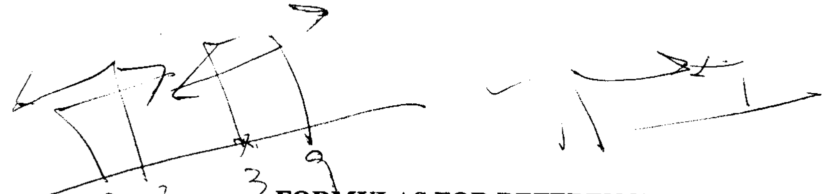

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Solve the inequality $\big||x|-6\big| \le 3$, where x is a real number.

   (b) Using the result of (a), or otherwise, solve the inequality $\big||1-2y|-6\big| \le 3$, where y is a real number.

(6 marks)

2. For any positive integer n , let C_k^n be the coefficient of x^k in the expansion of $(1+x)^n$. Evaluate

   (a) $C_1^n + C_2^n + C_3^n + \cdots + C_n^n$,

   (b) $C_1^n + 2C_2^n + 3C_3^n + \cdots + nC_n^n$,

   (c) $C_1^n + 2^2 C_2^n + 3^2 C_3^n + \cdots + n^2 C_n^n$.

   $n^2(n-1)2^{n-2}$

(6 marks)

Go on to the next page

3. (a) Resolve $\dfrac{5x-3}{x(x+1)(x+3)}$ into partial fractions.

(b) (i) Prove that $\displaystyle\sum_{k=1}^{n}\frac{5k-3}{k(k+1)(k+3)}<\frac{3}{2}$ for any positive integer n.

(ii) Evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{5k-3}{k(k+1)(k+3)}$.

(7 marks)

4. Let $\{x_n\}$ be a sequence of positive real numbers, where $x_1=2$ and $x_{n+1}=x_n^{\,2}-x_n+1$ for all $n=1,2,3,\ldots$.

Define $S_n=\displaystyle\sum_{i=1}^{n}\frac{1}{x_i}$ for all $n=1,2,3,\ldots$.

(a) Using mathematical induction, prove that for any positive integer n,

(i) $x_n>n$,

(ii) $S_n=1-\dfrac{1}{x_{n+1}-1}$.

converge?

(b) Using (a), or otherwise, prove that $\lim\limits_{n\to\infty}S_n$ exists.

(7 marks)

$\dfrac{-1}{x}+\dfrac{4}{x+1}-\dfrac{3}{x+3}$

$\dfrac{4}{x}-\dfrac{4}{x+3}$

$-(x^2+4x+3)+4(x)(x+3)-3(x)(x+1)$

$-\cancel{x^2}-4x-3+\cancel{4x^2}+12x-\cancel{3x^2}-3x$

$5x-3$

5. Let $\mathbf{m}$ and $\mathbf{n}$ be vectors in $\mathbf{R}^3$ and $\lambda\in\mathbf{R}$. It is given that

$$\begin{cases}\mathbf{u}=\lambda\mathbf{n}+(1-\lambda)\mathbf{m}\\ \mathbf{v}=2(1-\lambda)\mathbf{n}-\lambda\mathbf{m}\end{cases}.$$

(a) Prove that $\mathbf{u}\times\mathbf{v}=(3\lambda^2-4\lambda+2)\,\mathbf{m}\times\mathbf{n}$.

(b) Suppose $|\mathbf{m}|=4$, $|\mathbf{n}|=3$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $\dfrac{\pi}{6}$.

(i) Evaluate $|\mathbf{m}\times\mathbf{n}|$.

(ii) Find the smallest area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ as λ varies.

4

$\lambda=\frac{2}{3}$

(6 marks)

6. (a) Suppose the cubic equation $x^3+px^2+qx+r=0$, where p, q and r are real numbers, has three real roots.

Using relations between coefficients and roots, or otherwise, prove that the three roots form an arithmetic sequence if and only if $\dfrac{-p}{3}$ is a root of the equation.

(b) Find the two values of p such that the equation $x^3+px^2+21x+p=0$ has three real roots that form an arithmetic sequence.

(8 marks)

SECTION B (60 marks)

Answer any FOUR questions in this section. Each question carries 15 marks. Use a separate AL(D) answer book for each question.

7. (a) Consider the system of linear equations in x, y, z

$$(E): \begin{cases} x + ay - z = 0 \\ 2x - y + az = -2a \\ -x + 2a^2y + (a-3)z = 2a \end{cases},$$

where $a \in \mathbf{R}$.

(i) Find the range of values of a such that (E) has a unique solution. Solve (E) when (E) has a unique solution.

(ii) Solve (E) for

(1) $a = 1$,

(2) $a = -4$.

(10 marks)

(b) Suppose (x, y, z) satisfies

$$\begin{cases} x + y - z = 0 \\ 2x - y + z = -2 \\ -x + 2y - 2z = 2 \end{cases}.$$

Find the least value of $24x^2 + 3y^2 + 2z$ and the corresponding values of x, y, z.

(5 marks)

8. (a) If $\det\begin{pmatrix} -2-\alpha & \sqrt{3} \\ \sqrt{3} & -\alpha \end{pmatrix} = 0$, find the two values of α.

(2 marks)

(b) Let α_1 and α_2 be the values obtained in (a), where $\alpha_1 < \alpha_2$.

Find θ_1 and θ_2 such that

$$\begin{pmatrix} -2-\alpha_1 & \sqrt{3} \\ \sqrt{3} & -\alpha_1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0 \le \theta_1 < \pi,$$

$$\begin{pmatrix} -2-\alpha_2 & \sqrt{3} \\ \sqrt{3} & -\alpha_2 \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0 \le \theta_2 < \pi.$$

$\frac{2\pi}{3}$ $\frac{5\pi}{6}$ $\frac{\pi}{3}$ $\frac{\pi}{3}$

Let $P = \begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix}$. Evaluate P^n, where n is a positive integer.

Prove that $P^{-1}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}P$ is a matrix of the form $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

(8 marks)

(c) Evaluate $\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}^n$, where n is a positive integer.

(5 marks)

$\begin{pmatrix} 1 & -\frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{8} & \frac{3}{8} \end{pmatrix}$

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9. (a) Consider the vectors $\mathbf{a}=(p,q,0)$, $\mathbf{b}=(q,-p,0)$ and $\mathbf{c}=(0,0,r)$, where p, q and r are non-zero real numbers.

(i) Prove that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.

(ii) Let $\mathbf{d}$ be a vector in $\mathbf{R}^3$. Prove that

$$\mathbf{d}=\left(\frac{\mathbf{d}\cdot\mathbf{a}}{|\mathbf{a}|^2}\right)\mathbf{a}+\left(\frac{\mathbf{d}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}+\left(\frac{\mathbf{d}\cdot\mathbf{c}}{|\mathbf{c}|^2}\right)\mathbf{c}\ .$$

(6 marks)

(b) Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be linearly independent vectors in $\mathbf{R}^3$.

Define $\mathbf{u}=\mathbf{x}$ and $\mathbf{v}=\mathbf{y}-\left(\frac{\mathbf{y}\cdot\mathbf{u}}{|\mathbf{u}|^2}\right)\mathbf{u}$.

(i) Prove that $\mathbf{v}$ is a non-zero vector.

(ii) Define $\mathbf{w}=\mathbf{z}-\left(\frac{\mathbf{z}\cdot\mathbf{u}}{|\mathbf{u}|^2}\right)\mathbf{u}-\left(\frac{\mathbf{z}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$.

(1) Prove that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

(2) Describe the geometric relationship between $\mathbf{w}$ and the plane containing the vectors $\mathbf{x}$ and $\mathbf{y}$.

(9 marks)

10. (a) Let a and b be non-negative real numbers. Prove that

$$(a+b)^n \geq a^n + na^{n-1}b \qquad \text{for all } n=2,3,4,\dots\ .$$

Write down a necessary and sufficient condition for the equality to hold.

(3 marks)

(b) Let $\{a_1, a_2, a_3, \dots\}$ be a sequence of positive real numbers satisfying $a_1 \leq a_2 \leq a_3 \leq \cdots$.

For any positive integer n, define

$$A_n=\frac{a_1+a_2+a_3+\cdots+a_n}{n} \quad \text{and} \quad G_n=(a_1a_2a_3\cdots a_n)^{\frac{1}{n}}\ .$$

(i) Prove that $A_{k+1}\geq A_k$ for all $k=1,2,3,\dots$.

(ii) Using (a), prove that $A_{k+1}{}^{k+1}\geq A_k{}^k a_{k+1}$ for all $k=1,2,3,\dots$.

Hence prove that $A_n \geq G_n$ and

$A_n=G_n$ if and only if $a_1=a_2=a_3=\cdots=a_n$

for all $n=1,2,3,\dots$.

(8 marks)

(c) Let n be a positive integer. Using (b), prove that

$$\frac{n+2}{n+1} > \left(\frac{n+1}{n}\right)^{\frac{n}{n+1}}\ .$$

Hence deduce that $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$.

(4 marks)

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11. (a) Consider the equation

$$x^4 = ax^2 + bx + c \qquad \ldots\ldots\ldots\ldots\ldots (*),$$

where a, b and c are real numbers.

(i) Suppose $b = 0$. Solve $(*)$.

(ii) Suppose $b \neq 0$.

(1) Prove that $(*)$ can be written as

$$(x^2 - t)^2 = (a - 2t)x^2 + bx + (c + t^2) ,$$

where t is any real number.

(2) Prove that there exists a real number t_0 such that the equation

$$(a - 2t_0)x^2 + bx + (c + t_0{}^2) = 0$$

has a repeated root.

Hence, deduce that $(*)$ can be written as

$$(x^2 - t_0)^2 = (a - 2t_0)(x - \lambda)^2$$

for some real number λ.

(9 marks)

(b) Consider the equation

$$x^4 = 6x^2 + 12x + 8 \qquad \ldots\ldots\ldots\ldots\ldots (**).$$

Find a real value of t such that the equation

$$(6 - 2t)x^2 + 12x + (8 + t^2) = 0$$

has a repeated root.

Hence solve $(**)$.

(6 marks)

12. Let n be a positive integer.

(a) (i) Find all the roots of $z^{2n} + 1 = 0$.

(ii) By factorizing $z^{2n} + 1$ into a product of quadratic factors with real coefficients, or otherwise, prove that

$$z^n + \frac{1}{z^n} = \prod_{k=0}^{n-1}\left(z + \frac{1}{z} - 2\cos\frac{(2k+1)\pi}{2n}\right) \quad \text{for all } z \neq 0 .$$

(7 marks)

(b) Using (a), or otherwise, prove that

(i) $$\prod_{k=0}^{n-1}\left(\cos\theta - \cos\frac{(2k+1)\pi}{2n}\right) = \frac{\cos n\theta}{2^{n-1}} \quad \text{for any } \theta \in \mathbf{R} ,$$

(ii) $$\prod_{k=0}^{n-1}\cos\frac{(2k+1)\pi}{4n} = \frac{\sqrt{2}}{2^n} .$$

(8 marks)

END OF PAPER

FORMULAS FOR REFERENCE

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$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
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$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Evaluate

(a) $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{\tan x} \right)$,

(b) $\lim_{x \to \infty} \frac{x - \cos x}{x + \cos x}$.

(6 marks)

2. (a) Let $f(x) = x^{\frac{1}{x}}$ for all $x \geq 1$. Find the greatest value of $f(x)$.

(b) Using (a) or otherwise, find a positive integer m , such that $m^{\frac{1}{m}} \geq n^{\frac{1}{n}}$ for all positive integers n .

(7 marks)

Go on to the next page

3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \begin{cases} -1 & \text{when } x < 0, \\ 0 & \text{when } x = 0, \\ 1 & \text{when } x > 0. \end{cases}$

(a) Prove that f is an odd function.

(b) Is f an injective function? Explain your answer briefly.

(c) Sketch the graph of $y = f(x+1)$.

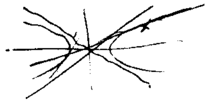
(d) Let $g(x) = f(x+1) + f(x-1)$ for all $x \in \mathbf{R}$.
Sketch the graph of $y = g(x)$. Write down the value(s) of x at which $g(x)$ is discontinuous.

(6 marks)

4. Using the substitution $t = \tan\frac{x}{2}$, evaluate $\int \frac{dx}{2+\cos x}$.

Hence, or otherwise, evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n\left(2+\cos(\frac{k\pi}{2n})\right)}$.

(7 marks)



5. Consider the hyperbola $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a and b are positive constants, with its asymptotes $L_1 : y = \frac{b}{a}x$ and $L_2 : y = \frac{-b}{a}x$. Let P be the point $(a\sec\theta, b\tan\theta)$, where $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

(a) Prove that P lies on H .

(b) The tangent to H at P cuts L_1 and L_2 at Q_1 and Q_2 respectively.

(i) Find the coordinates of Q_1 and Q_2 .

(ii) Find the areas of ΔOPQ_1 and ΔOQ_2P , where O is the origin.
Hence find $Q_1P : PQ_2$.

(7 marks)

6. Let Π be the plane containing (2, 1, 0), (1, 0, 1) and (3, 0, 1). Suppose L is the straight line passing through $A(0, 0, 2)$ and perpendicular to Π. Find

(a) the equation of Π,

(b) the coordinates of the point of intersection of L and Π,

(c) the distance from A to Π.

(7 marks)

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

7. Let $f(x)=\dfrac{x|x+1|}{x+2}$ $\quad (x \neq -2)$.

(a) (i) Find $f'(x)$ for $x \neq -1$.

(ii) Is f differentiable at -1 ? Explain your answer.

(iii) Find $f''(x)$ for $x \neq -1$.

(4 marks)

(b) Determine the range of values of x for each of the following cases:

(i) $f'(x)>0$,

(ii) $f'(x)<0$,

(iii) $f''(x)>0$,

(iv) $f''(x)<0$.

(3 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of $f(x)$.

(3 marks)

(d) Find the asymptote(s) of the graph of $f(x)$.

(3 marks)

(e) Sketch the graph of $f(x)$.

(2 marks)

8. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a non-constant function satisfying the following conditions:

(1) $f(x+y)=f(x)+f(y)+f(x)f(y)$ for all $x, y \in \mathbf{R}$;

(2) $\lim\limits_{h \to 0} \dfrac{f(h)}{h}=a$, where $a \in \mathbf{R}$.

(a) (i) Prove that $f(0)\big(1+f(x)\big)=0$ for all $x \in \mathbf{R}$.

(ii) Prove that $f(0)=0$ and that $f(x) \neq -1$ for all $x \in \mathbf{R}$.

(5 marks)

(b) By considering $f(x)=f\left(\dfrac{x}{2}+\dfrac{x}{2}\right)$, or otherwise, prove that $f(x)>-1$ for all $x \in \mathbf{R}$.

(2 marks)

(c) Prove that f is differentiable everywhere and that $f'(x)=a\big(1+f(x)\big)$ for all $x \in \mathbf{R}$. Deduce that $a \neq 0$.

(4 marks)

(d) By considering the derivative of the function $\ln\big(1+f(x)\big)$, prove that $f(x)=e^{ax}-1$ for all $x \in \mathbf{R}$.

(4 marks)

Go on to the next page

9. (a) Let $u: \mathbf{R} \to \mathbf{R}$ be a twice differentiable function satisfying the following conditions:

(1) $u''(x) = -u(x)$ for all $x \in \mathbf{R}$,

(2) $u(0) = 0$,

(3) $u'(0) = 1$.

Define $v(x) = u(x) - \sin x$ for all $x \in \mathbf{R}$.

By differentiating $w(x) = (v(x))^2 + (v'(x))^2$, prove that $u(x) = \sin x$ for all $x \in \mathbf{R}$.

(5 marks)

(b) Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be continuous functions such that

$f(x) = e^{-x} \int_0^x e^t g(t)\, dt$ and $g(x) = e^{-x} - e^{-x} \int_0^x e^t f(t)\, dt$ for all $x \in \mathbf{R}$.

(i) Prove that $f(x) + f'(x) = g(x)$ for all $x \in \mathbf{R}$.

(ii) (1) Prove that $f''(x) + 2f'(x) + 2f(x) = 0$ for all $x \in \mathbf{R}$.

(2) Let $h(x) = e^x f(x)$ for all $x \in \mathbf{R}$.
Prove that $h''(x) = -h(x)$. Using (a), find $f(x)$.

(iii) Find $g(x)$.

(10 marks)

10. (a) Let $f: [1, \infty) \to [0, \infty)$ be a continuous and decreasing function. For every $n = 1, 2, 3, \dots$, define $F_n = \int_1^n f(x)\, dx$ and $S_n = \sum_{j=1}^{n} f(j)$.

(i) Prove that $f(j+1) \le \int_j^{j+1} f(x)\, dx \le f(j)$ for all $j = 1, 2, 3, \dots$.
Hence deduce that $S_n - f(1) \le F_n \le S_{n-1}$ for all $n = 2, 3, 4, \dots$.

(ii) Using (a)(i), prove that the series $\sum_{j=1}^{\infty} f(j)$ is convergent if and only if there is a constant K (independent of n) such that $F_n \le K$ for all $n = 1, 2, 3, \dots$.

(7 marks)

(b) Using (a), prove that

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ is convergent,

(ii) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(4 marks)

(c) Use (a) to determine whether $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent. Explain your answer.

(4 marks)

Go on to the next page

11. Consider the parabola P: $y^2 = 4x$.
Let $A(a^2, 2a)$ be a point on P, L_1 be the tangent to P at A, and F be the point $(1, 0)$.

(a) (i) Find the equation of L_1.

(ii) Let F' be a point such that L_1 is the perpendicular bisector of FF'. Prove that the x-coordinate of F' is -1. Also find the y-coordinate of F'.

(7 marks)

(b) Suppose $a = 2$. The straight line $x = -1$ intersects L_1 at B. Let L_2 be the straight line passing through B and perpendicular to L_1.

(i) Prove that L_2 is tangent to P and find the point of contact.

(ii) Let L_3 be the tangent to P at the point $(1, 2)$. L_3 intersects L_1 and L_2 at C and D respectively. Prove that B, C, F and D are concyclic.

(8 marks)

12. (a) Let $f:(-1,1) \to \mathbf{R}$ be a function with derivatives of any order. For each $m = 1, 2, 3, \ldots$ and $x \in (-1, 1)$, define

$$I_m = \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1} f^{(m)}(t)\,dt .$$

(i) Prove that $I_{m+1} = I_m - \dfrac{f^{(m)}(0)}{m!}x^m$.

(ii) Using (a)(i), prove that

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!}x^k + I_m ,$$

where $f^{(0)} = f$.

(6 marks)

(b) Define $g(x) = \dfrac{1}{\sqrt{1-x^2}}$ for all $x \in (-1, 1)$. Let n be a positive integer.

(i) Prove that

$$(1-x^2)\,g'(x) - x\,g(x) = 0 .$$

Hence deduce that

$$(1-x^2)\,g^{(n+1)}(x) - (2n+1)\,x\,g^{(n)}(x) - n^2\,g^{(n-1)}(x) = 0 ,$$

where $g^{(0)} = g$.

(ii) Prove that $g^{(2n-1)}(0) = 0$ and $g^{(2n)}(0) = \left(\dfrac{(2n)!}{(2^n)(n!)}\right)^2$.

(iii) Using (a), prove that

$$g(x) = \sum_{k=0}^{n-1} \frac{C_k^{2k}}{2^{2k}}x^{2k} + \frac{1}{(2n-1)!}\int_0^x (x-t)^{2n-1} g^{(2n)}(t)\,dt .$$

(9 marks)

END OF PAPER