Solution	Marks
1. (a) $\lvert\lvert x\rvert-6\rvert\le 3$	
$-3\le\lvert x\rvert-6\le 3$	1A
$3\le\lvert x\rvert\le 9$	1A
$\lvert x\rvert\ge 3$ and $\lvert x\rvert\le 9$	
$(x\ge 3$ or $x\le -3)$ and $(-9\le x\le 9)$	1M
$\therefore\ -9\le x\le -3$ or $3\le x\le 9$	1A
Case 1: $\lvert x\rvert\ge 6$	1M for dividing cases
$\lvert x\rvert-6\le 3$	
$\lvert x\rvert\le 9$	
$-9\le x\le 9$	
$\therefore\ -9\le x\le -6$ or $6\le x\le 9$	1A
Case 2: $\lvert x\rvert<6$	
$-(\lvert x\rvert-6)\le 3$	
$\lvert x\rvert\ge 3$	
$x\ge 3$ or $x\le -3$	
$\therefore\ -6<x\le -3$ or $3\le x\le 6$	1A
Thus, the required solution is $-9\le x\le -3$ or $3\le x\le 9$.	1A
(b) Putting $x=1-2y$, we have	1M
$-9\le 1-2y\le -3$ or $3\le 1-2y\le 9$	
$-10\le -2y\le -4$ or $2\le -2y\le 8$	
$2\le y\le 5$ or $-4\le y\le -1$	1A
$\therefore\ -4\le y\le -1$ or $2\le y\le 5$.	
	----------(6)

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Solution	Marks
2. (a) $(1+x)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n$	1A
Putting $x=1$ and with the help of $C_0^n = 1$, we have $C_1^n + C_2^n + C_3^n + \cdots + C_n^n = 2^n - 1$	1A
(b) $(1+x)^n = C_0^n + C_1^n x + C_2^n x^2 + \cdots + C_n^n x^n$	
Differentiate both sides with respect to x, we have $C_1^n + 2C_2^n x + 3C_3^n x^2 + \cdots + nC_n^n x^{n-1} = n(1+x)^{n-1}$ ....(*)	1M
Putting $x=1$, we have $C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n = n(2^{n-1})$.	1A
Let $S = C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n$. Then, we have	
$2S = (0C_0^n + C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n) + (nC_n^n + (n-1)C_{n-1}^n + (n-2)C_{n-2}^n + \cdots + 0C_0^n)$	
$2S = (0C_0^n + C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n) + (nC_0^n + (n-1)C_1^n + (n-2)C_2^n + \cdots + 0C_n^n)$	1M for $C_k^n = C_{n-k}^n$
$2S = nC_0^n + nC_1^n + nC_2^n + \cdots + nC_n^n$	
$2S = n(C_0^n + C_1^n + C_2^n + \cdots + C_n^n)$	
$2S = n(1 + 2^n - 1)$ (by (a) with the help of $C_0^n = 1$)	
$2S = n(2^n)$	
$S = n(2^{n-1})$	1A
Thus, we have $C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n = n(2^{n-1})$.	
Note that $rC_r^n = \dfrac{r(n!)}{r!(n-r)!} = \dfrac{n(n-1)!}{(r-1)!(n-r)!} = nC_{r-1}^{n-1}$	1M
$\therefore$ $C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n$	
$= nC_0^{n-1} + nC_1^{n-1} + nC_2^{n-1} + \cdots + nC_{n-1}^{n-1}$	
$= n(C_0^{n-1} + C_1^{n-1} + C_2^{n-1} + \cdots + C_{n-1}^{n-1})$	
$= n(2^{n-1})$	1A

Solution	Marks
(c) Multiply both sides of (*) by x, we have $C_1^n x + 2C_2^n x^2 + 3C_3^n x^3 + \cdots + nC_n^n x^n = nx(1+x)^{n-1}$	
Differentiate both sides with respect to x, we have $C_1^n + 2^2 C_2^n x + 3^2 C_3^n x^2 + \cdots + n^2 C_n^n x^{n-1} = n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2}$	1A
Putting $x=1$, we have $C_1^n + 2^2 C_2^n + 3^2 C_3^n + \cdots + n^2 C_n^n = n(2^{n-1}) + n(n-1)(2^{n-2})$ $= 2^{n-2}(2n + n^2 - n)$	
$= 2^{n-2}(n^2 + n)$ $= n(n+1)(2^{n-2})$	1A
Differentiate both sides of (*) with respect to x, we have $2C_2^n + 3(2)C_3^n x + 4(3)C_4^n x^2 + \cdots + n(n-1)C_n^n x^{n-2} = n(n-1)(1+x)^{n-2}$ Putting $x=1$, we have $2C_2^n + 3(2)C_3^n + 4(3)C_4^n + \cdots + n(n-1)C_n^n = n(n-1)(2^{n-2})$	1A
By (b), we have $C_1^n + 2^2 C_2^n + 3^2 C_3^n + \cdots + n^2 C_n^n$ $= (C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n) + (2C_2^n + 3(2)C_3^n + 4(3)C_4^n + \cdots + n(n-1)C_n^n)$ $= n(2^{n-1}) + n(n-1)(2^{n-2})$ $= 2^{n-2}(2n + n^2 - n)$ $= 2^{n-2}(n^2 + n)$	1A
$= n(n+1)(2^{n-2})$	
$C_1^n + 2^2 C_2^n + 3^2 C_3^n + \cdots + n^2 C_n^n$ $= n(C_0^{n-1} + 2C_1^{n-1} + 3C_2^{n-1} + \cdots + nC_{n-1}^{n-1})$ ($\because rC_r^n = C_{r-1}^{n-1}$)	1A
$= n(C_0^{n-1} + C_1^{n-1} + C_2^{n-1} + \cdots + C_{n-1}^{n-1}) + n(C_1^{n-1} + 2C_2^{n-1} + 3C_3^{n-1} + \cdots + (n-1)C_{n-1}^{n-1})$ $= n(2^{n-1}) + n(n-1)(2^{n-2})$ (by (a) and (b)) $= 2^{n-2}(2n + n^2 - n)$ $= 2^{n-2}(n^2 + n)$	
$= n(n+1)(2^{n-2})$	1A
	---------(6)

Solution	Marks
3. (a) Let $\frac{5x-3}{x(x+1)(x+3)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+3}$.	1M
$5x-3\equiv A(x+1)(x+3)+Bx(x+3)+Cx(x+1)$	
Putting $x=0$, -1 , -3 , we have	
$A=-1$, $B=4$, $C=-3$	1A
$\therefore \quad \frac{5x-3}{x(x+1)(x+3)}=\frac{-1}{x}+\frac{4}{x+1}-\frac{3}{x+3}$.	
(b) (i) $\sum_{k=1}^{n}\frac{5k-3}{k(k+1)(k+3)}$	
$=\sum_{k=1}^{n}\left(-\frac{1}{k}+\frac{4}{k+1}-\frac{3}{k+3}\right)$	
$=\sum_{k=1}^{n}\left(-\frac{1}{k}+\frac{1}{k+1}\right)+3\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)$	1M
$=\frac{1}{n+1}-1+3\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right)$	1A
$=\frac{3}{2}+\frac{1}{n+1}-3\left(\frac{1}{n+2}+\frac{1}{n+3}\right)$	
$<\frac{3}{2}+\frac{1}{n+1}-\frac{3}{n+2}$ ($\because \quad \frac{3}{n+3}>0$)	
$=\frac{3}{2}-\frac{2n+1}{(n+1)(n+2)}$	
$<\frac{3}{2}$	1
$\sum_{k=1}^{n}\frac{5k-3}{k(k+1)(k+3)}$	
$=\sum_{k=1}^{n}\left(-\frac{1}{k}+\frac{4}{k+1}-\frac{3}{k+3}\right)$	
$=\sum_{k=1}^{n}\left(-\frac{1}{k}+\frac{1}{k+1}\right)+3\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)$	1M
$=\frac{1}{n+1}-1+3\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right)$	1A
$=\frac{3}{2}+\frac{1}{n+1}-3\left(\frac{1}{n+2}+\frac{1}{n+3}\right)$	
$=\frac{3}{2}-\frac{5n^2+16n+9}{(n+1)(n+2)(n+3)}$	
$<\frac{3}{2}$	1
(ii) $\sum_{k=1}^{\infty}\frac{5k-3}{k(k+1)(k+3)}$	
$=\lim_{n\to\infty}\left(\frac{3}{2}+\frac{1}{n+1}-\frac{3}{n+2}-\frac{3}{n+3}\right)$	1M
$=\frac{3}{2}+\lim_{n\to\infty}\frac{1}{n+1}-3\lim_{n\to\infty}\frac{1}{n+2}-3\lim_{n\to\infty}\frac{1}{n+3}$	
$=\frac{3}{2}$	1A
	----------(7)

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Solution	Marks
4. (a) (i) When $n=1$, $x_1=2>1$	
and $1-\frac{1}{x_2-1}=1-\frac{1}{3-1}=1-\frac{1}{2}=\frac{1}{2}=\frac{1}{x_1}=S_1$	
$\therefore$ the statements are true for $n=1$.	1
Assume that $x_k>k$ and $S_k=1-\frac{1}{x_{k+1}-1}$ for some $k\in\mathbf{N}$.	
Then, x_{k+1}	
$=x_k^2-x_k+1$	
$=x_k(x_k-1)+1$	1M
$>k(x_k-1)+1$ $(\because x_k>k\geq 1)$	
$\geq k+1$ $(\because x_k\geq 2$ since $x_k>1$ and x_k is an integer $)$	
$\therefore$ $x_{k+1}>k+1$	1
Then, x_{k+1}	
$=x_k^2-x_k+1$	
$=(x_k-1)^2+x_k$	1M
$>x_k$ $(\because x_k>1$ since $x_k>k\geq 1)$	
Note that x_n is an integer for all $n\in\mathbf{N}$.	
$\therefore$ $x_{k+1}\geq x_k+1$	
$\therefore$ $x_{k+1}>k+1$ $(\because x_k>k)$	1
(ii) Also, $S_{k+1}=S_k+\frac{1}{x_{k+1}}$	1M
$=1-\frac{1}{x_{k+1}-1}+\frac{1}{x_{k+1}}$	
$=1+\frac{-x_{k+1}+x_{k+1}-1}{x_{k+1}^2-x_{k+1}}$	
$=1-\frac{1}{x_{k+1}^2-x_{k+1}}$	
$=1-\frac{1}{x_{k+2}-1}$	1
$\therefore$ by mathematical induction, the statements are true for any positive integer n.	
(b) By (a)(i), $x_n>n\Rightarrow x_n>0\Rightarrow\{S_n\}$ is (strictly) increasing (as $S_n=\sum_{i=1}^{n}\frac{1}{x_i}$).	1M
Also, by (a)(ii), $S_n=1-\frac{1}{x_{n+1}-1}<1\Rightarrow\{S_n\}$ is bounded above by 1.	
Thus, $\lim_{n\to\infty}S_n$ exists.	1
By (a)(i), $x_n>n\Rightarrow x_n\to\infty$ as $n\to\infty$	1M
$\therefore$ $\frac{1}{x_{n+1}-1}\to 0$ as $n\to\infty$	
$\Rightarrow$ $\lim_{n\to\infty}(1-\frac{1}{x_{n+1}-1})=1$	
By (a)(ii), $\lim_{n\to\infty}S_n$ exists (and equals to 1).	1
	----------(7)

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Solution	Marks
5. (a) $\mathbf{u}\times\mathbf{v}$	
$= (\lambda\mathbf{n}+(1-\lambda)\mathbf{m})\times(2(1-\lambda)\mathbf{n}-\lambda\mathbf{m})$	
$= 2\lambda(1-\lambda)\mathbf{n}\times\mathbf{n}+2(1-\lambda)^2\mathbf{m}\times\mathbf{n}-\lambda^2\mathbf{n}\times\mathbf{m}-\lambda(1-\lambda)\mathbf{m}\times\mathbf{m}$	1A
$= 2(1-\lambda)^2\mathbf{m}\times\mathbf{n}-\lambda^2\mathbf{n}\times\mathbf{m}$ $\quad (\because\ \mathbf{n}\times\mathbf{n}=\mathbf{0}=\mathbf{m}\times\mathbf{m})$	
$= (2(1-\lambda)^2+\lambda^2)\mathbf{m}\times\mathbf{n}$ $\quad (\because\ \mathbf{n}\times\mathbf{m}=-\mathbf{m}\times\mathbf{n})$	
$= (3\lambda^2-4\lambda+2)\mathbf{m}\times\mathbf{n}$	1
(b) (i) $\lvert\mathbf{m}\times\mathbf{n}\rvert=\lvert\mathbf{m}\rvert\lvert\mathbf{n}\rvert\sin\frac{\pi}{6}$	
$= (4)(3)(\frac{1}{2})$	
$= 6$	1A
(ii) Area of the parallelogram	
$= \lvert\mathbf{u}\times\mathbf{v}\rvert$	1A
$= \lvert(2-4\lambda+3\lambda^2)\mathbf{m}\times\mathbf{n}\rvert$	
$= 6\lvert 3\lambda^2-4\lambda+2\rvert$	
Note that $3\lambda^2-4\lambda+2$	
$= 3(\lambda-\frac{2}{3})^2+\frac{2}{3}$	1M
$\geq \frac{2}{3}$ for any real λ	
Thus, the smallest area of the parallelogram	
$= 6(\frac{2}{3})$	
$= 4$	1A ----------(6)

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Solution	Marks
6. (a) "⇒" Let the three roots be $a-d$, a, $a+d$. Then,	1M
$a-d+a+a+d=-p$	1A (either one)
$3a=-p$	
$a=\frac{-p}{3}$	1
$\therefore$ $\frac{-p}{3}$ is a root of the equation.	
"⇐" Let α, $\frac{-p}{3}$, γ be the three roots. Then,	
$\alpha+\left(\frac{-p}{3}\right)+\gamma=-p$	
$\alpha+\gamma=2\left(\frac{-p}{3}\right)$	1
$\alpha-\left(\frac{-p}{3}\right)=\left(\frac{-p}{3}\right)-\gamma$	
$\therefore$ the three roots form an arithmetic sequence.	
(b) By (a), $\frac{-p}{3}$ is a root.	
$\frac{-p^3}{27}+\frac{p^3}{9}-7p+p=0$	1M
$p^3-81p=0$	
$p(p-9)(p+9)=0$	
$p=0$, 9 or -9	1A
When $p=0$, the equation becomes	
$x^3+21x=0$	
$x(x^2+21)=0$	
$x=0$ or $-\sqrt{21}i$ or $\sqrt{21}i$	
$\therefore$ $p=0$ is rejected since some of the roots are unreal.	1A
When $p=9$, the equation becomes	
$x^3+9x^2+21x+9=0$	
$(x+3)(x^2+6x+3)=0$	
$x=-3$ or $x=\frac{-6\pm\sqrt{24}}{2}$	
$x=-3$, $-3-\sqrt{6}$ or $-3+\sqrt{6}$	
When $p=-9$, the equation becomes	
$x^3-9x^2+21x-9=0$	
$(x-3)(x^2-6x+3)=0$	
$x=3$, $3-\sqrt{6}$ or $3+\sqrt{6}$	
Thus, the two values of p are -9 and 9.	1A
	----------(8)

Solution	Marks
7. (a) (i) (E) has a unique solution	
$\Leftrightarrow \Delta \neq 0$	1M
$\Leftrightarrow \Delta = \begin{vmatrix} 1 & a & -1 \\ 2 & -1 & a \\ -1 & 2a^2 & a-3 \end{vmatrix} \neq 0$	
$\Leftrightarrow -2a^3 - 7a^2 + 5a + 4 \neq 0$	1A
$\Leftrightarrow -(a-1)(2a+1)(a+4) \neq 0$	
$\Leftrightarrow a \neq 1,\ a \neq \frac{-1}{2}$ and $a \neq -4$	
$\Leftrightarrow a < -4,\ -4 < a < \frac{-1}{2},\ \frac{-1}{2} < a < 1$ or $a > 1$	1A
The augmented matrix of (E) is	
$\left(\begin{array}{ccc\|c} 1 & a & -1 & 0 \\ 2 & -1 & a & -2a \\ -1 & 2a^2 & a-3 & 2a \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & a & -1 & 0 \\ 0 & -2a-1 & a+2 & -2a \\ 0 & 2a^2+a & a-4 & 2a \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & a & -1 & 0 \\ 0 & -2a-1 & a+2 & -2a \\ 0 & 0 & (a+4)(a-1) & 2a(1-a) \end{array}\right)$	1A
(E) has a unique solution	
$\Leftrightarrow (a+4)(a-1) \neq 0$ and $-2a-1 \neq 0$	1M
$\Leftrightarrow a \neq 1,\ a \neq \frac{-1}{2}$ and $a \neq -4$	
$\Leftrightarrow a < -4,\ -4 < a < \frac{-1}{2},\ \frac{-1}{2} < a < 1$ or $a > 1$	1A
When (E) has a unique solution,	
$x = \frac{\begin{vmatrix} 0 & a & -1 \\ -2a & -1 & a \\ 2a & 2a^2 & a-3 \end{vmatrix}}{\Delta} = \frac{2a(4a+1)(a-1)}{\Delta}$	1M for Cramer's rule
$= \frac{-2a(4a+1)}{(2a+1)(a+4)}$	
$y = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 2 & -2a & a \\ -1 & 2a & a-3 \end{vmatrix}}{\Delta} = \frac{-4a(a-1)}{\Delta}$	
$= \frac{4a}{(2a+1)(a+4)}$	
$z = \frac{\begin{vmatrix} 1 & a & 0 \\ 2 & -1 & -2a \\ -1 & 2a^2 & 2a \end{vmatrix}}{\Delta} = \frac{2a(2a+1)(a-1)}{\Delta}$	
$= \frac{-2a}{a+4}$	1A+1A (1A for anyone, 1A for all)

Solution	Marks
When (E) has a unique solution, the augmented matrix of (E) becomes	
$\left(\begin{array}{ccc\|c} 1 & a & -1 & 0 \\ 0 & -2a-1 & a+2 & -2a \\ 0 & 0 & 1 & \frac{-2a}{a+4} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & a & -1 & 0 \\ 0 & 1 & \frac{-(a+2)}{2a+1} & \frac{2a}{2a+1} \\ 0 & 0 & 1 & \frac{-2a}{a+4} \end{array}\right)$	1M
$\sim \left(\begin{array}{ccc\|c} 1 & 0 & \frac{a^2-1}{2a+1} & \frac{-2a^2}{2a+1} \\ 0 & 1 & \frac{-(a+2)}{2a+1} & \frac{2a}{2a+1} \\ 0 & 0 & 1 & \frac{-2a}{a+4} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{-2a(4a+1)}{(2a+1)(a+4)} \\ 0 & 1 & 0 & \frac{4a}{(2a+1)(a+4)} \\ 0 & 0 & 1 & \frac{-2a}{a+4} \end{array}\right)$	
$\therefore \quad x=\frac{-2a(4a+1)}{(2a+1)(a+4)}$, $y=\frac{4a}{(2a+1)(a+4)}$, $z=\frac{-2a}{a+4}$.	1A+1A (1A for anyone, 1A for all)
(ii) (1) When $a=1$, the augmented matrix of (E) becomes	
$\left(\begin{array}{ccc\|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$	1A
$\therefore$ the solution set is $\left\{\left(\frac{-2}{3}, \frac{2+3t}{3}, t\right) : t \in \mathbf{R}\right\}$.	1A accept $\left\{\left(\frac{-2}{3}, t, \frac{3t-2}{3}\right) : t \in \mathbf{R}\right\}$
(2) When $a=-4$, the augmented matrix of (E) becomes	
$\left(\begin{array}{ccc\|c} 1 & -4 & -1 & 0 \\ 0 & 7 & -2 & 8 \\ 0 & 0 & 0 & -40 \end{array}\right)$	1A
$\therefore$ (E) is inconsistent.	1A
	---------(10)
(b) Putting $a=1$ into (E), by (a)(ii)(1), $x=\frac{-2}{3}$, $y=\frac{2+3t}{3}$ and $z=t$.	1M accept $x=\frac{-2}{3}$, $y=t$ and $z=\frac{3t-2}{3}$
Thus, $24x^2+3y^2+2z$	
$=24\left(-\frac{2}{3}\right)^2+3\left(\frac{2+3t}{3}\right)^2+2t$	1M accept $24\left(-\frac{2}{3}\right)^2+3(t^2)+2\left(\frac{3t-2}{3}\right)^2$
$=3t^2+6t+12$	
$=3(t+1)^2+9$	1A accept $3(t+\frac{1}{3})^2+9$
≥ 9 for all real values of t .	
$\therefore$ the least value of $24x^2+3y^2+2z$ is 9 and	1A
the corresponding values of x, y, z are $\frac{-2}{3}$, $\frac{-1}{3}$, -1 respectively.	1A
	---------(5)

Solution	Marks
8. (a) $\begin{vmatrix} -2-\alpha & \sqrt{3} \\ \sqrt{3} & -\alpha \end{vmatrix} = 0$	
$\Leftrightarrow \quad \alpha^2 + 2\alpha - 3 = 0$	1A
$\Leftrightarrow \quad (\alpha - 1)(\alpha + 3) = 0$	
$\Leftrightarrow \quad \alpha = 1 \text{ or } \alpha = -3$	1A
	----------(2)
(b) By (a), $\alpha_1 = -3$ and $\alpha_2 = 1$	
$\begin{pmatrix} -2-\alpha_1 & \sqrt{3} \\ \sqrt{3} & -\alpha_1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	
$\Leftrightarrow \quad \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	
$\Leftrightarrow \quad \tan\theta_1 = \frac{-1}{\sqrt{3}}$	1A
$\Leftrightarrow \quad \theta_1 = \frac{5\pi}{6} \qquad (\because \quad 0 \le \theta_1 < \pi)$	1A
$\begin{pmatrix} -2-\alpha_2 & \sqrt{3} \\ \sqrt{3} & -\alpha_2 \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	
$\Leftrightarrow \quad \begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	
$\Leftrightarrow \quad \tan\theta_2 = \sqrt{3}$	1A
$\Leftrightarrow \quad \theta_2 = \frac{\pi}{3} \qquad (\because \quad 0 \le \theta_2 < \pi)$	1A
$P = \begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$	
$P^2 = \begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1A
$P^n = \begin{cases} \begin{pmatrix} \frac{-\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} & \text{if } n = 1, 3, 5, \ldots \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } n = 2, 4, 6, \ldots \end{cases}$	1M for dividing into 2 cases + 1A for all correct answers

Solution	Marks
$P^{-1}\begin{pmatrix}-2 & \sqrt{3}\\ \sqrt{3} & 0\end{pmatrix}P=\begin{pmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}-2 & \sqrt{3}\\ \sqrt{3} & 0\end{pmatrix}\begin{pmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$ ($\because\ P^{-1}=P$)	
$=\begin{pmatrix}\frac{3\sqrt{3}}{2} & \frac{-3}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$	
$=\begin{pmatrix}-3 & 0\\ 0 & 1\end{pmatrix}$ which is of the required form.	1
	----------(8)
(c) By (b), $P^{-1}\begin{pmatrix}-2 & \sqrt{3}\\ \sqrt{3} & 0\end{pmatrix}P=\begin{pmatrix}-3 & 0\\ 0 & 1\end{pmatrix}$	
$\left(P^{-1}\begin{pmatrix}-2 & \sqrt{3}\\ \sqrt{3} & 0\end{pmatrix}P\right)^n=\begin{pmatrix}(-3)^n & 0\\ 0 & 1\end{pmatrix}$	1M+1A
$P^{-1}\begin{pmatrix}-2 & \sqrt{3}\\ \sqrt{3} & 0\end{pmatrix}^n P=\begin{pmatrix}(-3)^n & 0\\ 0 & 1\end{pmatrix}$	1A
$\begin{pmatrix}-2 & \sqrt{3}\\ \sqrt{3} & 0\end{pmatrix}^n=P\begin{pmatrix}(-3)^n & 0\\ 0 & 1\end{pmatrix}P^{-1}$	1A
$=\begin{pmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}(-3)^n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$	
$=\begin{pmatrix}\frac{-\sqrt{3}(-3)^n}{2} & \frac{1}{2}\\ \frac{(-3)^n}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$	
$=\begin{pmatrix}\frac{1+(-1)^n 3^{n+1}}{4} & \frac{3^{\frac{1}{2}}\left(1+(-1)^{n+1}3^n\right)}{4}\\ \frac{3^{\frac{1}{2}}\left(1+(-1)^{n+1}3^n\right)}{4} & \frac{3\left(1+(-1)^n 3^{n-1}\right)}{4}\end{pmatrix}$	1A
	----------(5)

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Solution	Marks
9. (a) (i) Suppose that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}=\mathbf{0}$ for some $\alpha, \beta, \gamma \in \mathbf{R}$.	
Then, $\alpha(p,q,0)+\beta(q,-p,0)+\gamma(0,0,r)=\mathbf{0}$. Therefore, we have	
(*) : $\begin{cases} p\alpha+q\beta+0\gamma=0 \\ q\alpha-p\beta+0\gamma=0 \\ 0\alpha+0\beta+r\gamma=0 \end{cases}$	
Note that $\begin{vmatrix} p & q & 0 \\ q & -p & 0 \\ 0 & 0 & r \end{vmatrix} = -r(p^2+q^2) \neq 0 \quad (\because \ p,q,r \neq 0)$	1M
$\therefore$ (*) has trivial solution only	
$\therefore$ $\alpha=\beta=\gamma=0$	
Thus, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.	1
Suppose that $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}=\mathbf{0}$ for some scalars α, β and γ.	
Then, $\mathbf{a}\cdot(\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c})=0$	1M
$(p^2+q^2)\alpha=0 \quad (\because \ \mathbf{a}\cdot\mathbf{a}=p^2+q^2 \text{ and } \mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}=0)$	
$\therefore \alpha=0 \quad (\because \ p,q\neq 0)$	
So, $\mathbf{b}\cdot(\beta\mathbf{b}+\gamma\mathbf{c})=0$	
$(q^2+p^2)\beta=0 \quad (\because \ \mathbf{b}\cdot\mathbf{b}=q^2+p^2 \text{ and } \mathbf{b}\cdot\mathbf{c}=0)$	
$\beta=0 \quad (\because \ p,q\neq 0)$	
Therefore, $\mathbf{c}\cdot(\gamma\mathbf{c})=0$	
$r^2\gamma=0 \quad (\because \ \mathbf{c}\cdot\mathbf{c}=r^2)$	
$\gamma=0 \quad (\because \ r\neq 0)$	
Thus, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.	1
(ii) Let $\mathbf{d}=(d_1,d_2,d_3)$. Consider the system of equations in α, β, γ	
(**) : $\begin{cases} p\alpha+q\beta+0\gamma=d_1 \\ q\alpha-p\beta+0\gamma=d_2 \\ 0\alpha+0\beta+r\gamma=d_3 \end{cases}$	
Note that $\begin{vmatrix} p & q & 0 \\ q & -p & 0 \\ 0 & 0 & r \end{vmatrix} = -r(p^2+q^2) \neq 0 \quad (\because \ p,q,r \neq 0)$	
$\therefore$ there exists unique solution (α, β, γ) which satisfies (**).	
$\therefore$ $\mathbf{d}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}$ for some $\alpha, \beta, \gamma \in \mathbf{R}$.	1A
Note that $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0$	1A
$\therefore$ $\mathbf{d}\cdot\mathbf{a}=\alpha\mathbf{a}\cdot\mathbf{a}=\alpha\|\mathbf{a}\|^2 \Rightarrow \alpha=\dfrac{\mathbf{d}\cdot\mathbf{a}}{\|\mathbf{a}\|^2} \quad (\because \ \|\mathbf{a}\|^2=p^2+q^2\neq 0)$	
Also, $\mathbf{d}\cdot\mathbf{b}=\beta\mathbf{b}\cdot\mathbf{b}=\beta\|\mathbf{b}\|^2 \Rightarrow \beta=\dfrac{\mathbf{d}\cdot\mathbf{b}}{\|\mathbf{b}\|^2} \quad (\because \ \|\mathbf{b}\|^2=q^2+p^2\neq 0)$	1M
Moreover, $\mathbf{d}\cdot\mathbf{c}=\gamma\mathbf{c}\cdot\mathbf{c}=\gamma\|\mathbf{c}\|^2 \Rightarrow \gamma=\dfrac{\mathbf{d}\cdot\mathbf{c}}{\|\mathbf{c}\|^2} \quad (\because \ \|\mathbf{c}\|^2=r^2\neq 0)$	
Thus, $\mathbf{d}=\left(\dfrac{\mathbf{d}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\right)\mathbf{a}+\left(\dfrac{\mathbf{d}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}+\left(\dfrac{\mathbf{d}\cdot\mathbf{c}}{\|\mathbf{c}\|^2}\right)\mathbf{c}$.	1

Solution	Marks
Let $\mathbf{d}=(d_1,d_2,d_3)$. Consider the system of equations in α, β, γ	
(**) : $\begin{cases} p\alpha+q\beta+0\gamma=d_1 \\ q\alpha-p\beta+0\gamma=d_2 \\ 0\alpha+0\beta+r\gamma=d_3 \end{cases}$	
Note that $\begin{vmatrix} p & q & 0 \\ q & -p & 0 \\ 0 & 0 & r \end{vmatrix}=-r(p^2+q^2)\neq 0 \quad (\because \ p,q,r\neq 0)$	
$\therefore$ there exists unique solution (α, β, γ) which satisfies (**).	
$\therefore$ $\mathbf{d}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}$ for some $\alpha, \beta, \gamma \in \mathbf{R}$.	1A
By Cramer's Rule,	
$\alpha=\dfrac{\begin{vmatrix} d_1 & q & 0 \\ d_2 & -p & 0 \\ d_3 & 0 & -r \end{vmatrix}}{-r(p^2+q^2)}=\dfrac{pd_1+qd_2}{p^2+q^2}=\dfrac{\mathbf{d}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert^2} \quad (\because \ \lvert\mathbf{a}\rvert^2=p^2+q^2)$	
$\beta=\dfrac{\begin{vmatrix} p & d_1 & 0 \\ q & d_2 & 0 \\ 0 & d_3 & r \end{vmatrix}}{-r(p^2+q^2)}=\dfrac{qd_1-pd_2}{q^2+p^2}=\dfrac{\mathbf{d}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert^2} \quad (\because \ \lvert\mathbf{b}\rvert^2=q^2+p^2)$	1M+1A
$\gamma=\dfrac{\begin{vmatrix} p & q & d_1 \\ q & -p & d_2 \\ 0 & 0 & d_3 \end{vmatrix}}{-r(p^2+q^2)}=\dfrac{d_3}{r}=\dfrac{rd_3}{r^2}=\dfrac{\mathbf{d}\cdot\mathbf{c}}{\lvert\mathbf{c}\rvert^2} \quad (\because \ \lvert\mathbf{c}\rvert^2=r^2)$	
Thus, $\mathbf{d}=\left(\dfrac{\mathbf{d}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert^2}\right)\mathbf{a}+\left(\dfrac{\mathbf{d}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)\mathbf{b}+\left(\dfrac{\mathbf{d}\cdot\mathbf{c}}{\lvert\mathbf{c}\rvert^2}\right)\mathbf{c}$.	1
Let $\mathbf{d}=(d_1,d_2,d_3)$. Then, we have	
$\left(\dfrac{\mathbf{d}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert^2}\right)\mathbf{a}+\left(\dfrac{\mathbf{d}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)\mathbf{b}+\left(\dfrac{\mathbf{d}\cdot\mathbf{c}}{\lvert\mathbf{c}\rvert^2}\right)\mathbf{c}$	
$=\left(\dfrac{pd_1+qd_2}{p^2+q^2}\right)(p,q,0)+\left(\dfrac{qd_1-pd_2}{p^2+q^2}\right)(q,-p,0)+\left(\dfrac{rd_3}{r^2}\right)(0,0,r)$	1M+1A
$=\left(\dfrac{p(pd_1+qd_2)+q(qd_1-pd_2)}{p^2+q^2}, \dfrac{q(pd_1+qd_2)-p(qd_1-pd_2)}{p^2+q^2}, d_3\right)$	1A
$=\left(\dfrac{(p^2+q^2)d_1}{p^2+q^2}, \dfrac{(p^2+q^2)d_2}{p^2+q^2}, d_3\right)$	
$=(d_1,d_2,d_3)$	
$=\mathbf{d}$	1
	---------(6)

Solution	Marks
(b) (i) Assume $\mathbf{v}=\mathbf{0}$. Then, we have	
$\mathbf{y}-\dfrac{\mathbf{y}\cdot\mathbf{x}}{\lvert\mathbf{x}\rvert^2}\mathbf{x}=0 \quad (\because\ \mathbf{u}=\mathbf{x})$	
$\Rightarrow\ -\dfrac{\mathbf{y}\cdot\mathbf{x}}{\lvert\mathbf{x}\rvert^2}\mathbf{x}+(1)\mathbf{y}+(0)\mathbf{z}=0$, where $1\neq 0$	1M
$\Rightarrow$ $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly dependent vectors. It contradicts that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent vectors. Thus, $\mathbf{v}$ is a non-zero vector.	1
(ii) (1) $\mathbf{u}\cdot\mathbf{v}=\mathbf{x}\cdot\left(\mathbf{y}-\left(\dfrac{\mathbf{y}\cdot\mathbf{x}}{\lvert\mathbf{x}\rvert^2}\right)\mathbf{x}\right)$	
$=\mathbf{x}\cdot\mathbf{y}-\dfrac{(\mathbf{y}\cdot\mathbf{x})(\mathbf{x}\cdot\mathbf{x})}{\lvert\mathbf{x}\rvert^2} \quad (\because\ \mathbf{u}=\mathbf{x})$	1M
$=\mathbf{x}\cdot\mathbf{y}-\dfrac{(\mathbf{x}\cdot\mathbf{y})(\lvert\mathbf{x}\rvert^2)}{\lvert\mathbf{x}\rvert^2}$	
$=\mathbf{x}\cdot\mathbf{y}-\mathbf{x}\cdot\mathbf{y}$	
$=0$	
$\mathbf{v}\cdot\mathbf{w}=\mathbf{v}\cdot\left(\mathbf{z}-\left(\dfrac{\mathbf{z}\cdot\mathbf{x}}{\lvert\mathbf{x}\rvert^2}\right)\mathbf{x}-\left(\dfrac{\mathbf{z}\cdot\mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}\right) \quad (\because\ \mathbf{u}=\mathbf{x})$	
$=\mathbf{v}\cdot\mathbf{z}-\dfrac{(\mathbf{z}\cdot\mathbf{x})(\mathbf{v}\cdot\mathbf{u})}{\lvert\mathbf{x}\rvert^2}-\dfrac{(\mathbf{z}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{v})}{\lvert\mathbf{v}\rvert^2} \quad (\because\ \mathbf{u}=\mathbf{x})$	1M
$=\mathbf{v}\cdot\mathbf{z}-\dfrac{(\mathbf{z}\cdot\mathbf{v})\lvert\mathbf{v}\rvert^2}{\lvert\mathbf{v}\rvert^2} \quad (\because\ \mathbf{v}\cdot\mathbf{u}=0)$	
$=\mathbf{v}\cdot\mathbf{z}-\mathbf{v}\cdot\mathbf{z} \quad (\because\ \mathbf{z}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{z})$	
$=0$	
$\mathbf{w}\cdot\mathbf{u}=\left(\mathbf{z}-\left(\dfrac{\mathbf{z}\cdot\mathbf{x}}{\lvert\mathbf{x}\rvert^2}\right)\mathbf{x}-\left(\dfrac{\mathbf{z}\cdot\mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}\right)\cdot\mathbf{u} \quad (\because\ \mathbf{u}=\mathbf{x})$	
$=\mathbf{z}\cdot\mathbf{u}-\dfrac{(\mathbf{z}\cdot\mathbf{x})(\mathbf{x}\cdot\mathbf{u})}{\lvert\mathbf{x}\rvert^2}-\dfrac{(\mathbf{z}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{u})}{\lvert\mathbf{v}\rvert^2}$	
$=\mathbf{z}\cdot\mathbf{u}-\dfrac{(\mathbf{z}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{u})}{\lvert\mathbf{u}\rvert^2} \quad (\because\ \mathbf{u}=\mathbf{x}$ and $\mathbf{v}\cdot\mathbf{u}=0)$	1M
$=\mathbf{z}\cdot\mathbf{u}-\mathbf{z}\cdot\mathbf{u} \quad (\because\ \mathbf{u}\cdot\mathbf{u}=\lvert\mathbf{u}\rvert^2)$	
$=0$	
Thus, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors.	1A+1A (1A for anyone, 1A for all)
(2) $\mathbf{w}$ is perpendicular to the plane containing $\mathbf{x}$ and $\mathbf{y}$.	2A
$\mathbf{w}$ is a normal vector of the plane containing $\mathbf{x}$ and $\mathbf{y}$.	2A
	---------(9)

Solution	Marks
10. (a) $(a+b)^n = a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \cdots + C_n^n b^n$	1A
$\because \quad a \ge 0$, $b \ge 0$, $n \ge 2$ and $C_1^n = n$	
$\therefore \quad (a+b)^n \ge a^n + na^{n-1}b$	1
The equality holds if and only if $C_2^n a^{n-2}b^2 = C_3^n a^{n-3}b^3 = \cdots = C_n^n b^n = 0$	
if and only if $b = 0$	1A ---------(3)
(b) (i) $A_{k+1} - A_k$	
$= \dfrac{a_1 + a_2 + a_3 + \cdots + a_{k+1}}{k+1} - \dfrac{a_1 + a_2 + a_3 + \cdots + a_k}{k}$	1M
$= \dfrac{k\,a_{k+1} - (a_1 + a_2 + a_3 + \cdots + a_k)}{k(k+1)}$	
$= \dfrac{(a_{k+1} - a_1) + (a_{k+1} - a_2) + (a_{k+1} - a_3) + \cdots + (a_{k+1} - a_k)}{k(k+1)}$	
$\ge 0 \qquad (\because \ a_1 \le a_2 \le a_3 \le \cdots \le a_{k+1})$	1
Thus, $A_{k+1} \ge A_k$ for all $k = 1, 2, 3, \ldots$.	
A_{k+1}	
$= \dfrac{kA_k + a_{k+1}}{k+1}$	1M
$= A_k + \dfrac{a_{k+1} - A_k}{k+1}$	
$\ge A_k + \dfrac{a_{k+1} - a_k}{k+1} \qquad (\because \ A_k \le a_k)$	
$\ge A_k + \dfrac{a_{k+1} - a_{k+1}}{k+1} \qquad (\because \ a_k \le a_{k+1})$	
$= A_k$ for all $k = 1, 2, 3, \ldots$.	1
(ii) By (b)(i), $A_{k+1} - A_k \ge 0$	
Note that $A_k > 0$ and $G_k > 0$ for all $k \in \mathbf{N}$.	
${A_{k+1}}^{k+1}$	
$= \left(A_k + (A_{k+1} - A_k)\right)^{k+1}$	
$\ge {A_k}^{k+1} + (k+1){A_k}^k (A_{k+1} - A_k) \qquad$ (by (a))	1A
$= {A_k}^k \left(A_k + (k+1)(A_{k+1} - A_k)\right)$	
$= {A_k}^k \left((k+1)A_{k+1} - kA_k\right)$	
$= {A_k}^k a_{k+1}$	1
Thus, ${A_{k+1}}^{k+1} \ge {A_k}^k a_{k+1}$ for all $k = 1, 2, 3, \ldots$.	

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Solution	Marks
Now, we are going to prove the latter result by mathematical induction.	
When $n=1$, $A_n = A_1 = a_1 = G_1 = G_n$.	
Therefore, the latter result is true for $n=1$.	
Assume that $A_k \ge G_k$ and	
$A_k = G_k$ if and only if $a_1 = a_2 = a_3 = \cdots = a_k$, where k is a positive integer.	
Then, we have	
$A_{k+1}{}^{k+1}$	
$\ge A_k{}^k a_{k+1}$	
$\ge G_k{}^k a_{k+1}$ (by induction assumption)	1M for using induction assumption
$= a_1 a_2 a_3 \cdots a_{k+1}$	
$= G_{k+1}{}^{k+1}$	1
Therefore, $A_{k+1} \ge G_{k+1}$.	
Moreover, if $a_1 = a_2 = a_3 = \cdots = a_{k+1}$, then	
$A_{k+1} = \dfrac{a_1 + a_2 + a_3 + \cdots + a_{k+1}}{k+1} = a_1$ and	
$G_{k+1} = (a_1 a_2 a_3 \cdots a_{k+1})^{\frac{1}{k+1}} = a_1$	
$\therefore \quad A_{k+1} = G_{k+1}$	1
Furthermore, if $A_{k+1} = G_{k+1}$, then	
$(A_k + (A_{k+1} - A_k))^{k+1} = A_k{}^{k+1} + (k+1)A_k{}^k (A_{k+1} - A_k)$ and $A_k{}^k = G_k{}^k$	
$\therefore \quad A_{k+1} - A_k = 0$ (by (a)) and $a_1 = a_2 = a_3 = \cdots = a_k$ (by induction assumption)	
$\therefore \quad a_1 = a_2 = a_3 = \cdots = a_k$ and $\dfrac{a_1 + a_2 + a_3 + \cdots + a_{k+1}}{k+1} = \dfrac{a_1 + a_2 + a_3 + \cdots + a_k}{k}$	
$\therefore \quad a_1 = a_2 = a_3 = \cdots = a_{k+1}$	1
$\therefore \quad A_{k+1} = G_{k+1}$ if and only if $a_1 = a_2 = a_3 = \cdots = a_{k+1}$.	
Thus, by mathematical induction, the latter result is true for all $n = 1, 2, 3, \ldots$.	
	---------(8)
(c) Put $a_1 = 1$, $a_2 = a_3 = a_4 = \cdots = a_n = a_{n+1} = \dfrac{n+1}{n}$.	1A
Note that $a_1, a_2, a_3, \cdots, a_{n+1}$ are not all equal because $a_1 < a_2 = a_3 = a_4 = \cdots = a_n$.	1M
By (b), we have	
$\dfrac{1 + n\left(\dfrac{n+1}{n}\right)}{n+1} > \left((1)\left(\dfrac{n+1}{n}\right)^n\right)^{\frac{1}{n+1}}$	
$\dfrac{n+2}{n+1} > \left(\dfrac{n+1}{n}\right)^{\frac{n}{n+1}}$	1
Therefore, $1 + \dfrac{1}{n+1} > \left(1 + \dfrac{1}{n}\right)^{\frac{n}{n+1}}$	
Thus, $\left(1 + \dfrac{1}{n+1}\right)^{n+1} > \left(1 + \dfrac{1}{n}\right)^n$	1 must show step(s)
	---------(4)

Solution	Marks
11. (a) (i) When $b=0$, (*) becomes $x^4-ax^2-c=0$. So, we have	
$x^2=\frac{a\pm\sqrt{a^2+4c}}{2}$	1M
$x=\pm\sqrt{\frac{a\pm\sqrt{a^2+4c}}{2}}$	1A
(ii) (1) $x^4=ax^2+bx+c$	
$\Leftrightarrow\quad x^4-2tx^2+t^2=ax^2-2tx^2+bx+c+t^2$	
$\Leftrightarrow\quad (x^2-t)^2=(a-2t)x^2+bx+(c+t^2)$	1
(2) Note that $(a-2t)x^2+bx+(c+t^2)=0$ has a repeated root	
$\Leftrightarrow\quad a-2t\neq 0$ and $b^2-4(a-2t)(c+t^2)=0$	1M+1M
Further note that $b^2-4(a-2t)(c+t^2)=0 \Rightarrow a-2t\neq 0$ ($\because\ b\neq 0$).	1A
So, $(a-2t)x^2+bx+(c+t^2)=0$ has a repeated root	
$\Leftrightarrow\quad b^2-4(a-2t)(c+t^2)=0$	
$\Leftrightarrow\quad 8t^3-4at^2+8ct+(b^2-4ac)=0$	1A
$\because$ the equation $8t^3-4at^2+8ct+(b^2-4ac)=0$ is of degree 3.	
$\therefore$ there exists a real number t_0 such that	
$8t_0^3-4at_0^2+8ct_0+(b^2-4ac)=0$.	
Hence, there exists a real number t_0 such that	
$(a-2t_0)x^2+bx+(c+t_0^2)=0$ has a repeated root λ.	1
Therefore, $(a-2t_0)x^2+bx+(c+t_0^2)=(a-2t_0)(x-\lambda)^2$	
Thus, by (1), (*) can be written as $(x^2-t_0)^2=(a-2t_0)(x-\lambda)^2$,	1 need to explain why λ is real
where λ is the repeated root which is real.	
	---------(9)

Solution	Marks
(b) $(6-2t)x^2+12x+(8+t^2)=0$ has a repeated root	
$\Leftrightarrow$ $(6-2t)\neq 0$ and $12^2-4(6-2t)(8+t^2)=0$	
$\Leftrightarrow$ $(6-2t)\neq 0$ and $2t^3-6t^2+16t-12=0$	
$\Leftrightarrow$ $2t^3-6t^2+16t-12=0$ $\quad(\because\ t\neq 3)$	1A
$\Leftrightarrow$ $t^3-3t^2+8t-6=0$	
$\Leftrightarrow$ $(t-1)(t^2-2t+6)=0$	
$\Leftrightarrow$ $t=1$ or $t=\dfrac{2\pm\sqrt{-20}}{2}$ (rejected)	
$\therefore$ a real value of t is 1 .	1A
Note that $4x^2+12x+9=0$ has a repeated root $\dfrac{-3}{2}$.	1A
By (a), (**) can be written as	
$(x^2-1)^2=4(x+\dfrac{3}{2})^2$	1A
$\Leftrightarrow$ $(x^2-1)^2=(2x+3)^2$	
$\Leftrightarrow$ $x^2-1=2x+3$ or $x^2-1=-2x-3$	1M
$\Leftrightarrow$ $x^2-2x-4=0$ or $x^2+2x+2=0$	
$\Leftrightarrow$ $x=\dfrac{2\pm\sqrt{20}}{2}$ or $x=\dfrac{-2\pm\sqrt{-4}}{2}$	
$\Leftrightarrow$ $x=1\pm\sqrt{5}$ or $x=-1\pm i$	1A for all roots being correct
Thus, all the roots of (**) are $1+\sqrt{5}$, $1-\sqrt{5}$, $-1+i$ and $-1-i$.	
	---------(6)

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Solution	Marks
12. (a) (i) $z^{2n}+1=0$	
$z^{2n}=-1$	
$z^{2n}=\cos\pi+i\sin\pi$	
$z=\cos\frac{(2k+1)\pi}{2n}+i\sin\frac{(2k+1)\pi}{2n}$, where $k=0,1,2,\cdots,2n-1$.	1A+1A
(ii) Let $z_k=\cos\frac{(2k+1)\pi}{2n}+i\sin\frac{(2k+1)\pi}{2n}$ for all $k=0,1,2,\cdots,2n-1$.	
Then, $z_{2n-k-1}=\cos\frac{\left(2(2n-k-1)+1\right)\pi}{2n}+i\sin\frac{\left(2(2n-k-1)+1\right)\pi}{2n}$	
$=\cos\left(2\pi-(\frac{2k+1}{2n})\pi\right)+i\sin\left(2\pi-(\frac{2k+1}{2n})\pi\right)$	
$=\cos\frac{(2k+1)\pi}{2n}-i\sin\frac{(2k+1)\pi}{2n}$	
$=\overline{z_k}$ for all $k=0,1,2,\cdots,n-1$	
$\therefore\quad z^{2n}+1=\prod_{k=0}^{2n-1}(z-z_k)$	1M
$=\prod_{k=0}^{n-1}\left((z-z_k)(z-\overline{z_k})\right)$	1A
$=\prod_{k=0}^{n-1}\left(z^2-(z_k+\overline{z_k})z+z_k\overline{z_k}\right)$	1A
$=\prod_{k=0}^{n-1}\left(z^2-2z\cos\frac{(2k+1)\pi}{2n}+1\right)$	1A
Thus, we have	
$\frac{1}{z^n}(z^{2n}+1)=\frac{1}{z^n}\prod_{k=0}^{n-1}\left(z^2-2z\cos\frac{(2k+1)\pi}{2n}+1\right)$ for all $z\neq 0$	
$z^n+\frac{1}{z^n}=\prod_{k=0}^{n-1}\left(z-2\cos\frac{(2k+1)\pi}{2n}+\frac{1}{z}\right)$ for all $z\neq 0$	
$z^n+\frac{1}{z^n}=\prod_{k=0}^{n-1}\left(z+\frac{1}{z}-2\cos\frac{(2k+1)\pi}{2n}\right)$ for all $z\neq 0$	1 must show steps
	---------(7)

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Solution	Marks
(b) (i) Put $z=\cos\theta+i\sin\theta$ into (a)(ii), we have	1M
$(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)=\prod_{k=0}^{n-1}\left((\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)-2\cos\frac{(2k+1)\pi}{2n}\right)$	
$2\cos n\theta=\prod_{k=0}^{n-1}\left(2\cos\theta-2\cos\frac{(2k+1)\pi}{2n}\right)$	1A
$2\cos n\theta=2^n\prod_{k=0}^{n-1}\left(\cos\theta-\cos\frac{(2k+1)\pi}{2n}\right)$	1M
$\prod_{k=0}^{n-1}\left(\cos\theta-\cos\frac{(2k+1)\pi}{2n}\right)=\frac{\cos n\theta}{2^{n-1}}$	1
	either one (the 1M above or the 1M's in (ii))
(ii) Put $z=-1$ into (a)(ii), we have	1M accept put $\theta=\pi$ into (b)(i)
$2(-1)^n=\prod_{k=0}^{n-1}\left(-2-2\cos\frac{(2k+1)\pi}{2n}\right)$	
$2(-1)^n=\prod_{k=0}^{n-1}\left((-2)\left(2\cos^2(\frac{(2k+1)\pi}{4n})\right)\right)$	1M
$2(-1)^n=(-2)^n(2^n)\prod_{k=0}^{n-1}\cos^2\left(\frac{(2k+1)\pi}{4n}\right)$	
$\prod_{k=0}^{n-1}\cos^2\left(\frac{(2k+1)\pi}{4n}\right)=\frac{2}{2^{2n}}$	1A
Note that $0<\frac{(2k+1)\pi}{4n}<\frac{\pi}{2}$ for all $k=0,1,2,\cdots,n-1$, we have	
$\cos\frac{(2k+1)\pi}{4n}>0$ for all $k=0,1,2,\cdots,n-1$.	
So, $\prod_{k=0}^{n-1}\cos\frac{(2k+1)\pi}{4n}>0$	
Thus, $\prod_{k=0}^{n-1}\cos\frac{(2k+1)\pi}{4n}=\frac{\sqrt{2}}{2^n}$	1 must show reasons
	---------(8)

只限教師參閱 FOR TEACHERS' USE ONLY

Typing mistakes in 2003 paper I Q. 8 P.5 line 2 last one word should be $\theta_1 = \frac{5\pi}{6}$

2003 Pure Maths Paper II suggested solution

1. (a) $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{\tan x}$

$= \lim_{x\to 0} \frac{1-\cos x}{\sin x}$

$= \lim_{x\to 0} \frac{\sin x}{\cos x}$ *(L' Hospital rule)*

$= 0$

(b) consider $0 \le \left|\frac{\cos x}{x}\right| \le \left|\frac{1}{x}\right|$ $\because |\cos x| \le 1$

$\lim_{x\to\infty} \left|\frac{1}{x}\right| = 0$ $\therefore \lim_{x\to\infty} \left|\frac{\cos x}{x}\right| = 0$

$\therefore \lim_{x\to\infty} \frac{\cos x}{x} = 0$

$$\lim_{x\to\infty} \frac{x-\cos x}{x+\cos x} = \lim_{x\to\infty} \frac{1-\frac{\cos x}{x}}{1+\frac{\cos x}{x}} = 1$$

2. (a) $f(x) = x^{\frac{1}{x}}$

$\ln f(x) = \frac{1}{x}\ln x$

$\Rightarrow \frac{1}{f(x)} f'(x) = \frac{1}{x^2} - \frac{1}{x^2}\ln x$

$f'(x) = x^{\frac{1}{x}-2}(1-\ln x) = 0$

$\Rightarrow x = e$

as $1 \le x < e$ $f'(x) > 0$

as $x > e$ $f'(x) < 0$

$\therefore$ f(x) attains absolute maximum at $x = e$ and f(e) = $e^{\frac{1}{e}}$

(b) by (a) and $2 < e < 3$ $\because 3^{\frac{1}{3}} > 2^{\frac{1}{2}}$ $\therefore m = 3$

3. (a) as $x > 0$, $f(-x) = -1 = -f(x)$

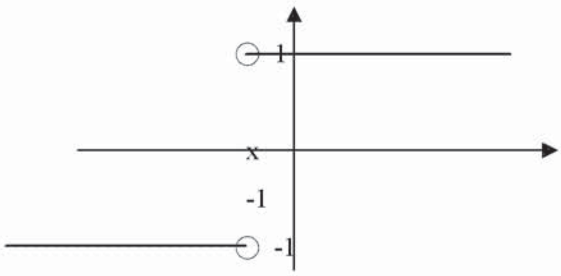
as $x < 0$, $f(-x) = 1 = -(-1) = -f(x)$

$f(0) = -f(0) = 0$

$\therefore$ $f(-x) = -f(x)$ odd function

(b) $f(1) = f(2)$ but $1 \neq 2$ f is not an injective function.

(c)



(d) $g(x) = \begin{cases} -2 & x < -1 \\ -1 & x = -1 \\ 0 & -1 < x < 1 \\ 1 & x = 1 \\ 2 & x > 1 \end{cases}$ it is easy to sketch $y = g(x)$

g(x) discontinous at $x = -1$ and 1

4. $\int \frac{dx}{2+\cos x}$ Let $t = \tan\frac{x}{2}$ $\cos x = \frac{1-t^2}{1+t^2}$ $dx = \frac{2dt}{1+t^2}$

$= \int \frac{1}{2+\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2}$

$= \int \frac{2dt}{3+t^2}$

$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}$

$= \frac{2\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}}{3}(\tan\frac{x}{2}) + c$

$\lim_{n\to\infty}\sum_{k-1}^{n}\frac{1}{n\left(2+\cos(\frac{k\pi}{2n})\right)} = \int_0^1 \frac{1}{2+\cos\frac{\pi}{2}x}dx$

$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{2+\cos t}dt$

$= \frac{2}{\pi}\cdot\frac{2\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}}{3}(\tan\frac{x}{2})\Big|_0^{\frac{\pi}{2}}$

$= \frac{2\sqrt{3}}{9}$

5. (a) $\dfrac{a^2\sec^2\theta}{a^2}-\dfrac{b^2\tan^2\theta}{b^2}=\sec^2\theta-\tan^2\theta=1$

$\therefore$ P lies on H.

(b) (i) tangent to H at P is $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$

solve $\begin{cases}\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1\\ y=\dfrac{b}{a}x\end{cases}$

$Q_1(\dfrac{a}{\sec\theta-\tan\theta},\dfrac{b}{\sec\theta-\tan\theta})$

similarly $Q_2(\dfrac{a}{\sec\theta+\tan\theta},\dfrac{-b}{\sec\theta+\tan\theta})$

(ii) area of $\Delta OPQ_1=\dfrac{1}{2}\left(\dfrac{ab\sec\theta}{\sec\theta-\tan\theta}-\dfrac{ab\tan\theta}{\sec\theta-\tan\theta}\right)=\dfrac{1}{2}ab$

area of $\Delta OPQ_2=\dfrac{1}{2}\left(\dfrac{ab\sec\theta}{\sec\theta+\tan\theta}+\dfrac{ab\tan\theta}{\sec\theta+\tan\theta}\right)=\dfrac{1}{2}ab$

$Q_1P:PQ_2=1:1$

6. (a) $\begin{vmatrix}x-2 & -1 & 1\\ y-1 & -1 & -1\\ z & 1 & 1\end{vmatrix}=0$

$\Rightarrow y+z-1=0$

(b) $L:\begin{cases}x=0\\ y=t\\ z=t+2\end{cases}$

$t+t+2-1=0$

$\Rightarrow 2t=-1$

$\Rightarrow t=-\dfrac{1}{2}$

point of intersection is $(0,-\dfrac{1}{2},\dfrac{3}{2})$

(c) distance $\sqrt{(0-0)^2+(-\dfrac{1}{2}-0)^2+(\dfrac{3}{2}-2)^2}=\dfrac{\sqrt{2}}{2}$

7. (a) $f(x)=\begin{cases}\dfrac{x(x+1)}{x+2} & x\ge -1\\ -\dfrac{x(x+1)}{x+2} & x<-1\end{cases}$

(i) $f'(x)=\begin{cases}\dfrac{x^2+4x+2}{(x+2)^2} & x>-1\\ -\dfrac{x^2+4x+2}{(x+2)^2} & x<-1\end{cases}$

(ii) $\lim_{h\to 0^+}\dfrac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0^+}\dfrac{(-1+h)h}{h}=-1$

$\lim_{h\to 0^-}\dfrac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0^-}\dfrac{-(-1+h)h}{h}=1$

$\lim_{h\to 0}\dfrac{f(-1+h)-f(-1)}{h}$ not exist

$\therefore$ f not differentiable at $x = -1$

(iii) $f''(x)=\begin{cases}\dfrac{4}{(x+2)^3} & x>-1\\ -\dfrac{4}{(x+2)^3} & x<-1\end{cases}$

(b) (i) as $x > -1$ $\quad \dfrac{x^2+4x+2}{(x+2)^2}>0$

$x>-2+\sqrt{2}$

as $x < -1$ $\quad -\dfrac{x^2+4x+2}{(x+2)^2}>0$

$-2-\sqrt{2}<x<-2$ or $-2<x<-1$

$f'(x)>0 \Rightarrow x>-2+\sqrt{2}$, $-2-\sqrt{2}<x<-2$ or $-2<x<-1$

(ii) $f'(x)<0 \Rightarrow x<-2-\sqrt{2}$ or $-1<x<-2+\sqrt{2}$

(iii) as $x > -1$ $\quad \dfrac{4}{(x+2)^3}>0$

$x > -1$

as $x < -1$ $\quad -\dfrac{4}{(x+2)^3}>0$

$x<-2$

$f''(x)>0 \Rightarrow x > -1$ or $x<-2$

(iv) $f''(x)>0 \Rightarrow -2<x<-1$

(c) $f'(x)=0 \Rightarrow x=-2-\sqrt{2}$ or $-2+\sqrt{2}$

$f''(-2-\sqrt{2})>0$ $\quad \therefore (-2-\sqrt{2}, f(-2-\sqrt{2}))$~(-3.4,5.8)relative minimum point

$f''(-2+\sqrt{2})>0$ $\quad \therefore (-2+\sqrt{2}, f(-2+\sqrt{2}))$~(-0.59,-0.17)relative minimum point

(-1,0) relative maximum point and also point of inflection.

(d) vertical asy. $x = -2$

$a = \lim_{x\to\infty} \frac{x(x+1)}{x(x+2)} = 1$

$a = \lim_{x\to-\infty} -\frac{x(x+1)}{x(x+2)} = -1$

$b = \lim_{x\to\infty} \frac{x(x+1)}{x+2} - x = \lim_{x\to\infty} \frac{-x}{x+2} = -1$

$b = \lim_{x\to-\infty} \frac{-x(x+1)}{x+2} + x = \lim_{x\to-\infty} \frac{x}{x+2} = 1$

asymptotes $y = x - 1$ or $y = -x + 1$

(e) students sketch it yourself --easy sketch

8. (a) (i) put $y = 0$ f(x) = f(x) + f(0) + f(x)f(0)

$\therefore$ f(0)(1+f(x)) = 0

(ii) by (i) f(0) = 0 or f(x) = -1 for all x

Suppose $f(0) \neq 0$, then f(x) = -1 for all x

but f(x) is non-constant

$\therefore$ f(0) = 0

Suppose f(r) = -1 for some real no r

f($x + r$) = f(x) + f(r) +f(x)f(r) for all x

f($x + r$) = -1

Put $x = -r$ f(0) = -1 contradiction

$\therefore f(x) \neq -1$

(b) $f(x) = f(\frac{x}{2} + \frac{x}{2})$

$= f(\frac{x}{2}) + f(\frac{x}{2}) + f(\frac{x}{2})\, f(\frac{x}{2})$

$= \left(f(\frac{x}{2}) + 1\right)^2 - 1 > -1 \qquad \because f(x) \neq -1$

(c) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{f(h) + f(x)f(h)}{h}$

$= \lim_{h\to 0} \frac{f(h)(1 + f(x))}{h}$

$= a\,(\,1 + f(x)\,)$

$\therefore$ f is differentiable everywhere

Suppose $a = 0$ i.e. f′(x) = 0 for all x.

Then f is a constant function.

$\because$ f is a non-constant function $\therefore a \neq 0$

(d) $[\ln(1+f(x))]' = \dfrac{f'(x)}{1+f(x)} = a$ by (c)

$\therefore [\ln(1+f(x))] = ax + c$

by putting $x = 0$, get $c = 0$

$\therefore [\ln(1+f(x))] = ax$

$\therefore f(x) = e^{ax} - 1$

9. (a) $w'(x) = 2v(x)v'(x) + 2v'(x)v''(x)$

$= 2(u(x) - \sin x)(u'(x) - \cos x) + 2(u'(x) - \cos x)(u''(x) + \sin x)$

$= 2(u(x) - \sin x)(u'(x) - \cos x) + 2(u'(x) - \cos x)(-u(x) + \sin x)$

$= 0$

$\therefore w(x)$ is a constant function

put $x = 0$ $v(0) = u(0) - 0 = 0$

$v'(0) = u'(0) - 1 = 0$

$\therefore w(x) = 0$

$(v(x))^2 + (v'(x))^2 = 0$

$\Rightarrow v(x) = 0$

$\Rightarrow u(x) = \sin x$

(b) (i) $f'(x) = e^{-x}(e^x g(x)) - e^{-x}\int_0^x e^t g(t)dt$

$= g(x) - f(x)$

$f(x) + f'(x) = g(x)$

(ii) (1) $f''(x) = g'(x) - f'(x)$

$g'(x) = -e^{-x} + e^{-x}\int_0^x e^t f(t)dt - e^{-x}(e^x f(x))$

$= -g(x) - f(x)$

$f''(x) = -g(x) - f(x) - f'(x)$

$= -(f(x) + f'(x)) - f(x) - f'(x)$

$\therefore f''(x) + 2f'(x) + 2f(x) = 0$

(2) Let $h(x) = e^x f(x)$

$h'(x) = e^x f'(x) + e^x f(x)$

$h''(x) = e^x f''(x) + e^x f'(x) + e^x f'(x) + e^x f(x)$

$= e^x[f''(x) + 2f'(x) + f(x)]$

$= e^x[-f(x)]$

$= -h(x)$

$h(0) = 0$ $\because f(0) = 0$

$h'(0) = f'(0) + f(0) = g(0) = 1$

by (a) $h(x) = \sin x$

$\therefore \mathrm{f}(x) = e^{-x} \sin x$

(iii) $\mathrm{g}(x) = e^{-x} \sin x + \mathrm{f}'(x)$

$= e^{-x} \sin x + e^{-x} \cos x - e^{-x} \sin x$

$= e^{-x} \cos x$

10. (a) (i) $\mathrm{f}(j+1) \le \mathrm{f}(x) \le \mathrm{f}(j)$ for $j \le x \le j+1$ ($\because$ f is decreasing)

$$\int_j^{j+1} \mathrm{f}(j+1)dx \le \int_j^{j+1} \mathrm{f}(x)dx \le \int_j^{j+1} \mathrm{f}(j)dx$$

$$\Rightarrow \mathrm{f}(j+1) \le \int_j^{j+1} \mathrm{f}(x)dx \le \mathrm{f}(j)$$

$$\sum_{j=1}^{n-1} \mathrm{f}(j+1) \le \sum_{j=1}^{n-1} \int_j^{j+1} \mathrm{f}(x)dx \le \sum_{j=1}^{n-1} \mathrm{f}(j)$$

$$\Rightarrow S_n - \mathrm{f}(1) \le \int_1^n \mathrm{f}(x)dx \le S_{n-1}$$

$$\Rightarrow S_n - \mathrm{f}(1) \le F_n \le S_{n-1} \text{ ....(*)}$$

(ii) $\{\Leftarrow\}$

obviously $\{S_n\}$ is increasing $\mathrm{f}: [1,\infty) \to [0,\infty)$

by (*) $\mathrm{f}(1) + F_n \ge S_n \ge F_n + \mathrm{f}(n)$

$\Rightarrow \mathrm{f}(1) + K \ge S_n$

$\therefore \{S_n\}$ is bounded

$\therefore \{S_n\}$ is convergent

$\therefore \sum_{j=1}^{\infty} \mathrm{f}(j)$ is convergent

$\{\Rightarrow\}$

$\mathrm{f}(1) + F_n \ge S_n \ge F_n + \mathrm{f}(n)$

$\because \lim_{n\to\infty} S_n$ exist $\therefore \{S_n\}$ is bounded

$S_n \ge F_n + \mathrm{f}(n)$ $\therefore \{F_n\}$ is bounded

$\therefore$ there is a constant K independent of n such that $F_n \le K$ for all $n = 1,2,3\ldots.$

(b) (i) Let $\mathrm{f}(x) = \dfrac{1}{\sqrt[3]{x}}$ obviously f is continuous and decreasing when $x \ge 1$

$$F_n = \int_1^n \frac{1}{\sqrt[3]{x}}dx = \left. -2x^{-\frac{1}{2}} \right|_1^n = 2(1 - \frac{1}{\sqrt{n}}) \le 2$$

By (a) (ii) $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt[3]{n}}$ is convergent

(ii) Let $\mathrm{f}(x) = \dfrac{1}{x}$ obviously f is continuous and decreasing when $x \ge 1$

$$F_n = \int_1^n \frac{1}{x}dx = \ln x\Big|_1^n = \ln n$$

as $n \to \infty$ $\quad F_n \to \infty$

there does not exist a constant K independent of n such that $F_n \le K$

$\therefore$ by (a)(ii) $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent

(c) $$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2} = \sum_{n=1}^{\infty}\frac{1}{(n+1)(\ln(n+1))^2}$$

Put $f(x) = \frac{1}{(x+1)(\ln(x+1))^2}$ obviously f is continuous and decreasing when $x \ge 1$

$$F_n = \int_1^n \frac{1}{(x+1)(\ln(x+1))^2}dx = \frac{-1}{\ln(x+1)}\Big|_1^n = \frac{1}{\ln 2} - \frac{1}{\ln(n+1)} \le \frac{1}{\ln 2}$$

by (a)(ii) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ is convergent

12. (a) (i) $$I_{m+1} = \frac{1}{m!}\int_0^x (x-t)^m f^{(m+1)}(t)dt$$

$$= \frac{1}{m!}[(x-t)^m f^{(m)}(t)\Big|_0^x + m\int_0^x (x-t)^{m-1} f^{(m)}(t)dt\,]$$

$$= -\frac{1}{m!}x^m f^{(m)}(0) + \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1} f^{(m)}(t)dt$$

$$\therefore I_{m+1} = I_m - \frac{f^{(m)}(0)}{m!}x^m$$

(ii) by (i) $$I_m = I_{m-1} - \frac{f^{(m-1)}(0)}{(m-1)!}x^{m-1}$$

$$= I_{m-2} - \frac{f^{(m-2)}(0)}{(m-2)!}x^{m-2} - \frac{f^{(m-1)}(0)}{(m-1)!}x^{m-1}$$

$$= I_{m-3} - \frac{f^{(m-3)}(0)}{(m-3)!}x^{m-3} - \frac{f^{(m-2)}(0)}{(m-2)!}x^{m-2} - \frac{f^{(m-1)}(0)}{(m-1)!}x^{m-1}$$

:

:

$$= I_1 - \sum_{k=1}^{m-1}\frac{f^{(k)}(0)}{k!}x^k$$

where $I_1 = \int_0^x f'(t)dt = f(x) - f(0)$

$$\therefore I_m = f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(0)}{k!}x^k$$

(b) (i) $g'(x) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x) = x(1-x^2)^{-\frac{3}{2}}$

$(1-x^2)g'(x) - xg(x)$

$= (1-x^2)x(1-x^2)^{-\frac{3}{2}} - x(1-x^2)^{-\frac{1}{2}}$

$= 0$

diff. both side w.r.t. x n times

$(1-x^2)g^{(n+1)}(x) - C_1^n(-2x)g^{(n)}(x) + C_2^n(-2)g^{(n-1)}(x) - xg^{(n)}(x) - C_1^n g^{(n-1)}(x) = 0$

$\Rightarrow (1-x^2)g^{(n+1)}(x) - (2n+1)xg^{(n)}(x) - n^2 g^{(n-1)}(x) = 0$

(ii) Put $x = 0$ in (i)

$g^{(n+1)}(0) - n^2 g^{(n-1)}(0) = 0$

$\Rightarrow g^{(n+1)}(0) = n^2 g^{(n-1)}(0)$

$$\begin{aligned} g^{(2n-1)}(0) &= (2n-2)^2 g^{(2n-3)}(0) \\ &= (2n-2)^2(2n-4)^2 g^{(2n-5)}(0) \\ &\ \vdots \\ &= (2n-2)^2(2n-4)^2 \ldots\ldots\ldots 2^2 g^{(1)}(0) \\ &= 0 \qquad (\because g'(0) = 0) \end{aligned}$$

similarly

$$\begin{aligned} g^{(2n)}(0) &= (2n-1)^2(2n-3)^2 \ldots\ldots\ldots 1^2 g(0) \\ &= (2n-1)^2(2n-3)^2 \ldots\ldots\ldots 1^2 \qquad (\because g(0) = 1) \\ &= \frac{[(2n)!]^2}{2^2 4^2 \ldots\ldots (2n)^2} \\ &= \left(\frac{(2n)!}{2^n n!}\right)^2 \end{aligned}$$

(iii) by (a) put f(x) = g(x) and $m = 2n$

$$g(x) = \sum_{k=0}^{2n-1} \frac{g^{(k)}(0)}{k!} x^k + \frac{1}{(2n-1)!}\int_0^x (x-t)^{2n-1} g^{(2n)}(t)\,dt$$

[$\because g^{(k)}(0) = 0$ when k is odd and $g^{(2n)}(0) = \frac{(2n)!}{2^{2n}} C_n^{2n}$]

$$\therefore g(x) = \sum_{k=0}^{n} \frac{C_k^{2k}}{2^{2k}} x^{2k} + \frac{1}{(2n-1)!}\int_0^x (x-t)^{2n-1} g^{(2n)}(t)\,dt$$