1. (a) $\dfrac{1}{(2x-1)(2x+1)(2x+3)}=\dfrac{A}{2x-1}+\dfrac{B}{2x+1}+\dfrac{C}{2x+3}$

$1=A(2x+1)(2x+3)+B(2x-1)(2x+3)+C(2x-1)(2x+1)$

put $x=\dfrac{1}{2}$ $\quad A=\dfrac{1}{8}$

put $x=-\dfrac{1}{2}$ $\quad B=-\dfrac{1}{4}$

put $x=-\dfrac{3}{2}$ $\quad C=\dfrac{1}{8}$

$\therefore \dfrac{1}{(2x-1)(2x+1)(2x+3)}=\dfrac{1}{8(2x-1)}-\dfrac{1}{4(2x+1)}+\dfrac{1}{8(2x+3)}$

(b) $\therefore \sum_{k=1}^{n}\dfrac{1}{(2k-1)(2k+1)(2k+3)}=\dfrac{1}{8}\sum_{k=1}^{n}\dfrac{1}{(2k-1)}-\dfrac{1}{(2k+1)}+\dfrac{1}{(2k+3)}-\dfrac{1}{(2k+1)}$ by (a)

$=\dfrac{1}{8}(1-\dfrac{1}{2n+1}-\dfrac{1}{3}+\dfrac{1}{2n+3})$

$=\dfrac{1}{12}-\dfrac{1}{8(2n+1)}+\dfrac{1}{8(2n+3)}$

$\therefore \sum_{k=10}^{\infty}\dfrac{1}{(2k-1)(2k+1)(2k+3)}=\lim_{n\to\infty}\dfrac{1}{12}-\dfrac{1}{8(2n+1)}+\dfrac{1}{8(2n+3)}-(\dfrac{1}{12}-\dfrac{1}{8(2(9)+1)}+\dfrac{1}{8(2(9)+3)})$

$=\dfrac{1}{1596}$

2. (a) $a_1=1\le 3$

Suppose $a_k\le 3$

$a_{k+1}-3=\dfrac{12a_k+12}{a_k+13}-3=\dfrac{9a_k-27}{a_k+13}=\dfrac{9(a_k-3)}{a_k+13}\le 0$

$\therefore$ By the principle of M.I. $a_n\le 3$

(b) $a_{n+1}-a_n=\dfrac{12a_n+12}{a_n+13}-a_n=\dfrac{(3-a_n)(a_n+4)}{a_n+13}\ge 0$

$\therefore \{a_n\}$ is increasing and bounded

$\therefore \{a_n\}$ is convergent

$\lim_{n\to\infty}a_n$ exist

Let $\lim_{n\to\infty}a_n=l$

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\dfrac{12a_{n-1}+12}{a_{n-1}+13}$

$\Rightarrow l = \dfrac{12l+12}{l+13}$

$\Rightarrow l^2 + l - 12 = 0$

$\Rightarrow l = 3$ or -4 (rej.)

3. (a) (i) $R = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

$R^6 = \begin{pmatrix} \cos 360^\circ & -\sin 360^\circ \\ \sin 360^\circ & \cos 360^\circ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $A^{-1}RA = \dfrac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{pmatrix}$

$= \dfrac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix}$

$= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

(b) $\left(A^{-1}RA\right)^2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

$\therefore M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

where $M \neq I$ and $M^3 = I$

4. (a) $f(x) = (x^2 + q)(x + \frac{r}{q})$

$= x^3 + \frac{r}{q}x^2 + qx + r$

compare the coefficient of x^2

$p = \frac{r}{q}$

$\Rightarrow r = pq$

(b) (i) $f(x) = (x^2 - a^2)(x + p)$ by (a)

$= (x-a)(x+a)(x+p)$

(ii) $f(x+a) = x(x+2a)(x+a+p)$

compare with (i) $p = 2a$ or $-2a$

5. (a) (i) $1 + a^{k+1} - a - a^k = a^k(a-1) + 1 - a$

$= (a-1)(a^k-1)$

$= (a-1)^2(a^{k-1}+a^{k-2}+.....+a+1) \geq 0$

(ii) When $n = 1$ obviously it is true

Suppose $(1+a)^k \leq 2^{k-1}(1+a^k)$

When $n = k+1$

$(1+a)^{k+1} \leq (1+a)2^{k-1}(1+a^k)$

$= 2^{k-1}(1+a^k+a+a^{k+1})$

$\leq 2^{k-1}(1+a^{k+1}+1+a^{k+1})$ by (i)

$= 2^k(1+a^{k+1})$

$\therefore$ By the principle of M.I. it is true.

(b) put $a = \dfrac{y}{x}$

$$\left(1+\frac{y}{x}\right)^n \leq 2^{n-1}\left(1+\left(\frac{y}{x}\right)^n\right)$$

$$\Rightarrow (x+y)^n \leq 2^{n-1}\left(x^n+y^n\right)$$

$$\Rightarrow \left(\frac{x+y}{2}\right)^n \leq \frac{x^n+y^n}{2}$$

6. (a) $x_{n+1}x_1 - x_{n+2} = \mathrm{f}(\theta)x_n$

$\Rightarrow (\sin^{n+1}\theta+\cos^{n+1}\theta)(\sin\theta+\cos\theta)-(\sin^{n+2}\theta+\cos^{n+2}\theta) = \mathrm{f}(\theta)(\sin^n\theta+\cos^n\theta)$

$\Rightarrow \sin^{n+1}\theta\cos\theta+\sin\theta\cos^{n+2}\theta = \mathrm{f}(\theta)(\sin^n\theta+\cos^n\theta)$

$\Rightarrow \cos\theta\sin\theta = \mathrm{f}(\theta)$

$$\mathrm{f}(\theta) = \frac{(\sin\theta+\cos\theta)^2-1}{2} = \frac{x_1^{\ 2}-1}{2}$$

(b) x_1 is rational

$x_2 = \sin^2\theta+\cos^2\theta = 1$ is also rational

Suppose x_k, x_{k+1} are rational numbers

$x_{k+2} = x_{k+1}x_1 - \dfrac{x_1^{\ 2}-1}{2}x_k$ is rational

$\therefore$ by the principle of M.I. x_n is a rational number for every n.

7. (a) (i) (E) has a unique solution

$$\Leftrightarrow \begin{vmatrix} 1 & a-2 & a \\ 1 & 2 & 4 \\ a & -1 & 3 \end{vmatrix} \neq 0$$

$\Leftrightarrow 2a^2-12a+16 \neq 0$

$\Leftrightarrow (a-2)(a-4) \neq 0$

$\Leftrightarrow a \neq 2$ and $a \neq 4$

$$x=\frac{\begin{vmatrix}1 & a-2 & a\\ 1 & 2 & 4\\ b & -1 & 3\end{vmatrix}}{2(a-2)(a-4)}=\frac{2ab-8b-4a+16}{2(a-2)(a-4)}=\frac{b-2}{a-2}$$

$$y=\frac{\begin{vmatrix}1 & 1 & a\\ 1 & 1 & 4\\ a & b & 3\end{vmatrix}}{2(a-2)(a-4)}=\frac{ab-4b+4a-a^2}{2(a-2)(a-4)}=\frac{b-a}{2(a-2)}$$

$$y=\frac{\begin{vmatrix}1 & a-2 & 1\\ 1 & 2 & 1\\ a & -1 & b\end{vmatrix}}{2(a-2)(a-4)}=\frac{a^2-4a+4b-ab}{2(a-2)(a-4)}=\frac{a-b}{2(a-2)}$$

(ii) (1) $\begin{pmatrix}1 & 0 & 2 & 1\\ 1 & 2 & 4 & 1\\ 2 & -1 & 3 & b\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 2 & 1\\ 5 & 0 & 10 & 1+2b\\ 2 & -1 & 3 & b\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 2 & 1\\ 0 & 0 & 0 & 2b-4\\ 2 & -1 & 3 & b\end{pmatrix}$

$\therefore b = 2$ for which (E) is consistent.

$\begin{pmatrix}1 & 0 & 2 & 1\\ 2 & -1 & 3 & 2\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 2 & 1\\ 0 & -1 & -1 & 0\end{pmatrix}$

s.s. = $\{(1-2t,-t,t):t\in R\}$

(2) $\begin{pmatrix}1 & 2 & 4 & 1\\ 1 & 2 & 4 & 1\\ 4 & -1 & 3 & b\end{pmatrix}\sim\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 2 & 4 & 1\\ 4 & -1 & 3 & b\end{pmatrix}$

b any real number for which (E) is consistent

$\begin{pmatrix}9 & 0 & 10 & 1+2b\\ 4 & -1 & 3 & b\end{pmatrix}\sim\begin{pmatrix}9 & 0 & 10 & 1+2b\\ 13 & -10 & 0 & 4b-3\end{pmatrix}$

s.s. $=\left\{(t,\frac{4b-3-13t}{-10},\frac{1+2b-9t}{10}):t\in R\right\}$

(b) by (a) the solution of the system is $\{(1-2t,-t,t):t\in R\}$

$k((1-2t)^2-3)>-t^2$

$\Rightarrow (4k+1)t^2-4kt-2k>0$

$\Delta<0$ and 4k+1 > 0

$\Rightarrow 16k^2-4(4k+1)(-2k)<0$ and $k > -0.25$

$\Rightarrow 6k^2+k<0$ and $k > -0.25$

$\Rightarrow \frac{-1}{6}<x<0$

8. (a) $XY = \dfrac{1}{(\alpha-\beta)(\beta-\alpha)}(A-\beta I)(A-\alpha I)$

$$= \frac{1}{(\alpha-\beta)(\beta-\alpha)}\begin{pmatrix}\alpha-k-\beta & \alpha-\beta-k\\ k & k\end{pmatrix}\begin{pmatrix}-k & \alpha-\beta-k\\ k & \beta+k-\alpha\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$YX = \frac{1}{(\alpha-\beta)(\beta-\alpha)}\begin{pmatrix}-k & \alpha-\beta-k\\ k & \beta+k-\alpha\end{pmatrix}\begin{pmatrix}\alpha-k-\beta & \alpha-\beta-k\\ k & k\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$X+Y = \begin{pmatrix}\dfrac{\alpha-k-\beta}{\alpha-\beta} & \dfrac{\alpha-k-\beta}{\alpha-\beta}\\ \dfrac{k}{\alpha-\beta} & \dfrac{k}{\alpha-\beta}\end{pmatrix} + \begin{pmatrix}\dfrac{-k}{\beta-\alpha} & \dfrac{\alpha-k-\beta}{\beta-\alpha}\\ \dfrac{k}{\beta-\alpha} & \dfrac{\beta+k-\alpha}{\beta-\alpha}\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$X^2 = \frac{1}{(\alpha-\beta)^2}\begin{pmatrix}\alpha-k-\beta & \alpha-\beta-k\\ k & k\end{pmatrix}\begin{pmatrix}\alpha-k-\beta & \alpha-\beta-k\\ k & k\end{pmatrix}$$

$$= \frac{1}{(\alpha-\beta)^2}\begin{pmatrix}(\alpha-k-\beta)(\alpha-\beta) & (\alpha-\beta-k)(\alpha-\beta)\\ k(\alpha-\beta) & k(\alpha-\beta)\end{pmatrix}$$

$$= \frac{1}{(\alpha-\beta)}\begin{pmatrix}\alpha-k-\beta & \alpha-\beta-k\\ k & k\end{pmatrix}$$

$$Y^2 = \frac{1}{(\beta-\alpha)^2}\begin{pmatrix}-k & \alpha-\beta-k\\ k & \beta+k-\alpha\end{pmatrix}\begin{pmatrix}-k & \alpha-\beta-k\\ k & \beta+k-\alpha\end{pmatrix}$$

$$= \frac{1}{\beta-\alpha}\begin{pmatrix}-k & \alpha-\beta-k\\ k & \beta+k-\alpha\end{pmatrix}$$

(b) when $n=1$

$$\alpha X+\beta Y = \frac{\alpha}{\alpha-\beta}(A-\beta I)+\frac{\beta}{\beta-\alpha}(A-\alpha I)$$

$$= \frac{\alpha}{\alpha-\beta}A+\frac{\beta}{\beta-\alpha}A$$

$$= A$$

Suppose $A^k = \alpha^k X+\beta^k Y$

$A^{k+1} = A^k A = (\alpha^k X+\beta^k Y)(\alpha X+\beta Y)$

$= \alpha^{k+1}X+\beta^{k+1}Y$ by (a) $XY = YX = 0$

$\therefore$ By the principle of M.I. it is true.

(c) put $k=2$, $\alpha=7, \beta=1$

then $A^{2004} = 7^{2004} \cdot \frac{1}{6}\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} + 1^{2004} \cdot \frac{1}{-6}\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}$

$$= \begin{pmatrix} \frac{2\cdot 7^{2004}+1}{3} & \frac{2\cdot 7^{2004}-2}{3} \\ \frac{7^{2004}-1}{3} & \frac{7^{2004}+2}{3} \end{pmatrix}$$

(d) $A^{-1} = \frac{1}{\alpha\beta}(\beta X + \alpha Y)$

$AA^{-1} = \frac{1}{\alpha\beta}(\alpha X + \beta Y)(\beta X + \alpha Y)$

$= \frac{1}{\alpha\beta}(\alpha\beta X^2 + \alpha\beta Y^2 + \alpha^2 XY + \beta^2 YX)$

$= X^2 + Y^2$ by (a) $XY = YX = 0$

$= I$ by (a)

9. (a) $\{\Rightarrow\}$Let f(x) = $(x - r)^2$ q(x)

Obviously f(r) = 0 and f '(x) = 2($x - r$)q(x) + $(x - r)^2$ q'(x)

$\therefore$ f '(r) = 0

$\{\Leftarrow\}$f(r) = 0 $\Rightarrow$ f(x) = ($x - r$)q(x)

$\Rightarrow$ f '(x) = q'(x)($x - r$) + q(x)

$\Rightarrow$ f '(r) = q(r) = 0

$\Rightarrow$ q(x) = ($x - r$)h(x)

$\therefore$ f(x) = $(x - r)^2$h(x) r is a repeated root of f(x) = 0

(b) Suppose g(x) = 0 has a repeated r

g′(r) = 0 by (a)

g '(x) = $3x^2 + 2ax + b$

for the equation $3x^2 + 2ax + b = 0$

$\Delta = 4a^2 - 12b < 0$ $\because a^2 < 3b$

$\therefore$ g '(x) = 0 has no real roots then the repeated root r is a complex number.

$\because$ a , b and c are real coefficients

$\therefore$ r is the repeated root then $\bar{r}$ also the repeated roots. But g(x) = 0 is a degree 3 polynomial equation has at most 3 roots. Contradiction

$\therefore$ all the roots of g(x) = 0 are distinct.

(c) Let r be the positive repeated roots.

Let f(x) = $12x^3 - 8x^2 - x + k$

f′(x) = $36x^2 - 16x - 1$

by (a) $36r^2 - 16r - 1 = 0$

$\Rightarrow (18r+1)(2r-1)=0$

$\Rightarrow r=\frac{1}{2}$ or $-\frac{1}{18}$ (rej.)

by (a) $12\left(\frac{1}{2}\right)^3-8\left(\frac{1}{2}\right)^2-\frac{1}{2}+k=0$

$\Rightarrow k=1$

$12x^3-8x^2-x+1=0$

$\Rightarrow (2x-1)^2(3x+1)=0$

$\Rightarrow x=\frac{1}{2}$ or $-\frac{1}{3}$

10. (a) (i) $f'(x)=px^{p-1}-p=0$

$\Rightarrow x=1$

as $0<x<1$ $f'(x)>0$ $f(x)$ is strictly increasing

as $x>1$ $f'(x)<0$ $f(x)$ is strictly decreasing

$\therefore$ $f(x)$ attains its absolute maximum at $x=1$

$f(x)\le f(1)=1-p$ for $x>0$

$\therefore x^p\le px+1-p$

(ii) put $x=\frac{a}{b}$

$$\left(\frac{a}{b}\right)^p\le p\left(\frac{a}{b}\right)+1-p$$

$$\Rightarrow a^pb^{1-p}\le pa+b-pb=pa+(1-p)b$$

(iii) put $a=\frac{a_i}{\sum_{i=1}^n a_i}$, $b=\frac{b_i}{\sum_{i=1}^n b_i}$

$$\left(\frac{a_i}{\sum_{i=1}^n a_i}\right)^p\left(\frac{b_i}{\sum_{i=1}^n b_i}\right)^{1-p}\le p\frac{a_i}{\sum_{i=1}^n a_i}+(1-p)\frac{b_i}{\sum_{i=1}^n b_i}$$

$$\Rightarrow \sum_{i=1}^n\left(\frac{a_i}{\sum_{i=1}^n a_i}\right)^p\left(\frac{b_i}{\sum_{i=1}^n b_i}\right)^{1-p}\le \sum_{i=1}^n[p\frac{a_i}{\sum_{i=1}^n a_i}+(1-p)\frac{b_i}{\sum_{i=1}^n b_i}]=1$$

$$\Rightarrow \sum_{i=1}^n a_i{}^p b_i{}^{1-p}\le(\sum_{i=1}^n a_i)^p(\sum_{i=1}^n b_i)^{1-p}$$

(b) (i) by (a)(iii) put $p=\frac{1}{2}$, $a_i=x_i{}^s y_i{}^{2-s}$, $b_i=x_i{}^{2-s}y_i{}^s$

$$\therefore \sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^{\ s} y_i^{\ 2-s}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} x_i^{\ 2-s} y_i^{\ s}\right)^{\frac{1}{2}}$$

(ii) by (a)(iii) put $p = \dfrac{s}{2}$, $a_i = x_i^{\ 2}$, $b_i = y_i^{\ 2}$

$$\sum_{i=1}^{n} (x_i^{\ 2})^{\frac{s}{2}} (y_i^{\ 2})^{1-\frac{s}{2}} \le (\sum_{i=1}^{n} x_i^{\ 2})^{\frac{s}{2}} (\sum_{i=1}^{n} y_i^{\ 2})^{1-\frac{s}{2}}$$

$$\Rightarrow \sum_{i=1}^{n} x_i^{\ s} y_i^{\ 2-s} \le (\sum_{i=1}^{n} x_i^{\ 2})^{\frac{s}{2}} (\sum_{i=1}^{n} y_i^{\ 2})^{1-\frac{s}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

by (a)(iii) again put $p = \dfrac{s}{2}$, $a_i = y_i^{\ 2}$, $b_i = x_i^{\ 2}$

$$\Rightarrow \sum_{i=1}^{n} y_i^{\ s} x_i^{\ 2-s} \le (\sum_{i=1}^{n} y_i^{\ 2})^{\frac{s}{2}} (\sum_{i=1}^{n} x_i^{\ 2})^{1-\frac{s}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

by (1) and (2) $\because$ all are positive numbers

$$\therefore (\sum_{i=1}^{n} x_i^{\ s} y_i^{\ 2-s})(\sum_{i=1}^{n} x_i^{\ 2-s} y_i^{\ s}) \le (\sum_{i=1}^{n} x_i^{\ 2})^{\frac{s}{2}} (\sum_{i=1}^{n} y_i^{\ 2})^{1-\frac{s}{2}} (\sum_{i=1}^{n} y_i^{\ 2})^{\frac{s}{2}} (\sum_{i=1}^{n} x_i^{\ 2})^{1-\frac{s}{2}}$$

$$\Rightarrow (\sum_{i=1}^{n} x_i^{\ s} y_i^{\ 2-s})(\sum_{i=1}^{n} x_i^{\ 2-s} y_i^{\ s}) \le (\sum_{i=1}^{n} x_i^{\ 2})(\sum_{i=1}^{n} y_i^{\ 2})$$

11. (a) (i) Let $\vec{w} = xi + yj + zk$

$$\begin{cases} \vec{w} \cdot \vec{u} = 9 \\ \vec{w} \cdot \vec{v} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x + y + z = 9 \\ 3x + 4y - z = 1 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & 1 & 9 \\ 3 & 4 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 3 & 4 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & -5 \end{pmatrix}$$

$$\therefore \vec{w} = (2-t)i + tj + (5+t)z \qquad \text{where } t \in R$$

(ii) $\left|\vec{w}\right| = \sqrt{(2-t)^2 + t^2 + (5+t)^2}$

$= \sqrt{3t^2 + 6t + 29}$

$= \sqrt{3(t+1)^2 + 26}$

$$\therefore \left|\overrightarrow{w_0}\right| = \sqrt{26}$$

$$\overrightarrow{w_0} = 3i - j + 4k$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & -1 \\ 3 & -1 & 4 \end{vmatrix} = 2(15) - 15 - 15 = 0$$

$\therefore \vec{u}, \vec{v}$ and $\overrightarrow{w_0}$ are linear dependent

(b) (i) $(\vec{r} - \vec{c}) \cdot \vec{a} = \vec{r} \cdot \vec{a} - \vec{c} \cdot \vec{a} = p - p = 0$

$(\vec{r} - \vec{c}) \cdot \vec{b} = \vec{r} \cdot \vec{b} - \vec{c} \cdot \vec{b} = q - q = 0$

$\therefore \vec{r} - \vec{c} \perp \vec{a}$ and $\vec{b}$

$\because$ $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$

$\therefore$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar

$\because \vec{r} - \vec{c} \perp \vec{a}$ and $\vec{b}$

$\therefore \vec{c}$ is the orthogonal projection of $\vec{r}$ on the plane containing $\vec{a}$ and $\vec{b}$.

$\therefore |\vec{r}| \geq |\vec{c}|$

(ii) $\begin{cases} \vec{c} \cdot \vec{a} = p \\ \vec{c} \cdot \vec{b} = q \end{cases}$

$\Rightarrow s\vec{a} \cdot \vec{a} + t\vec{b} \cdot \vec{a} = p$ ……….(1)

and $s\vec{a} \cdot \vec{b} + t\vec{b} \cdot \vec{b} = q$ ………(2)

(1)x($\vec{b} \cdot \vec{b}$) – (2)x($\vec{a} \cdot \vec{b}$)

$s[(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2] = p(\vec{b} \cdot \vec{b}) - q(\vec{a} \cdot \vec{b})$

$$\Rightarrow s = \frac{p(\vec{b} \cdot \vec{b}) - q(\vec{a} \cdot \vec{b})}{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$$

sub into (1) $t = \dfrac{p(\vec{a} \cdot \vec{b}) - q(\vec{a} \cdot \vec{a})}{(\vec{a} \cdot \vec{b})^2 - (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})}$

12. (a) (i)
$$\frac{(\cos\theta + i\sin\theta)^n - 1}{\cos\theta + i\sin\theta - 1} = \frac{\cos n\theta + i\sin n\theta - 1}{\cos\theta + i\sin\theta - 1}$$
$$= \frac{-2\sin^2\frac{n\theta}{2} + i2\cos\frac{n\theta}{2}\sin\frac{n\theta}{2}}{-2\sin^2\frac{\theta}{2} + i2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}$$

$$= \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \cdot \frac{-\sin\frac{n\theta}{2} + i\cos\frac{n\theta}{2}}{\sin\frac{\theta}{2} + i2\cos\frac{\theta}{2}}$$

$$= \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \left(\frac{\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}} \right)$$

$$= \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \left(\cos\frac{(n-1)\theta}{2} + i\sin\frac{(n-1)\theta}{2} \right)$$

(ii) using the given identity let $z = \cos\theta + i\sin\theta$

$$\sum_{k=1}^{n} (\cos\theta + i\sin\theta)^k = (\cos\theta + i\sin\theta)\left(\frac{(\cos\theta + i\sin\theta)^n - 1}{\cos\theta + i\sin\theta - 1} \right)$$

$$\Rightarrow \sum_{k=1}^{n} (\cos k\theta + i\sin k\theta) = (\cos\theta + i\sin\theta)\ \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \left(\cos\frac{(n-1)\theta}{2} + i\sin\frac{(n-1)\theta}{2} \right) \text{ by(i)}$$

$$\Rightarrow \sum_{k=1}^{n} (\cos k\theta + i\sin k\theta) = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \left(\cos\frac{(n+1)\theta}{2} + i\sin\frac{(n+1)\theta}{2} \right)$$

$$\Rightarrow \sum_{k=1}^{n} \cos k\theta = \frac{\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} \quad \text{and} \quad \sum_{k=1}^{n} \sin k\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

(iii) by (ii)

$$\sum_{k=1}^{n} \cos 2k\theta = \frac{\sin\frac{2n\theta}{2}\cos\frac{2(n+1)\theta}{2}}{\sin\frac{2\theta}{2}}$$

$$\Rightarrow \sum_{k=1}^{n} (2\cos^2 k\theta - 1) = \frac{\sin n\theta\cos(n+1)\theta}{\sin\theta}$$

$$\Rightarrow \sum_{k=1}^{n} 2\cos^2 k\theta - n = \frac{\sin n\theta\cos(n+1)\theta}{\sin\theta}$$

$$\Rightarrow \sum_{k=1}^{n} \cos^2 k\theta = \frac{n}{2} + \frac{\sin n\theta\cos(n+1)\theta}{2\sin\theta}$$

(b) (i) by (a)(iii) $\sum_{k=1}^{n} (1 - \sin^2 k\theta) = \frac{n}{2} + \frac{\sin n\theta\cos(n+1)\theta}{2\sin\theta}$

$$\Rightarrow n - \sum_{k=1}^{n} \sin^2 k\theta = \frac{n}{2} + \frac{\sin n\theta\cos(n+1)\theta}{2\sin\theta}$$

$$\Rightarrow \sum_{k=1}^{n} \sin^2 k\theta = \frac{n}{2} - \frac{\sin n\theta \cos(n+1)\theta}{2\sin\theta}$$

$$\therefore \sum_{k=1}^{n} \sin^2 \frac{k\pi}{n} = \frac{n}{2}$$

(ii) $$\sum_{k=1}^{n} (\sin\frac{k\pi}{n} + \cos\frac{k\pi}{n})^2 = \sum_{k=1}^{n} (\sin^2\frac{k\pi}{n} + \cos^2\frac{k\pi}{n} + 2\sin\frac{k\pi}{n}\cos\frac{k\pi}{n})$$

$$= \sum_{k=1}^{n} (1 + \sin\frac{2k\pi}{n})$$

$$= n + \sum_{k=1}^{n} \sin\frac{2k\pi}{n}$$

$= n$ by(a)(ii) $\sum_{k=1}^{n} \sin\frac{2k\pi}{n} = 0$

1. (a) Let $y=(\tan 3x+\cos 4x)^{\frac{1}{x}}$

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(\tan 3x+\cos 4x)}{x}$$

$$=\lim_{x\to 0}\frac{\dfrac{3\sec^2 3x-4\sin 4x}{\tan 3x+\cos 4x}}{1} \qquad (l\text{'hospital rule})$$

$$=3$$

$$\lim_{x\to 0}y=e^3$$

(b) $$\lim_{x\to\infty}(\cos\sqrt{2004+x}-\cos\sqrt{x})=\lim_{x\to\infty}-2\sin\frac{\sqrt{2004+x}+\sqrt{x}}{2}\sin\frac{\sqrt{2004+x}-\sqrt{x}}{2}$$

$$=\lim_{x\to\infty}-2\sin\frac{\sqrt{2004+x}+\sqrt{x}}{2}\sin\frac{2004+x-x}{2(\sqrt{2004+x}+\sqrt{x})}$$

$$=\lim_{x\to\infty}-2\sin\frac{\sqrt{2004+x}+\sqrt{x}}{2}\sin\frac{2004}{2(\sqrt{2004+x}+\sqrt{x})}$$

$$=0 \qquad [\because x\to\infty \quad \sin\frac{2004}{\sqrt{2004+x}-\sqrt{x}}\to 0$$

$$\text{and } -1\le\sin\frac{\sqrt{2004+x}+\sqrt{x}}{2}\le 1]$$

2. f is differentiable at 1 $\therefore$ f is continuous at 1

$$\lim_{x\to 1^-}\mathrm{f}(x)=\lim_{x\to 1^-}x^2+ax+b=1+a+b$$

$$\lim_{x\to 1^+}\mathrm{f}(x)=\lim_{x\to 1^+}\frac{\sin\pi x}{\pi}=0$$

$$\therefore 1+a+b=0$$

$$\lim_{h\to 0^+}\frac{\mathrm{f}(1+h)-\mathrm{f}(1)}{h}=\lim_{h\to 0^+}\frac{\dfrac{\sin\pi(1+h)}{\pi}-(1+a+b)}{h}=\lim_{h\to 0^+}\frac{\sin\pi(1+h)}{\pi h}$$

$$=\lim_{h\to 0^+}\frac{\pi\cos\pi(1+h)}{\pi} \qquad (l\text{'hospital rule})$$

$$=-1$$

$$\lim_{h\to 0^-}\frac{\mathrm{f}(1+h)-\mathrm{f}(1)}{h}=\lim_{h\to 0^-}\frac{(1+h)^2+a(1+h)+b-(1+a+b)}{h}=\lim_{h\to 0^+}2+h+a=2+a$$

$$\therefore 2+a=-1\Rightarrow a=-3 \quad \text{and} \quad b=2$$

3. (a) Let $u = \sec\theta \qquad dv = \sec^2\theta d\theta$

$du = \sec\theta\tan\theta d\theta \quad v = \tan\theta$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta$$

$$\Rightarrow 2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta$$

$$\Rightarrow \int \sec^3\theta d\theta = \frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + c$$

(b) $\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \int_0^1 \sqrt{1+x^2}dx \qquad$ Let $x = \tan\theta \qquad dx = \sec^2\theta d\theta$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left|\sqrt{2}+1\right|$$

4. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2}dx$

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}dx = \int_{2}^{\frac{1}{2}} \frac{\ln\frac{1}{u}}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}\right)du$$

$$= -\int_{\frac{1}{2}}^{2} \frac{\ln u}{1+u^2}du$$

$$\Rightarrow 2\int_{\frac{1}{2}}^{2} \frac{\ln u}{1+u^2}du = 0$$

$$\Rightarrow \int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}du = 0$$

$$\lim_{n\to\infty}\sum_{k=1}^{3n} \frac{\ln\left(2\left(\frac{1}{2}+\frac{k}{2n}\right)\right)}{2\left(1+\left(\frac{1}{2}+\frac{k}{2n}\right)^2\right)}\frac{1}{n} = \int_0^3 \frac{\ln 2\left(\frac{1}{2}+\frac{1}{2}x\right)}{2\left(1+\left(\frac{1}{2}+\frac{1}{2}x\right)^2\right)}dx$$

Let $u = \frac{1}{2} + \frac{1}{2}x \qquad du = \frac{1}{2}dx$

$$= \int_{\frac{1}{2}}^{2} \frac{\ln 2u}{1+u^2}du = \int_{\frac{1}{2}}^{2} \frac{\ln 2}{1+u^2}du + \int_{\frac{1}{2}}^{2} \frac{\ln u}{1+u^2}du = \ln 2\left[\tan^{-1}x\right]_{\frac{1}{2}}^{2} = \left[\tan^{-1}2 - \tan^{-1}\frac{1}{2}\right]\ln 2$$

5. (a) (i) $I_0 = \int_0^\infty e^{-x} dx = \lim_{l \to \infty} -e^{-x}\Big|_0^l = 1$

(ii) $I_{n+1} = \int_0^\infty e^{-x} x^{n+1} dx$

$= \lim_{l \to \infty} \left[-x^{n+1} e^{-x}\right]_0^l + (n+1)\int_0^l e^{-x} x^n dx$

consider $\lim_{l \to \infty} -\frac{l^{n+1}}{e^l} = \lim_{l \to \infty} -\frac{(n+1)!}{e^l}$ by apply l'hospital rule n times

$= 0$

$\because I_n$ is convergent

$I_{n+1} = (n+1)I_n$

(iii) $I_n = nI_{n-1}$

$= n(n-1)I_{n-2}$

$\vdots$

$= n(n-1)(n-1).......1I_0$

$= n!$

(b) $J_n = \int_0^\infty e^{-x}(1 - \frac{2x^2}{n+1} + \frac{x^4}{(n+1)^2})dx$

$= \int_0^\infty e^{-x} dx - \int_0^\infty e^{-x} \frac{2x^2}{n+1} dx + \int_0^\infty e^{-x} \frac{x^4}{(n+1)^2} dx$

$= I_0 - \frac{2}{n+1} I_2 + \frac{1}{(n+1)^2} I_4$

$= 1 - \frac{2}{n+1} 2! + \frac{1}{(n+1)^2} 4!$

$= 24(\frac{1}{n+1} - \frac{1}{12})^2 + \frac{5}{6}$

minimum J_n attained when $n = 11$

$\therefore m = 11$

6. (a) $\begin{cases} x = \frac{3}{2} - t \\ y = -\frac{1}{2} \\ z = t \end{cases}$

(b) Let the equation of plane $x + y + z - 1 + k(x - y + z - 2) = 0$

i.e. $(k+1)x + (1-k)y + (k+1)z - 1 - 2k = 0$

the angle between the plane with π_1 and π_2 is equal

$$\left|\frac{k+1+1-k+k-1}{\sqrt{(k+1)^2+(1-k)^2+(k+1)^2}\cdot\sqrt{1^2+1^2+1^2}}\right|=\left|\frac{k+1-1+k+k+1}{\sqrt{(k+1)^2+(1-k)^2+(k+1)^2}\cdot\sqrt{1^2+1^2+1^2}}\right|$$

$\Rightarrow |k+3|=|3k+1|$

$\Rightarrow k=1$ or -1

$\therefore$ the equations of plane is $2x+2z-3=0$ or $2y+1=0$

7. $$f(x)=\begin{cases}\dfrac{x^4}{x^3-2} & x\ge 0\\ -\dfrac{x^4}{x^3-2} & x<0\end{cases}$$

(a) (i) $f'(x)=\dfrac{x^6-8x^3}{(x^3-2)^2}$ $\qquad f''(x)=\dfrac{12x^5+48x^2}{(x^3-2)^3}$

(ii) $f'(x)=-\dfrac{x^6-8x^3}{(x^3-2)^2}$ $\qquad f''(x)=-\dfrac{12x^5+48x^2}{(x^3-2)^3}$

(iii) $$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{|h|h^3}{(h^3-2)h}$$
$$=\lim_{h\to 0}\frac{|h|h^2}{(h^3-2)}$$
$$=0$$

(iv) $$f''(0)=\lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h}$$

$$\lim_{h\to 0^+}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0^+}\frac{h^6-8h^3}{(h^3-2)^2h}$$
$$=\lim_{h\to 0^+}\frac{h^5-8h^2}{(h^3-2)^2}$$
$$=0$$

$$\lim_{h\to 0^-}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0^-}-\frac{h^6-8h^3}{(h^3-2)^2h}$$
$$=\lim_{h\to 0^-}-\frac{h^5-8h^2}{(h^3-2)^2}$$
$$=0$$

$\therefore f''(0)=0$ exist.

(b) (i) $f'(x)>0 \Rightarrow x>2$

(ii) $f'(x)<0 \Rightarrow x<2$ and $x\ne\sqrt[3]{2}$

(iii) $f''(x)>0 \Rightarrow \sqrt[3]{-4}<x<0$ or $x>\sqrt[3]{2}$

(iv) $f''(x)>0 \Rightarrow x<\sqrt[3]{-4}$ or $0<x<\sqrt[3]{2}$

(c) $f'(x) = 0 \Rightarrow x = 0$ or 2

$f''(2) > 0 \quad \therefore (2, \frac{8}{3})$ relative minimum point.

$f''(x) = 0 \Rightarrow x = 0$ or $\sqrt[3]{-4}$

$\therefore (0,0), (\sqrt[3]{-4}, f(\sqrt[3]{-4}))$ are points of inflexion.

(d) vertical asymptote $x = \sqrt[3]{2}$

$$a = \lim_{x\to+\infty} \frac{|x|x^3}{x(x^3-2)} = \lim_{x\to+\infty} \frac{x^4}{x(x^3-2)} = 1$$

$$b = \lim_{x\to+\infty} \frac{x^4}{x^3-2} - x = \lim_{x\to+\infty} \frac{2x}{x^3-2} = 0$$

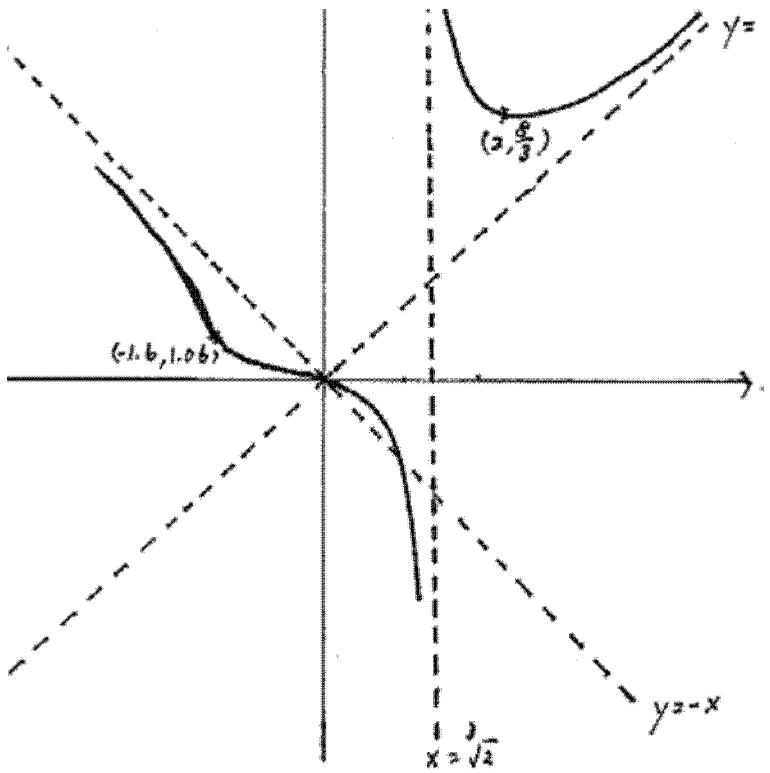
$\therefore y = x$ is a asymptote

$$a = \lim_{x\to-\infty} \frac{|x|x^3}{x(x^3-2)} = \lim_{x\to-\infty} -\frac{x^4}{x(x^3-2)} = -1$$

$$b = \lim_{x\to-\infty} -\frac{x^4}{x^3-2} + x = \lim_{x\to-\infty} -\frac{2x}{x^3-2} = 0$$

$\therefore y = -x$ is a asymptote

(e)



8. (a) Consider $I_{m+1,+n+1}(x)-\frac{\cos^{m+1}x\sin(n+1)x}{m+n+2}-I_{m,n}$

$$\frac{d}{dx}I_{m+1,+n+1}(x)-\frac{\cos^{m+1}x\sin(n+1)x}{m+n+2}-\frac{m+1}{m+n+2}I_{m,n}$$

$$=\cos^{m+1}x\cos(n+1)x-\frac{(n+1)\cos^{m+1}x\cos(n+1)x-(m+1)\cos^{m}x\sin x\sin(n+1)x}{m+n+2}-\frac{m+1}{m+n+2}\cos^{m}x\cos nx$$

$$=\frac{(m+1)\cos^{m+1}x\cos(n+1)x+(m+1)\cos^{m}x\sin x\sin(n+1)x}{m+n+2}-\frac{m+1}{m+n+2}\cos^{m}x\cos nx$$

$$=\frac{(m+1)\cos^{m}x[\cos x\cos(n+1)x+\sin x\sin(n+1)x]}{m+n+2}-\frac{m+1}{m+n+2}\cos^{m}x\cos nx$$

$$=\frac{(m+1)}{m+n+2}\cos^{m}x\cos nx-\frac{m+1}{m+n+2}\cos^{m}x\cos nx$$

$$=0$$

$$\therefore I_{m+1,+n+1}(x)-\frac{\cos^{m+1}x\sin(n+1)x}{m+n+2}-I_{m,n}\text{ is a constant}$$

put $x=0$ $\quad I_{m+1,+n+1}(x)-\frac{\cos^{m+1}x\sin(n+1)x}{m+n+2}-I_{m,n}=0$

$$\therefore I_{m+1,+n+1}(x)=\frac{\cos^{m+1}x\sin(n+1)x}{m+n+2}+I_{m,n}$$

(b) by (a)

$$I_{4,3}\left(\frac{\pi}{2}\right)=\frac{\cos^4\frac{\pi}{2}\sin\frac{3\pi}{2}}{7}+\frac{4}{7}I_{3,2}\left(\frac{\pi}{2}\right)$$

$$=\frac{4}{7}\left(\frac{\cos^2\frac{\pi}{2}\sin\pi}{5}+\frac{3}{5}I_{2,1}\left(\frac{\pi}{2}\right)\right)$$

$$=\frac{4}{7}\cdot\frac{3}{5}\left(\frac{\cos\frac{\pi}{2}\sin\frac{\pi}{2}}{3}+\frac{2}{3}I_{1,0}\left(\frac{\pi}{2}\right)\right)$$

$$=\frac{4}{7}\cdot\frac{3}{5}\cdot\frac{2}{3}\int_0^{\frac{\pi}{2}}\cos\theta d\theta$$

$$=\frac{8}{35}\sin\theta\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{8}{35}$$

(c)

$$\int_0^{\frac{\pi}{2}}\sin^4\theta\cos 3\theta d\theta=\int_0^{\frac{\pi}{2}}(1-\cos^2\theta)^2\cos 3\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}(1-2\cos^2\theta+\cos^4\theta)\cos 3\theta d\theta$$

$$=\int_0^{\frac{\pi}{2}}\cos 3\theta d\theta-2\int_0^{\frac{\pi}{2}}\cos^2\theta\cos 3\theta d\theta+\int_0^{\frac{\pi}{2}}\cos^4\theta\cos 3\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos 3\theta d\theta - 2I_{2,3}(\frac{\pi}{2}) + I_{4,3}(\frac{\pi}{2})$$

$$\int_0^{\frac{\pi}{2}} \cos 3\theta d\theta = \left.\frac{\sin 3\theta}{3}\right|_0^{\frac{\pi}{2}} = -\frac{1}{3}$$

$$I_{2,3}(\frac{\pi}{2}) = \frac{2}{5} I_{1,2}(\frac{\pi}{2})$$

$$= \frac{2}{5}\cdot\frac{1}{3}\int_0^{\frac{\pi}{2}} \cos\theta d\theta$$

$$= \frac{2}{15} \sin\theta \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{15}$$

$$\int_0^{\frac{\pi}{2}} \sin^4\theta \cos 3\theta d\theta = -\frac{1}{3} - 2\left(\frac{2}{15}\right) + \frac{8}{35} = -\frac{13}{35}$$

10. (a) (i) $\int \frac{dx}{1-\sqrt{2}x + x^2} = \int \frac{dx}{(x-\frac{\sqrt{2}}{2})^2 + \frac{1}{2}} = \sqrt{2}\tan^{-1}\sqrt{2}(x - \frac{\sqrt{2}}{2}) + c$

(ii) $\int_0^{\frac{1}{\sqrt{2}}} \frac{2\sqrt{2}dx}{1-\sqrt{2}x+x^2} = 4\tan^{-1}\sqrt{2}(x-\frac{\sqrt{2}}{2})\Big|_0^{\frac{1}{\sqrt{2}}} = 4(-\tan^{-1}(-1)) = -4(-\frac{\pi}{4}) = \pi$

(iii) $\int_0^{\frac{1}{\sqrt{2}}} \frac{2\sqrt{2}+4x+2\sqrt{2}x^2 dx}{1+x^4} = \int_0^{\frac{1}{\sqrt{2}}} \frac{2\sqrt{2}(1+\sqrt{2}x+x^2)dx}{1+x^4}$

$= \int_0^{\frac{1}{\sqrt{2}}} \frac{2\sqrt{2}dx}{1-\sqrt{2}x+x^2} = \pi$ by (a)(ii)

(b) (i) $\left|\sum_{n=0}^{k} (-1)^n x^{4n} - \frac{1}{1+x^4}\right| = \left|\frac{1-(-x^4)^{k+1}}{1-(-x^4)} - \frac{1}{1+x^4}\right|$ sun of a G.P

$= \left|\frac{x^{4k+4}}{1+x^4}\right| \le \left|x^{4k+4}\right| = x^{4k+4}$

$\therefore -x^{4k+4} \le \sum_{n=0}^{k} (-1)^n x^{4n} - \frac{1}{1+x^4} \le x^{4k+4}$

(ii) Let $p(x) = 2\sqrt{2} + 4x + 2\sqrt{2}x^2$

$-x^{4k+4}p(x) \le [\sum_{n=0}^{k}(-1)^n x^{4n} - \frac{1}{1+x^4}]p(x) \le x^{4k+4}p(x)$ by(a)(iii) for $x \in [0, \frac{1}{\sqrt{2}}]$

$\Rightarrow \int_0^{\frac{1}{\sqrt{2}}} [-x^{4k+4}p(x)]dx \le \sum_{n=0}^{k}(-1)^n \int_0^{\frac{1}{\sqrt{2}}} [2\sqrt{2}x^{4n} + 4x^{4n+1} + 2\sqrt{2}x^{4n+2} - \frac{p(x)}{1+x^4}]dx \le \int_0^{\frac{1}{\sqrt{2}}} [x^{4k+4}p(x)]dx$

$$\Rightarrow \int_0^{\frac{1}{\sqrt{2}}}[-x^{4k+4}p(x)]dx \le \sum_{n=0}^{k}(-1)^n\left[\frac{2\sqrt{2}x^{4n+1}}{4n+1}+\frac{4x^{4n+2}}{4n+2}+\frac{2\sqrt{2}x^{4n+3}}{4n+3}\right]_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}}[\frac{p(x)}{1+x^4}]dx \le \int_0^{\frac{1}{\sqrt{2}}}[x^{4k+4}p(x)]dx$$

$$\Rightarrow \int_0^{\frac{1}{\sqrt{2}}}[-x^{4k+4}p(x)]dx \le \sum_{n=0}^{k}(-1)^n\frac{1}{4^n}\left[\frac{2}{4n+1}+\frac{2}{4n+2}+\frac{1}{4n+3}\right]-\pi \le \int_0^{\frac{1}{\sqrt{2}}}[x^{4k+4}p(x)]dx$$

by (a)(iii)

$$\because \lim_{k\to\infty}\int_0^{\frac{1}{\sqrt{2}}}[-x^{4k+4}p(x)]dx = \lim_{k\to\infty}\int_0^{\frac{1}{\sqrt{2}}}[x^{4k+4}p(x)]dx = 0$$

$$\Rightarrow \lim_{k\to\infty}\sum_{n=0}^{k}(-1)^n\frac{1}{4^n}\left[\frac{2}{4n+1}+\frac{2}{4n+2}+\frac{1}{4n+3}\right]-\pi = 0$$

$$\Rightarrow \lim_{k\to\infty}\sum_{n=0}^{k}(\frac{-1}{4})^n\left[\frac{2}{4n+1}+\frac{2}{4n+2}+\frac{1}{4n+3}\right] = \pi$$

11. (a) (i) $f(x) = \frac{1}{2}$ when x is an integer

$\Rightarrow f(x+1) = \frac{1}{2}$ when $x+1$ is an integer[$\because x$ is an integer, $x+1$ is also an integer]

$f(x) = x-[x]-\frac{1}{2}$ when x is not integer

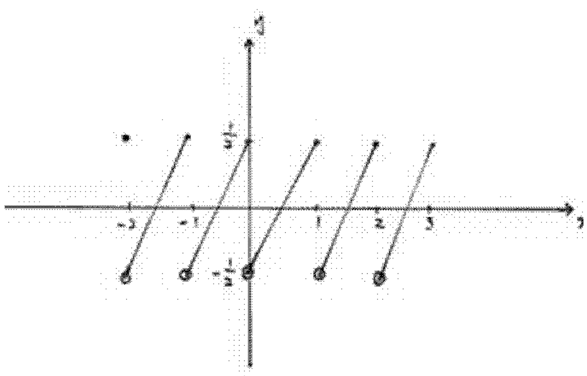
$\Rightarrow f(x+1) = x+1-[x+1]-\frac{1}{2}$ when $x+1$ is not integer

$\Rightarrow f(x+1) = x+1-[x]-1-\frac{1}{2} = x-[x]-\frac{1}{2}$

$\therefore$ $f(x+1)= f(x)$

f is a periodic function period 1.

(ii)



(iii) f is discontinuous for all $x \in Z$

(b) (i) For $0<x<1$ $\quad F(x)=\int_0^x (t-\frac{1}{2})dt$

$$= \left[\frac{t^2}{2}-\frac{t}{2}\right]_0^x$$

$$= \frac{x^2-x}{2}$$

$\therefore 0 \le x \le 1 \quad F(x) = \dfrac{x^2-x}{2}$

(ii) $F(x+1) = \int_0^{x+1} f(t)dt$

$$= \int_0^1 f(t)dt + \int_1^{x+1} f(t)dt$$

$$= \left[\frac{t^2-t}{2}\right]_0^1 + \int_1^{x+1} f(t)dt$$

$$= \int_1^{x+1} f(t)dt$$

$$= \int_0^x f(u+1)du \qquad \text{[let } t = u + 1]$$

$$= \int_0^x f(u)du \qquad \text{by (a)(ii)}$$

$= F(x)$ for all $x \in R$

$\therefore$ F(x) is a periodic function period 1.

(iii) $\int_0^\pi F(x)dx = \int_0^3 F(x)dx + \int_3^\pi F(x)dx$

$$= 3\int_0^1 F(x)dx + \int_3^\pi F(x)dx \qquad \text{by (b)(ii)}$$

$$= 3\int_0^1 F(x)dx + \int_0^{3-\pi} F(u+3)du \qquad \text{[let } x = u + 3]$$

$$= 3\int_0^1 F(x)dx + \int_0^{3-\pi} F(u)du$$

$$= 3\int_0^1 \frac{x^2-x}{2}dx + \int_0^{3-\pi} \frac{x^2-x}{2}dx$$

$$= 3\left[\frac{x^3}{6}-\frac{x^2}{4}\right]_0^1 + \left[\frac{x^3}{6}-\frac{x^2}{4}\right]_0^{3-\pi}$$

$$= -\frac{1}{4}+\frac{(\pi-3)^3}{6}-\frac{(\pi-3)^2}{4}$$

12. (a) (i) $h(a) = 0$

$\Rightarrow f(a) - f(b) - f'(a)(a-b) - k(a-b)^2 = 0$

$$\Rightarrow k = \frac{f(a) - f(b) - f'(a)(a-b)}{(a-b)^2}$$

$h(b) = f(b) - f(b) = 0$

(ii) By mean value theorem there exist $c \in I$ such that $h'(c) = \frac{h(b) - h(a)}{b-a}$

By (a)(i) $h'(c) = 0$

$h'(x) = f'(x) - f''(x)(x-b) - f'(x) - 2k(x-b)$

$h'(c) = f'(c) - f''(c)(c-b) - f'(c) - 2k(c-b) = 0$

$\Rightarrow f''(c) - 2k = 0$

subs $k = \frac{f(a) - f(b) - f'(a)(a-b)}{(a-b)^2}$ by (i)

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2$$

(b) (i) by(a) put $f(x) = g(x)$, $a = \beta$ and $b = 1$

$g(1) = g(\beta) + g'(\beta)(1-\beta) + \frac{g''(\gamma)}{2}(1-\beta)^2$ for some $\gamma \in (\beta, 1)$

$\because$ g is differentiable and g attain relatively maximum at β $\therefore g'(\beta) = 0$

$g(1) = 1 + \frac{g''(\gamma)}{2}(1-\beta)^2$ for some $\gamma \in (0,1)$ since $\beta \in (0,1)$

(c) (ii) by (a) put $f(x) = g(x)$, $a = \beta$ and $b = 0$

$g(0) = g(\beta) + g'(\beta)(0-\beta) + \frac{g''(\alpha)}{2}(0-\beta)^2$ for some $\alpha \in (0, \beta)$

$\therefore g(0) = 1 + \frac{g''(\alpha)}{2}\beta^2$ for some $\alpha \in (0,1)$

$$g(1) + g(0) = 2 + \frac{g''(\gamma)}{2}(1-\beta)^2 + \frac{g''(\alpha)}{2}\beta^2$$

$\geq 2 - (1-\beta)^2 - \beta^2$ [$\because g''(x) \geq -2$ for $x \in (0,1)$]

$= 1 + 2\beta - 2\beta^2$

$= 1 + 2\beta(1-\beta)$

≥ 1 $\because \beta \in (0,1)$