_{n\to\infty}\cos^n\alpha_n=0$ (by (b)(iii)).	
So, we have $\lim_{n\to\infty}\cos^{n+1}\alpha_n=(\lim_{n\to\infty}\cos\alpha_n)\left(\lim_{n\to\infty}\cos^n\alpha_n\right)=0$.	
Hence, we have $\frac{(\cos\alpha)(1-0)}{1-\cos\alpha}=1$.	1A withhold 1A for no justification
Therefore, we have $\cos\alpha=1-\cos\alpha$.	
Hence, we have $\cos\alpha=\frac{1}{2}$.	
Solving, we have $\alpha=\frac{\pi}{3}$ ($\because$ $0\le\alpha\le\frac{\pi}{2}$ since $0\le\alpha_n\le\frac{\pi}{2}$).	1A
Thus, we have $\lim_{n\to\infty}\alpha_n=\frac{\pi}{3}$.	
	---------(11)

Solution	Marks
12. (a) (i) When $m=0$, RS $=\int_0^x f'(t)\,dt=\left[f(t)\right]_{t=0}^{t=x}=f(x)-f(0)=E_0(x)=$ LS	1
When $m\geq 1$, we have	
RS	
$=\frac{1}{m!}\int_0^x (x-t)^m f^{(m+1)}(t)\,dt$	
$=\frac{1}{m!}\int_0^x (x-t)^m\, df^{(m)}(t)$	1M
$=\frac{1}{m!}\left\{\left[(x-t)^m f^{(m)}(t)\right]_{t=0}^{t=x}+\int_0^x m(x-t)^{m-1}f^{(m)}(t)\,dt\right\}$	1A
$=\frac{-f^{(m)}(0)}{m!}x^m+\frac{1}{(m-1)!}\int_0^x (x-t)^{m-1}f^{(m)}(t)\,dt$	
$=\cdots$	
$=\frac{-f^{(m)}(0)}{m!}x^m-\frac{f^{(m-1)}(0)}{(m-1)!}x^{m-1}-\cdots-\frac{f'(0)}{1!}x+\int_0^x f'(t)\,dt$	
$=\frac{-f^{(m)}(0)}{m!}x^m-\frac{f^{(m-1)}(0)}{(m-1)!}x^{m-1}-\cdots-\frac{f'(0)}{1!}x+f(x)-f(0)$	
$=f(x)-\sum_{k=0}^{m}\frac{f^{(k)}(0)}{k!}x^k$	
$=E_m(x)$	
$=$ LS	1
When $m=0$, RS $=\int_0^x f'(t)\,dt=\left[f(t)\right]_{t=0}^{t=x}=f(x)-f(0)=E_0(x)=$ LS	1
Assume that $E_l(x)=\frac{1}{l!}\int_0^x (x-t)^l f^{(l+1)}(t)\,dt$ for some $l\geq 0$. Then,	
$E_{l+1}(x)$	
$=f(x)-\sum_{k=0}^{l+1}\frac{f^{(k)}(0)}{k!}x^k$	
$=f(x)-\sum_{k=0}^{l}\frac{f^{(k)}(0)}{k!}x^k-\frac{f^{(l+1)}(0)}{(l+1)!}x^{l+1}$	
$=E_l(x)-\frac{f^{(l+1)}(0)}{(l+1)!}x^{l+1}$	
$=\frac{1}{l!}\int_0^x (x-t)^l f^{(l+1)}(t)\,dt-\frac{f^{(l+1)}(0)}{(l+1)!}x^{l+1}$ (by induction assumption)	
$=\frac{-1}{(l+1)!}\int_0^x f^{(l+1)}(t)\,d(x-t)^{l+1}-\frac{f^{(l+1)}(0)}{(l+1)!}x^{l+1}$	1M
$=\frac{-1}{(l+1)!}\left\{\left[f^{(l+1)}(t)(x-t)^{l+1}\right]_{t=0}^{t=x}-\int_0^x (x-t)^{l+1}f^{(l+2)}(t)\,dt\right\}-\frac{f^{(l+1)}(0)}{(l+1)!}x^{l+1}$	1A
$=\frac{f^{(l+1)}(0)}{(l+1)!}x^{l+1}+\frac{1}{(l+1)!}\int_0^x (x-t)^{l+1}f^{(l+2)}(t)\,dt-\frac{f^{(l+1)}(0)}{(l+1)!}x^{l+1}$	
$=\frac{1}{(l+1)!}\int_0^x (x-t)^{l+1}f^{(l+2)}(t)\,dt$	1
Thus, by induction, the statement is true for all $m=0,1,2,\ldots$.	

Solution	Marks
(ii) By (a)(i), we have	
$\lvert \mathrm{E}_m(x) \rvert$	
$= \dfrac{1}{m!}\left\lvert \int_0^x (x-t)^m \mathrm{f}^{(m+1)}(t)\,\mathrm{d}t \right\rvert$	
$\le \dfrac{1}{m!}\left\lvert \int_0^x \lvert (x-t)^m \rvert\lvert \mathrm{f}^{(m+1)}(t) \rvert \mathrm{d}t \right\rvert$	1M
$\le \dfrac{C}{m!}\left\lvert \int_0^x \lvert (x-t)^m \rvert \mathrm{d}t \right\rvert$	
$= \dfrac{C}{m!}\left\lvert \int_0^x (x-t)^m \,\mathrm{d}t \right\rvert$	
$= \dfrac{C}{m!}\left\lvert \left[\dfrac{-(x-t)^{m+1}}{m+1} \right]_{t=0}^{t=x} \right\rvert$	
$= \dfrac{C}{(m+1)!}\lvert x^{m+1} \rvert$	
$= \dfrac{C}{(m+1)!}\lvert x \rvert^{m+1}$	1
	----------(6)
(b) (i) Let $\mathrm{f}(x) = \sin x$ for all $x \in \mathbf{R}$.	
Then, we have $\mathrm{f}^{(k)}(x) = \begin{cases} \sin x & \text{if } k = 0, 4, 8, \ldots \\ \cos x & \text{if } k = 1, 5, 9, \ldots \\ -\sin x & \text{if } k = 2, 6, 10, \ldots \\ -\cos x & \text{if } k = 3, 7, 11, \ldots \end{cases}$.	1A for all the four cases correct
Therefore, we have $\mathrm{f}^{(2k)}(x) = (-1)^k \sin x$ and $\mathrm{f}^{(2k+1)}(x) = (-1)^k \cos x$ for all $k = 0, 1, 2, \ldots$.	
Note that $\mathrm{f}^{(2k)}(0) = 0$ and $\mathrm{f}^{(2k+1)}(0) = (-1)^k$ for all $k = 0, 1, 2, \ldots$.	1A for both cases correct
Also note that $\sin x$ has derivatives of any order.	
Therefore, by (a)(i), we have	
$\sin x$	
$= \mathrm{f}(x)$	
$= \sum_{k=0}^{2n+1} \dfrac{\mathrm{f}^{(k)}(0)}{k!} x^k + \dfrac{1}{(2n+1)!}\int_0^x (x-t)^{2n+1} \mathrm{f}^{(2n+2)}(t)\,\mathrm{d}t$	1M for putting $m = 2n+1$ in (a)
$= \sum_{k=0}^{n} \dfrac{\mathrm{f}^{(2k)}(0)}{(2k)!} x^{2k} + \sum_{k=0}^{n} \dfrac{\mathrm{f}^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} + \dfrac{1}{(2n+1)!}\int_0^x (x-t)^{2n+1} \mathrm{f}^{(2n+2)}(t)\,\mathrm{d}t$	1M
$= \sum_{k=0}^{n} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!} + \dfrac{(-1)^{n+1}}{(2n+1)!}\int_0^x (x-t)^{2n+1} \sin t\,\mathrm{d}t$	1
(ii) Let $\mathrm{R}_{2n+1}(x) = \dfrac{(-1)^{n+1}}{(2n+1)!}\int_0^x (x-t)^{2n+1} \sin t\,\mathrm{d}t$. Then, by (b)(i),	
$\sin x = \sum_{k=0}^{n} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!} + \mathrm{R}_{2n+1}(x)$.	

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Solution	Marks

So, we have

$$\frac{3}{4}\sin\frac{1}{3}=\frac{3}{4}\sum_{k=0}^{n}\frac{(-1)^k\left(\frac{1}{3}\right)^{2k+1}}{(2k+1)!}+\frac{3}{4}R_{2n+1}\left(\frac{1}{3}\right)$$ 1M (either one)

$$=\frac{1}{4}\sum_{k=0}^{n}\frac{(-1)^k\left(\frac{1}{9^k}\right)}{(2k+1)!}+\frac{3}{4}R_{2n+1}\left(\frac{1}{3}\right) \quad \text{and}$$

$$\frac{1}{4}\sin 1=\frac{1}{4}\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!}+\frac{1}{4}R_{2n+1}(1) .$$

Therefore, we have

$$\frac{3}{4}\sin\frac{1}{3}-\frac{1}{4}\sin 1=\frac{1}{4}\sum_{k=0}^{n}\frac{(-1)^{k+1}\left(1-\frac{1}{9^k}\right)}{(2k+1)!}+\frac{3}{4}R_{2n+1}\left(\frac{1}{3}\right)-\frac{1}{4}R_{2n+1}(1) .$$

Hence, we have

$$\sin^3\frac{1}{3}=\frac{1}{4}\sum_{k=0}^{n}\frac{(-1)^{k+1}\left(1-\frac{1}{9^k}\right)}{(2k+1)!}+\frac{3}{4}R_{2n+1}\left(\frac{1}{3}\right)-\frac{1}{4}R_{2n+1}(1) .$$

So, we have

$$\left|\sin^3\frac{1}{3}-\frac{1}{4}\sum_{k=0}^{n}\frac{(-1)^{k+1}\left(1-\frac{1}{9^k}\right)}{(2k+1)!}\right|$$

$$=\left|\frac{3}{4}R_{2n+1}\left(\frac{1}{3}\right)-\frac{1}{4}R_{2n+1}(1)\right|$$ 1A

$$\le\frac{3}{4}\left|R_{2n+1}\left(\frac{1}{3}\right)\right|+\frac{1}{4}\left|R_{2n+1}(1)\right|$$ 1M

Note that $|f^{(k)}(x)|\le 1$ for all $k=0,1,2,\ldots$ and $x\in\mathbf{R}$.

By (a)(ii), we have

$$\left|R_{2n+1}\left(\frac{1}{3}\right)\right|\le\frac{\frac{1}{3^{2n+2}}}{(2n+2)!} \quad \text{and} \quad \left|R_{2n+1}(1)\right|\le\frac{1}{(2n+2)!} .$$

Thus, we have

$$\left|\sin^3\frac{1}{3}-\frac{1}{4}\sum_{k=0}^{n}\frac{(-1)^{k+1}\left(1-\frac{1}{9^k}\right)}{(2k+1)!}\right|$$

$$\le\frac{\frac{1}{3^{2n+1}}}{4(2n+2)!}+\frac{1}{4(2n+2)!}$$

$$=\frac{1}{4(2n+2)!}\left(1+\frac{1}{3^{2n+1}}\right)$$ 1

--------(9)

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