FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. For each positive integer n, define $S_n = (1+\sqrt{5})^n + (1-\sqrt{5})^n$.
   Prove that

   (a) $S_{n+2} = 2S_{n+1} + 4S_n$,

   (b) S_n is divisible by 2^n.

(6 marks)

2. For any two positive integers k and n, let T_r be the rth term in the expansion of $(1+x)^{kn}$ in ascending powers of x, i.e. $T_r = C_{r-1}^{kn} x^{r-1}$.

   (a) Suppose $x = \dfrac{2}{k}$. Find, in terms of k and n, the range of values of r such that $T_{r+1} \le T_r$.

   (b) Suppose $x = \dfrac{2}{3}$. Using the result of (a), find the greatest term in the expansion of $(1+x)^{51}$.

(6 marks)

(a) By considering the function $f(x) = x - \ln(x+1)$, or otherwise, prove that $x \geq \ln(x+1)$ for all $x > -1$.

(b) Using (a), prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(7 marks)

Let $f(x)$ be a polynomial of degree 4 with real coefficients. When $f(x)$ is divided by $x-2$, the remainder is 4. When $f(x)$ is divided by $x+3$, the remainder is -6. Let $r(x)$ be the remainder when $f(x)$ is divided by $(x-2)(x+3)$.

(a) Find $r(x)$.

(b) Let $g(x) = f(x) - r(x)$. It is known that $g(x)$ is divisible by x^2+1 and $g(1) = -16$. Find $g(x)$.

(7 marks)

5. Let T_1 be the transformation which transforms a vector $\mathbf{x}$ to a vector $\mathbf{y} = A\mathbf{x}$, where $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

(a) (i) Find $\mathbf{y}$ when $\mathbf{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

(ii) Describe the geometric meaning of the transformation T_1.

(iii) Find A^{2005}.

(b) For every integer n greater than 1, let T_n be the transformation which transforms a vector $\mathbf{x}$ to a vector $\mathbf{y} = A^n \mathbf{x}$.

Is there a positive integer m such that the transformation T_m transforms $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$? Explain your answer.

(7 marks)

6. Let θ be a real number.

(a) Solve the quadratic equation $z^2 + 2z\cos 6\theta + 1 = 0$.

(b) Using the result of (a), express $x^6 + 2x^3 \cos 6\theta + 1$ as a product of quadratic polynomials with real coefficients.

(7 marks)

B (60 marks)

y FOUR questions in this section. Each question carries 15 marks.

r answers in the AL(A) answer book.

Consider the system of linear equations in x , y , z

$$(E):\begin{cases} x + ay + z = b \\ 2x + (a+3)y + (a-1)z = 0 \\ 3x + a^2y + (4a+1)z = -b \end{cases},$$

where $a, b \in \mathbf{R}$.

(i) Find the range of values of a for which (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) For each of the following cases, find the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b .

(1) $a = 1$,

(2) $a = -2$.

(10 marks)

) Suppose that a real solution of $\begin{cases} x - 2y + z = b \\ 2x + y - 3z = 0 \\ 3x + 4y - 7z = -b \end{cases}$

satisfies $x^2 + y^2 + z^2 = b + 3$, where $b \in \mathbf{R}$. Find the range of values of b .

(5 marks)

8. (a) Let $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, where $p, q, r, s \in \mathbf{R}$.

(i) Suppose $\det M = 0$.

Prove that $M^{n+1} = (p+s)^n M$ for any positive integer n .

(ii) Suppose $qr > 0$. Let α and β be the roots of the quadratic equation $x^2 - (p+s)x + \det M = 0$.

Denote the 2×2 identity matrix by I .

(1) Prove that α and β are two distinct real numbers.

(2) Prove that $M^2 - (\alpha+\beta)M + \alpha\beta I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(3) Define $A = M - \alpha I$ and $B = M - \beta I$.

Prove that $AB = BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\det A = \det B = 0$.

Find real numbers λ and μ , in terms of α and β , such that $M = \lambda A + \mu B$.

(11 marks)

(b) Evaluate $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^n$, where n is a positive integer.

Candidates may use the fact, without proof, that if X and Y are 2×2 matrices satisfying $XY = YX = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $(X+Y)^n = X^n + Y^n$ for any positive integer n .

(4 marks)

9. Let a_1 and b_1 be real numbers satisfying $a_1 b_1 > 0$.
For each $n = 1, 2, 3, \ldots$, define

$$a_{n+1} = \frac{a_n{}^2 + b_n{}^2}{a_n + b_n} \quad \text{and} \quad b_{n+1} = \frac{2\,a_n\,b_n}{a_n + b_n} \ .$$

(a) Suppose $a_1 \geq b_1 > 0$.

(i) Prove that $a_n \geq b_n$ for all $n = 1, 2, 3, \ldots$.

(ii) Prove that the sequence $\{ a_n \}$ is monotonic decreasing and that the sequence $\{ b_n \}$ is monotonic increasing.

(iii) Prove that $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ both exist.

(iv) Prove that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

(v) Find $\lim\limits_{n\to\infty} (a_n + b_n)$ and $\lim\limits_{n\to\infty} a_n$ in terms of a_1 and b_1 .

(12 marks)

(b) Suppose $a_1 \leq b_1 < 0$.
Do the limits of the sequences $\{ a_n \}$ and $\{ b_n \}$ exist? Explain your answer.

(3 marks)

10. (a) Let $f(x) = x^4 + 2ax^2 + 4bx + c$, where $a, b, c \in \mathbf{R}$ with $b \neq 0$. It is known that $f(x) \equiv (x^2 + 2tx + r)(x^2 - 2tx + s)$, where $r, s, t \in \mathbf{R}$.

(i) Prove that $t \neq 0$.

(ii) Express r and s in terms of a , b and t .

(iii) Prove that $4t^6 + 4at^4 + (a^2 - c)t^2 - b^2 = 0$.

(6 marks)

(b) Consider the equation

$$y^4 + 4y^3 - 2y^2 + 52y + 9 = 0 \qquad \ldots\ldots\ldots\ldots\ (*) .$$

(i) Find a constant h such that when $y = x + h$, (*) can be written as

$$x^4 - 8x^2 + 64x - 48 = 0 \qquad \ldots\ldots\ldots\ldots\ (**) .$$

(ii) Using the results of (a), solve (**) in (b)(i).
Hence write down all the roots of (*) .

(9 marks)

11. (a) For any positive integer n, prove that

$$t^n - 1 \geq n(t-1) \quad \text{for all } t > 0 .$$

(3 marks)

(b) (i) Let a, b and c be positive real numbers.

By putting $n = 3$ and $t = \dfrac{\sqrt[3]{abc}}{\sqrt{ab}}$ in (a), prove that

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \geq \frac{2}{3}\left(\frac{a+b}{2} - \sqrt{ab}\right) .$$

(ii) Let $y_1, y_2, \ldots, y_{k+1}$ be positive real numbers, where k is a positive integer. Using (a), prove that

$$y_{k+1} \geq (k+1)G_{k+1} - kG_k ,$$

where $G_k = \sqrt[k]{y_1 y_2 \cdots y_k}$ and $G_{k+1} = \sqrt[k+1]{y_1 y_2 \cdots y_{k+1}}$.

(iii) Using mathematical induction and (b)(ii), prove that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

for any n positive real numbers $x_1, x_2, \ldots, x_n$.

(8 marks)

(c) Let n be a positive integer. Using (b)(iii), prove that

$$n^n \geq (1)(3)(5)\cdots(2n-1) .$$

Hence prove that $(n^2 + n)^n \geq (2n)!$.

(4 marks)

12. (a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbf{R}^3$.

(i) Prove that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

Hence deduce that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.

(ii) Suppose $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \neq 0$. Prove that

$$\mathbf{x} = \left(\frac{\mathbf{x}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\right)\mathbf{a} + \left(\frac{\mathbf{x}\cdot(\mathbf{c}\times\mathbf{a})}{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}\right)\mathbf{b} + \left(\frac{\mathbf{x}\cdot(\mathbf{a}\times\mathbf{b})}{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}\right)\mathbf{c}$$

for any vector $\mathbf{x}$ in $\mathbf{R}^3$.

(iii) Suppose $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = 1$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$.

(1) Prove that $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = 1$.

(2) Using (a)(ii), prove that

$$\mathbf{x} = (\mathbf{x}\cdot\mathbf{a})\mathbf{a} + (\mathbf{x}\cdot\mathbf{b})\mathbf{b} + (\mathbf{x}\cdot\mathbf{c})\mathbf{c}$$

for any vector $\mathbf{x}$ in $\mathbf{R}^3$.

(12 marks)

(b) Let $\mathbf{u} = \dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\mathbf{v} = \dfrac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{k})$, $\mathbf{w} = \dfrac{1}{\sqrt{6}}(\mathbf{i}-2\mathbf{j}+\mathbf{k})$ and $\mathbf{r} = 6\mathbf{i} - \mathbf{j} + 10\mathbf{k}$.

Find real numbers α, β and γ such that $\mathbf{r} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$.

(3 marks)

END OF PAPER

FORMULAS FOR REFERENCE

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$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

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SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Let $f(x) = x^x$ for all $x > 0$.

   Prove that $f'(x) = x^x(1 + \ln x)$.

   Hence evaluate $\lim_{x \to 1} \frac{x^x - x}{\ln x - x + 1}$.

   (b) Evaluate $\lim_{x \to 0} \frac{\int_0^x t\sin(\sin t)\,dt}{x^3}$.

(7 marks)

2. Let a be a constant and $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \frac{x^2}{2\pi} - x + a & \text{when } x < \pi, \\ a\cos x & \text{when } x \geq \pi. \end{cases}$$

It is known that f is continuous everywhere.

(a) Prove that $a = \frac{\pi}{4}$.

(b) Prove that f is differentiable at π.

(c) Is f' continuous at π? Explain your answer.

(7 marks)

3. (a) Resolve $\dfrac{3-5x}{(1+x)(1-x+2x^2)}$ into partial fractions.

(b) Evaluate $\displaystyle\int_0^{\infty} \frac{3-5x}{(1+x)(1-x+2x^2)}\,\mathrm{d}x$.

(6 marks)

4. (a) (i) Evaluate $\displaystyle\int \sqrt{1+x}\,\mathrm{d}x$.

(ii) Prove that $\displaystyle\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-x}}\,\mathrm{d}x = 2\sqrt{6}-4-\frac{\sqrt{2}\,\pi}{6}$.

(b) Consider the curve $C: y=\sqrt{1-x^2}-\arcsin x$, $0 \le x \le \dfrac{1}{2}$.

Find the area of the surface generated by rotating C about the x-axis.

(7 marks)

5. Let $P: y^2 = 80x$ be a parabola.

(a) Prove that the straight line $y = mx + c$ is a tangent to P if and only if $mc = 20$.

(b) Consider the curve $\Gamma: \begin{cases} x = 3\cos\theta , \\ y = \sin\theta , \end{cases} \quad \dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$.

Find the coordinates of the two points on Γ at which the normals to Γ are tangents to P .

(7 marks)

6. Let $L_1 : \dfrac{x+2}{3} = \dfrac{y-3}{4} = z-2$ and $L_2 : x-3 = 5-y = 1-z$ be two straight lines.

(a) Prove that L_1 and L_2 intersect at a point and find the coordinates of the point of intersection.

(b) Find the acute angle between L_1 and L_2 .

(c) Find the equation of the plane containing L_1 and L_2 .

(6 marks)

SECTION B (60 marks)

Answer any FOUR questions in this section. Each question carries 15 marks.
Write your answers in the AL(A) answer book.

7. Define $f(x) = -x + |x|\sqrt{\dfrac{x}{x+2}}$ for $x < -2$ or $x > 0$.

(a) (i) Find $f'(x)$ and $f''(x)$ for $x > 0$.

(ii) Write down $f'(x)$ and $f''(x)$ for $x < -2$.

(4 marks)

(b) Solve each of the following inequalities:

(i) $f'(x) > 0$,

(ii) $f''(x) > 0$.

(3 marks)

(c) Find the relative extreme point(s) of the graph of $y = f(x)$.

(2 marks)

(d) Find the asymptote(s) of the graph of $y = f(x)$.

(4 marks)

(e) Sketch the graph of $y = f(x)$.

(2 marks)

8. Let $f : \mathbf{R} \to \mathbf{R}$ be a function satisfying the following conditions:

(1) $f(x+y) = e^x f(y) + e^y f(x)$ for all $x, y \in \mathbf{R}$;

(2) $\lim_{h\to 0} \dfrac{f(h)}{h} = 2005$.

(a) Find $f(0)$.

(1 mark)

(b) Find $\lim_{h\to 0} f(h)$. Hence prove that f is a continuous function.

(4 marks)

(c) (i) Prove that f is differentiable everywhere and that $f'(x) = 2005e^x + f(x)$ for all $x \in \mathbf{R}$.

(ii) Let n be a positive integer. Using (c)(i), find $f^{(n)}(0)$.

(6 marks)

(d) By considering the derivative of the function $\dfrac{f(x)}{e^x}$, find $f(x)$.

(4 marks)

9. Let $H_1 : xy = 4$ and $H_2 : xy = 1$ be two hyperbolas, and L be the tangent to H_1 at $P(2t, \frac{2}{t})$.

(a) Find the equation of L .

(2 marks)

(b) Let $A(\alpha, \frac{1}{\alpha})$ and $B(\beta, \frac{1}{\beta})$ be the two distinct points where L intersects H_2 .

(i) Prove that $\alpha + \beta = 4t$ and $\alpha\beta = t^2$.

(ii) Prove that the length of chord AB is $\sqrt{12\left(t^2 + \frac{1}{t^2}\right)}$.

(6 marks)

(c) It is given that the tangents to H_2 at A and B intersect at Q, where A and B are the points described in (b).

(i) Find the coordinates of Q in terms of t .

(ii) Using (b)(ii), or otherwise, prove that the area of ΔQAB is independent of t .

(7 marks)

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10. Define $g_0(x) = 1$ and $g_n(x) = (x^2 - 1)^n$ for every positive integer n .

(a) For every non-negative integer n , let $I_n = \int_{-1}^{1} g_n(x)\,dx$.

Express I_{n+1} in terms of I_n .

Hence prove that $I_{n+1} = \frac{(-1)^{n+1} 2^{2n+3} ((n+1)!)^2}{(2n+3)!}$.

(5 marks)

(b) Prove that $g_{n+1}{}^{(k)}(-1) = g_{n+1}{}^{(k)}(1) = 0$ for all $k = 0, 1, \ldots, n$,

where n is a non-negative integer.

(3 marks)

(c) For every non-negative integer n , let $h_n(x) = \frac{g_n{}^{(n)}(x)}{2^n \, n!}$.

(i) Using (b), prove that

$$\int_{-1}^{1} p(x) h_{n+1}(x)\,dx = \frac{(-1)^{n+1}}{2^{n+1}(n+1)!} \int_{-1}^{1} p^{(n+1)}(x) g_{n+1}(x)\,dx$$

for any polynomial $p(x)$.

(ii) Using (c)(i), or otherwise, evaluate $\int_{-1}^{1} x\, h_n(x) h_{n+1}(x)\,dx$.

(7 marks)

Note: For any function f, $f^{(0)} = f$.

11. Let $f_n : \mathbf{R} \to \mathbf{R}$ be defined by $f_n(x) = \cos x + \cos^2 x + \cdots + \cos^{n+1} x$,

where $n = 1, 2, 3, \ldots$.

(a) (i) Prove that f_n is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$.

(ii) Prove that the equation $f_n(x) = 1$ has one and only one root in $\left(0, \frac{\pi}{2}\right)$.

(4 marks)

(b) For each $n = 1, 2, 3, \ldots$, let α_n be the root of the equation $f_n(x) = 1$ in $\left(0, \frac{\pi}{2}\right)$.

(i) Prove that $\cos \alpha_1 \le \frac{2}{3}$.

(ii) Is the sequence $\{ \alpha_n \}$ monotonic? Explain your answer.

(iii) Find $\lim_{n \to \infty} \cos^n \alpha_n$.

(iv) Prove that the sequence $\{ \alpha_n \}$ is convergent and find its limit.

(11 marks)

12. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be a function with derivatives of any order. For each $m = 0, 1, 2, \ldots$ and $x \in \mathbf{R}$, define

$$E_m(x) = f(x) - \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} x^k \text{ , where } f^{(0)} = f \text{ .}$$

(i) Using integration by parts, prove that

$$E_m(x) = \frac{1}{m!} \int_0^x (x-t)^m f^{(m+1)}(t) \, dt \text{ .}$$

(ii) Suppose that there is a constant C such that $| f^{(k)}(x) | \le C$ for all $k = 0, 1, 2, \ldots$ and $x \in \mathbf{R}$. Using (a)(i), prove that

$$| E_m(x) | \le \frac{C}{(m+1)!} | x |^{m+1} \text{ .}$$

(6 marks)

(b) Let n be a non-negative integer.

(i) Using (a)(i), prove that

$$\sin x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^{n+1}}{(2n+1)!} \int_0^x (x-t)^{2n+1} \sin t \, dt$$

for all $x \in \mathbf{R}$.

(ii) Using (b)(i) and the identity $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$, prove that

$$\left| \sin^3 \frac{1}{3} - \frac{1}{4} \sum_{k=0}^{n} \frac{(-1)^{k+1} \left(1 - \frac{1}{9^k} \right)}{(2k+1)!} \right| \le \frac{1}{4(2n+2)!} \left(1 + \frac{1}{3^{2n+1}} \right) \text{ .}$$

(9 marks)

END OF PAPER