2004-AL
P MATH

PAPER 1

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2004

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A and any FOUR questions in Section B.

3. You are provided with one AL(E) answer book and four AL(D) answer books.

   Section A : Write your answers in the AL(E) answer book.

   Section B : Use a separate AL(D) answer book for each question and put the question number on the front cover of each answer book.

4. The four AL(D) answer books should be tied together with the green tag provided. The AL(E) answer book and the four AL(D) answer books must be handed in separately at the end of the examination.

5. Unless otherwise specified, all working must be clearly shown.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Resolve $\dfrac{1}{(2x-1)(2x+1)(2x+3)}$ into partial fractions.

   (b) Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)(2k+3)} = \frac{1}{12} - \frac{1}{8(2n+1)} + \frac{1}{8(2n+3)}$ for all positive integers n.

   Hence or otherwise, evaluate $\displaystyle\sum_{k=10}^{\infty} \frac{1}{(2k-1)(2k+1)(2k+3)}$.

(6 marks)

2. Let $\{ a_n \}$ be a sequence of positive real numbers, where

   $a_1 = 1$ and

   $$a_n = \frac{12 a_{n-1} + 12}{a_{n-1} + 13}, \quad n = 2, 3, 4, \ldots .$$

   (a) Prove that $a_n \le 3$ for all positive integers n.

   (b) Prove that $\{ a_n \}$ is convergent and find its limit.

(6 marks)

Go on to the next page

3. (a) Let R be the matrix representing the rotation in the Cartesian plane anticlockwise about the origin by $60°$.

(i) Write down R and R^6 .

(ii) Let $A = \begin{pmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{pmatrix}$. Verify that $A^{-1}RA$ is a matrix in which all the elements are integers.

(b) Using the results of (a), or otherwise, find a 2×2 matrix M, in which all the elements are integers, such that $M^3 = I$ but $M \neq I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(7 marks)

4. Let $f(x) = x^3 + px^2 + qx + r$, where p , q and r are non-zero real numbers.

(a) If $f(x)$ is divisible by $x^2 + q$, find r in terms of p and q .

(b) Suppose that $f(x)$ is divisible by both $x - a$ and $x + a$, where a is a non-zero real number.

(i) Factorize $f(x)$ as a product of three linear polynomials with real coefficients.

(ii) If $f(x)$ and $f(x+a)$ have a non-constant common factor, find p in terms of a .

(7 marks)

5. Let n be a positive integer.

(a) Let $a > 0$.

(i) If k is a positive integer, prove that $a + a^k \leq 1 + a^{k+1}$.

(ii) Prove that $(1+a)^n \leq 2^{n-1}(1+a^n)$.

(b) Let x and y be positive real numbers. Using (a)(ii), or otherwise, prove that $\left(\dfrac{x+y}{2}\right)^n \leq \dfrac{x^n + y^n}{2}$.

(7 marks)

6. Let $\theta \in \mathbf{R}$. For each $n \in \mathbf{N}$, define $x_n = \sin^n \theta + \cos^n \theta$.

(a) Find a function $f(\theta)$, which is independent of n , such that $x_{n+1}\,x_1 - x_{n+2} = f(\theta)\,x_n$. Also express $f(\theta)$ in terms of x_1 .

(b) Suppose that x_1 is a rational number. Using mathematical induction, prove that x_n is a rational number for every n .

(7 marks)

Go on to the next page

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

7. (a) Consider the system of linear equations

$$(E):\begin{cases} x + (a-2)y + az = 1 \\ x + 2y + 4z = 1 \\ ax - y + 3z = b \end{cases},$$

where $a, b \in \mathbf{R}$.

(i) Prove that (E) has a unique solution if and only if $a \neq 2$ and $a \neq 4$. Solve (E) in this case.

(ii) For each of the following cases, determine the value(s) of b for which (E) is consistent, and solve (E) for such value(s) of b.

(1) $a = 2$,

(2) $a = 4$.

(10 marks)

(b) If all solutions (x, y, z) of $\begin{cases} x + 2z = 1 \\ x + 2y + 4z = 1 \\ 2x - y + 3z = 2 \end{cases}$ satisfy $k(x^2 - 3) > yz$, find the range of values of k.

(5 marks)

8. Let $A = \begin{pmatrix} \alpha - k & \alpha - \beta - k \\ k & \beta + k \end{pmatrix}$, where $\alpha, \beta, k \in \mathbf{R}$ with $\alpha \neq \beta$.

Define $X = \dfrac{1}{\alpha - \beta}(A - \beta I)$ and $Y = \dfrac{1}{\beta - \alpha}(A - \alpha I)$, where I is the 2×2 identity matrix.

(a) Evaluate XY, YX, $X + Y$, X^2 and Y^2.

(4 marks)

(b) Prove that $A^n = \alpha^n X + \beta^n Y$ for all positive integers n.

(4 marks)

(c) Evaluate $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}^{2004}$.

(4 marks)

(d) If α and β are non-zero real numbers, guess an expression for A^{-1} in terms of α, β, X and Y, and verify it.

(3 marks)

Go on to the next page

9. (a) Let $f(x)$ be a polynomial with real coefficients. Prove that a real number r is a repeated root of $f(x)=0$ if and only if $f(r)=f'(r)=0$.

(5 marks)

(b) Let $g(x)=x^3+ax^2+bx+c$, where a , b and c are real numbers. If $a^2<3b$, prove that all the roots of $g(x)=0$ are distinct.

(6 marks)

(c) Let k be a real constant. If the equation $12x^3-8x^2-x+k=0$ has a positive repeated root, find all the roots of the equation.

(4 marks)

10. Let n be a positive integer.

(a) Suppose $0<p<1$.

(i) By considering the function $f(x)=x^p-px$ on $(0,\infty)$, or otherwise, prove that $x^p \le px+1-p$ for all $x>0$.

(ii) Using (a)(i), or otherwise, prove that $a^p b^{1-p} \le pa+(1-p)b$ for all $a, b>0$.

(iii) Let $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ be positive real numbers. Using (a)(ii), or otherwise, prove that

$$\sum_{i=1}^{n} a_i^{\,p} b_i^{\,1-p} \le \left(\sum_{i=1}^{n} a_i\right)^p \left(\sum_{i=1}^{n} b_i\right)^{1-p} .$$

(9 marks)

(b) Suppose $0<s<2$. Let $x_1, x_2, x_3, \ldots, x_n$ and $y_1, y_2, y_3, \ldots, y_n$ be positive real numbers. Prove that

(i) $$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^{\,s} y_i^{\,2-s}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} x_i^{\,2-s} y_i^{\,s}\right)^{\frac{1}{2}} ,$$

(ii) $$\left(\sum_{i=1}^{n} x_i^{\,s} y_i^{\,2-s}\right)\left(\sum_{i=1}^{n} x_i^{\,2-s} y_i^{\,s}\right) \le \left(\sum_{i=1}^{n} x_i^{\,2}\right)\left(\sum_{i=1}^{n} y_i^{\,2}\right) .$$

(6 marks)

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11. (a) Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ be vectors in $\mathbf{R}^3$.

(i) Find all vectors $\mathbf{w}$ in $\mathbf{R}^3$ such that $\mathbf{w} \cdot \mathbf{u} = 9$ and $\mathbf{w} \cdot \mathbf{v} = 1$.

(ii) Find the vector $\mathbf{w}_0$ which has the least magnitude among the vectors obtained in (a)(i).

Prove that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}_0$ are linearly dependent.

(7 marks)

(b) Suppose that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent vectors in $\mathbf{R}^3$. Let $\mathbf{c}$ be a linear combination of $\mathbf{a}$ and $\mathbf{b}$, $p = \mathbf{c} \cdot \mathbf{a}$ and $q = \mathbf{c} \cdot \mathbf{b}$.

(i) For any vector $\mathbf{r}$ in $\mathbf{R}^3$ satisfying $\mathbf{r} \cdot \mathbf{a} = p$ and $\mathbf{r} \cdot \mathbf{b} = q$, prove that $\mathbf{r} - \mathbf{c}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Hence deduce that $|\mathbf{r}| \geq |\mathbf{c}|$.

(ii) If $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$, express s and t in terms of p, q, $\mathbf{a}$ and $\mathbf{b}$.

(8 marks)

12. Let n be a positive integer.

(a) Assume that θ is not an integral multiple of π.

(i) Prove that

$$\frac{(\cos\theta + i\sin\theta)^n - 1}{\cos\theta + i\sin\theta - 1} = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\left(\cos\frac{(n-1)\theta}{2} + i\sin\frac{(n-1)\theta}{2}\right).$$

(ii) Using the identity $\sum_{k=1}^{n} z^k = z\left(\frac{z^n - 1}{z - 1}\right)$ for $z \neq 1$, or otherwise, prove that $\sum_{k=1}^{n} \cos k\theta = \dfrac{\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$ and

$$\sum_{k=1}^{n} \sin k\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

(iii) Using (a)(ii), or otherwise, prove that

$$\sum_{k=1}^{n} \cos^2(k\theta) = \frac{n}{2} + \frac{\sin n\theta\cos(n+1)\theta}{2\sin\theta}.$$

(9 marks)

(b) For $n > 1$, evaluate

(i) $\displaystyle\sum_{k=1}^{n} \sin^2\left(\frac{k\pi}{n}\right)$,

(ii) $\displaystyle\sum_{k=1}^{n} \left(\sin\frac{k\pi}{n} + \cos\frac{k\pi}{n}\right)^2$.

(6 marks)

END OF PAPER

2004-AL
P MATH
PAPER 2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 2004

PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A and any FOUR questions in Section B.

3. You are provided with one AL(E) answer book and four AL(D) answer books.

   Section A : Write your answers in the AL(E) answer book.

   Section B : Use a separate AL(D) answer book for each question and put the question number on the front cover of each answer book.

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5. Unless otherwise specified, all working must be clearly shown.

FORMULAS FOR REFERENCE

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$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
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$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Evaluate

   (a) $\lim_{x \to 0} (\tan 3x + \cos 4x)^{\frac{1}{x}}$,

   (b) $\lim_{x \to \infty} (\cos \sqrt{2004 + x} - \cos \sqrt{x})$.

(7 marks)

2. Let $f(x) = \begin{cases} x^2 + ax + b & \text{when } x \le 1 , \\ \dfrac{\sin \pi x}{\pi} & \text{when } x > 1 . \end{cases}$

   If f is differentiable at 1 , find a and b .

(6 marks)

Go on to the next page

3. (a) Evaluate $\int \sec^3 \theta \, d\theta$.

(b) Consider the curve $C: x^2 = 2y$, $0 \le x \le 1$.

Find the length of C .

(7 marks)

4. Using the substitution $u = \frac{1}{x}$, prove that $\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2} dx = 0$.

Hence, or otherwise, evaluate $\lim_{n\to\infty} \sum_{k=1}^{3n} \frac{\ln\left(2(\frac{1}{2}+\frac{k}{2n})\right)}{2n\left(1+(\frac{1}{2}+\frac{k}{2n})^2\right)}$.

(6 marks)

5. (a) For each non-negative integer n, define $I_n = \int_0^\infty e^{-x}x^n \, dx$.

(i) Evaluate I_0 .

(ii) Prove that if I_n is convergent, then $I_{n+1} = (n+1) I_n$.

(iii) Express I_n in terms of n .

(b) For each non-negative integer n, define $J_n = \int_0^\infty e^{-x}\left(1 - \frac{x^2}{n+1}\right)^2 dx$.

Find a non-negative integer m such that $J_m \le J_n$ for all n .

(8 marks)

6. Consider the two planes $\Pi_1 : x+y+z=1$ and $\Pi_2 : x-y+z=2$.

(a) Find a parametric equation of the line of intersection of Π_1 and Π_2 .

(b) Find the equation(s) of the plane(s) containing all the points which are equidistant from Π_1 and Π_2 .

(6 marks)

$m = \frac{n+1}{2}$

$2m = n+1$

$2m-1 = n$

Go on to the next page

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

7. Let $f(x) = \dfrac{|x|x^3}{x^3 - 2}$ $(x \neq \sqrt[3]{2})$. ≈ 1.259

(a) (i) Find $f'(x)$ and $f''(x)$ for $x > 0$.

(ii) Write down $f'(x)$ and $f''(x)$ for $x < 0$.

(iii) Prove that $f'(0)$ exists.

(iv) Does $f''(0)$ exist? Explain your answer.

(5 marks)

(b) Determine the range of values of x for each of the following cases:

(i) $f'(x) > 0$,

(ii) $f'(x) < 0$,

(iii) $f''(x) > 0$,

(iv) $f''(x) < 0$.

(3 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of $f(x)$.

(2 marks)

(d) Find the asymptote(s) of the graph of $f(x)$.

(3 marks)

(e) Sketch the graph of $f(x)$.

(2 marks)

8. (a) For any non-negative integers m and n , define

$$I_{m,n}(x) = \int_0^x \cos^m \theta \cos n\theta \, d\theta \quad \text{for all } x \in \mathbf{R} .$$

Prove that

$$I_{m+1,n+1}(x) = \frac{\cos^{m+1} x \sin(n+1)x}{m+n+2} + \frac{m+1}{m+n+2} I_{m,n}(x) .$$

(6 marks)

(b) Evaluate $\int_0^{\frac{\pi}{2}} \cos^4 \theta \cos 3\theta \, d\theta$.

(4 marks)

(c) Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 \theta \cos 3\theta \, d\theta$.

(5 marks)

Go on to the next page

9. Consider the ellipse $E: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where a and b are two positive constants with $a>b$. Let P be the point $(a\cos\theta,\ b\sin\theta)$, where $0<\theta<\dfrac{\pi}{2}$.

(a) Prove that P lies on E.

(1 mark)

(b) Let L be the tangent to E at P. L cuts the x-axis and the y-axis at P_1 and P_2 respectively. Find

(i) the equation of L,

(ii) the coordinates of P_1 and P_2.

(4 marks)

(c) Consider the two circles $C_1: x^2+y^2=a^2$ and $C_2: x^2+y^2=b^2$. Also consider the two points P_1 and P_2 described in (b). For $k=1, 2$, let L_k be the tangent to C_k from P_k, with the point of contact Q_k lying in the first quadrant.

(i) Prove that L_1 is parallel to L_2.

(ii) Find the coordinates of Q_1 and Q_2.

(iii) Let l be the straight line passing through Q_1 and Q_2. Is l a common normal to C_1 and C_2? Explain your answer.

(10 marks)

10. (a) (i) Evaluate $\displaystyle\int \frac{dx}{1-\sqrt{2}x+x^2}$.

(ii) Prove that

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{2\sqrt{2}}{1-\sqrt{2}x+x^2}\,dx = \pi .$$

(iii) Using (a)(ii), deduce that

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{2\sqrt{2}+4x+2\sqrt{2}x^2}{1+x^4}\,dx = \pi .$$

(6 marks)

$=0,1,2$

(b) (i) Let k be a non-negative integer. Prove that

$$-x^{4k+4} \le \sum_{n=0}^{k}(-1)^n x^{4n} - \frac{1}{1+x^4} \le x^{4k+4}$$

for all real numbers x.

(ii) Using (b)(i) and (a)(iii), or otherwise, prove that

$$\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\left(\frac{2}{4n+1}+\frac{2}{4n+2}+\frac{1}{4n+3}\right) = \pi .$$

Candidates may use the fact, without proof, that for any given polynomial $p(x)$, $\displaystyle\lim_{k\to\infty}\int_0^{\frac{1}{\sqrt{2}}} x^k\, p(x)\,dx = 0$.

(9 marks)

11. For any real number x, let $[x]$ denote the greatest integer not greater than x. Let $\mathrm{f}:\mathbf{R}\to\mathbf{R}$ be defined by

$$\mathrm{f}(x)=\begin{cases}\dfrac{1}{2} & \text{when } x \text{ is an integer,}\\[2ex] x-[x]-\dfrac{1}{2} & \text{when } x \text{ is not an integer.}\end{cases}$$

(a) (i) Prove that f is a periodic function with period 1.

(ii) Sketch the graph of $\mathrm{f}(x)$, where $-2\le x\le 3$.

(iii) Write down all the real number(s) x at which f is discontinuous.

(6 marks)

(b) Define $\mathrm{F}(x)=\int_0^x \mathrm{f}(t)\,\mathrm{d}t$ for all real numbers x.

(i) If $0\le x\le 1$, prove that $\mathrm{F}(x)=\dfrac{x^2-x}{2}$.

(ii) Is F a periodic function? Explain your answer.

(iii) Evaluate $\int_0^{\pi}\mathrm{F}(x)\,\mathrm{d}x$.

(9 marks)

12. (a) Let $\mathrm{f}:\mathbf{R}\to\mathbf{R}$ be a twice differentiable function. Assume that a and b are two distinct real numbers.

(i) Find a constant k (independent of x) such that the function $\mathrm{h}(x)=\mathrm{f}(x)-\mathrm{f}(b)-\mathrm{f}'(x)(x-b)-k(x-b)^2$ satisfies $\mathrm{h}(a)=0$. Also find $\mathrm{h}(b)$.

$I \in (a, b)$

(ii) Let I be the open interval with end points a and b. Using Mean Value Theorem and (a)(i), prove that there exists a real number $c\in I$ such that

$$\mathrm{f}(b)=\mathrm{f}(a)+\mathrm{f}'(a)(b-a)+\frac{\mathrm{f}''(c)}{2}(b-a)^2 .$$

(7 marks)

(b) Let $\mathrm{g}:\mathbf{R}\to\mathbf{R}$ be a twice differentiable function. Assume that there exists a real number $\beta\in(0,1)$ such that $\mathrm{g}(x)\le\mathrm{g}(\beta)=1$ for all $x\in(0,1)$.

$a=0$ $b=1$

(i) Using (a)(ii), prove that there exists a real number $\gamma\in(0,1)$ such that $\mathrm{g}(1)=1+\dfrac{\mathrm{g}''(\gamma)}{2}(1-\beta)^2$.

(ii) If $\mathrm{g}''(x)\ge -2$ for all $x\in(0,1)$, prove that $\mathrm{g}(0)+\mathrm{g}(1)\ge 1$.

(8 marks)

END OF PAPER