香港考試局
HONG KONG EXAMINATIONS AUTHORITY

2002年香港高級程度會考
HONG KONG ADVANCED LEVEL EXAMINATION 2002

純粹數學　高級程度　試卷一
PURE MATHEMATICS　A-LEVEL　PAPER 1

本評卷參考乃考試局專爲今年本科考試而編寫，供閱卷員參考之用。閱卷員在完成閱卷工作後，若將本評卷參考提供其任教會考班的本科同事參閱，本局不表反對，但須切記，在任何情況下均不得容許本評卷參考落入學生手中。學生若索閱或求取此等文件，閱卷員/教師應嚴詞拒絕，因學生極可能將評卷參考視爲標準答案，以致但知硬背死記，活剝生吞。這種落伍的學習態度，既不符現代教育原則，亦有違考試着重理解能力與運用技巧之旨。因此，本局籲請各閱卷員/教師通力合作，堅守上述原則。

This marking scheme has been prepared by the Hong Kong Examinations Authority for markers' reference. The Examinations Authority has no objection to markers sharing it, after the completion of marking, with colleagues who are teaching the subject. However, under no circumstances should it be given to students because they are likely to regard it as a set of model answers. Markers/teachers should therefore firmly resist students' requests for access to this document. Our examinations emphasise the testing of understanding, the practical application of knowledge and the use of processing skills. Hence the use of model answers, or anything else which encourages rote memorisation, should be considered outmoded and pedagogically unsound. The Examinations Authority is counting on the co-operation of markers/teachers in this regard.

考試結束後，各科評卷參考將存放於教師中心，供教師參閱。
After the examinations, marking schemes will be available for reference at the teachers' centre.

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Advanced Level Pure Mathematics

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Makers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. In the marking scheme, steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

For Section A:

7. Marks may be deducted for poor presentation (*pp*). The symbol (*pp-1*) should be used to denote 1 mark deducted for *pp*.
   a. At most deduct 1 mark for *pp* in each question, up to a maximum of 2 marks for section A.
   b. For similar *pp*, deduct 1 mark for the first time that it occurs. Do not penalize candidates twice in the paper for the same *pp*.
   c. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

8. Marks entered in the Page Total Box should be the NET total scored on that page.

For Section B:

9. Markers should devise a detailed marking scheme for his/her question after reading a number of live scripts and trial mark 30 scripts based on this scheme before attending the markers' meeting.

10. The Chief Examiner would checkmark these 30 scripts and finalize the marking scheme with the marker.

11. Each marker must hand in a marker's report together with a copy of the revised marking scheme for his/her question. Any changes to the marking scheme must be highlighted and notes should be added to make it clear what each mark is for and the treatment of any special considerations.

Solution	Marks
1. When $n=1$,	
(L1) $\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}=\frac{(1+\sqrt{2})+(1-\sqrt{2})}{2}=1=a_1$.	
When $n=2$,	
$\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}=\frac{(1+\sqrt{2})^2+(1-\sqrt{2})^2}{2}$	
(L2) $=\frac{(3+2\sqrt{2})+(3-2\sqrt{2})}{2}=3=a_2$.	
Assume	
(L3) $a_k=\frac{(1+\sqrt{2})^k+(1-\sqrt{2})^k}{2}$ and	
(L4) $a_{k+1}=\frac{(1+\sqrt{2})^{k+1}+(1-\sqrt{2})^{k+1}}{2}$ for some positive integer k .	3
3 marks if (L1), (L2), (L3) and (L4) are correct. 2 marks if three of (L1), (L2), (L3) and (L4) are correct. 1 mark if giving (L1) and (L3) only.	
Then, $a_{k+2}=2a_{k+1}+a_k$	
$=2\left[\frac{(1+\sqrt{2})^{k+1}+(1-\sqrt{2})^{k+1}}{2}\right]+\frac{(1+\sqrt{2})^k+(1-\sqrt{2})^k}{2}$	
$=\frac{2(1+\sqrt{2})^{k+1}+(1+\sqrt{2})^k}{2}+\frac{2(1-\sqrt{2})^{k+1}+(1-\sqrt{2})^k}{2}$	
$=\frac{(1+\sqrt{2})^k[2(1+\sqrt{2})+1]}{2}+\frac{(1-\sqrt{2})^k[2(1-\sqrt{2})+1]}{2}$	1
$=\frac{(1+\sqrt{2})^k(1+\sqrt{2})^2+(1-\sqrt{2})^k(1-\sqrt{2})^2}{2}$	
$=\frac{(1+\sqrt{2})^{k+2}+(1-\sqrt{2})^{k+2}}{2}$	1
$\therefore$ $a_n=\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}$ for any positive integer n .	
	----------(5)
2. (a) $\left\|\frac{z-1}{z-4}\right\|=2$	
$\|z-1\|^2=4\|z-4\|^2$	
$(z-1)(\bar{z}-1)=4(z-4)(\bar{z}-4)$	1M for $\|\alpha\|^2=\alpha\bar{\alpha}$.
$z\bar{z}-z-\bar{z}+1=4(z\bar{z}-4z-4\bar{z}+16)$	
$z\bar{z}-5z-5\bar{z}+21=0$	1A
$(z-5)(\bar{z}-5)=4$	
$\|z-5\|=2$	1A
Let $z=x+yi$, where $x,y\in\mathbf{R}$.	
Then $\|(x-1)+yi\|=2\|(x-4)+yi\|$	
$(x-1)^2+y^2=4[(x-4)^2+y^2]$	1M
$x^2+y^2-10x+21=0$	1A
$(x-5)^2+y^2=4$	
$\|z-5\|=2$	1A

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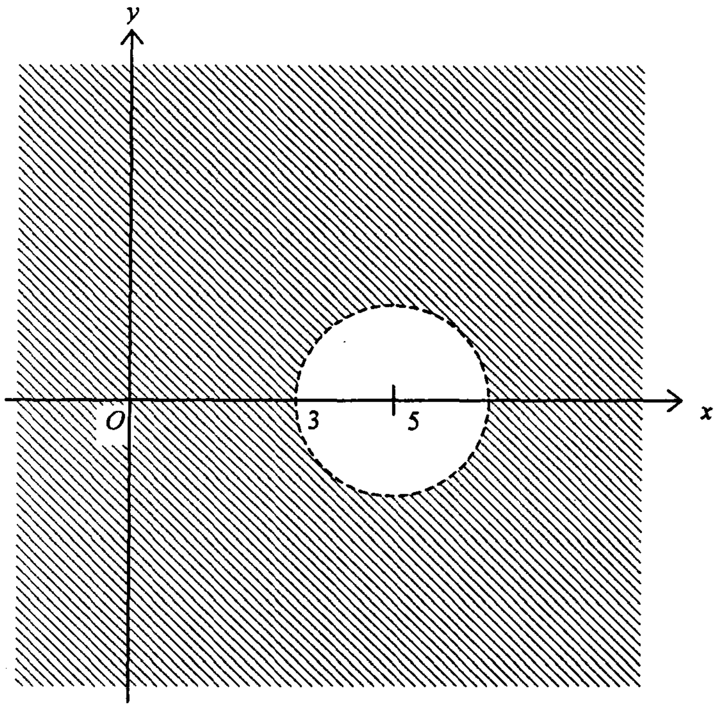
Solution	Marks
(b) For $\left\lvert\dfrac{z-1}{z-4}\right\rvert<2$, we get $\lvert z-5\rvert>2$	1A
The region is shown below (the shaded part):	
	1A
	----------(5)
3. (a) $A=\begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix}$	1A
$=\dfrac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$	
(b) $B=\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}$	1A
(c) $\because\ BA=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$,	
$V=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x-y\\ x+y\end{pmatrix}$	1A
$\because\ V^t\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}V=-4$	
$\therefore\ \begin{pmatrix}x-y & x+y\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x-y\\ x+y\end{pmatrix}=-4$	1A
$(x-y)^2-(x+y)^2=-4$	
$y=\dfrac{1}{x}\quad (x\neq 0)$	1A

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Solution	Marks
$\because \quad BA=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$	
and $V^t=(BAX)^t=X^t(BA)^t$,	1A
for $V^t\begin{pmatrix}1&0\\0&-1\end{pmatrix}V=-4$,	
$X^t\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}X=-4$	
$\begin{pmatrix}x&y\end{pmatrix}\begin{pmatrix}0&-2\\-2&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=-4$	1A
$-4xy=-4$	
$xy=1$	
$y=\frac{1}{x}\quad (x\neq 0)$	1A
	----------(5)
4. (a) $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&1&1\end{vmatrix}=\mathbf{i}-\mathbf{j}+\mathbf{k}$	1A
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{i}\cdot\mathbf{i}=1\neq 0$	
$\therefore$ **a** is not perpendicular to $\mathbf{b}\times\mathbf{c}$.	1
(b) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\1&-1&1\end{vmatrix}=-\mathbf{j}-\mathbf{k}$	1M
$\mathbf{u}=\frac{1}{\sqrt{2}}(0,-1,-1)$ or $\mathbf{u}=\frac{1}{\sqrt{2}}(0,1,1)$.	1A $\mathbf{u}=\frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})$, $\mathbf{u}=\frac{1}{\sqrt{2}}(-\mathbf{j}-\mathbf{k})$
(c) Let θ the angle between **a** and $\mathbf{b}\times\mathbf{c}$.	
$\cos\theta=\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\times\mathbf{c}\rvert}=\frac{1}{\sqrt{3}}$.	1M
Since $\frac{1}{2}<\frac{1}{\sqrt{3}}<\frac{1}{\sqrt{2}}$,	
$\therefore\ \frac{\pi}{4}<\theta<\frac{\pi}{3}$.	1
$\theta=\cos^{-1}\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\times\mathbf{c}\rvert}=\cos^{-1}\frac{1}{1\cdot\sqrt{3}}\approx 54.74^\circ$	1M
$\therefore\ \frac{\pi}{4}<\theta<\frac{\pi}{3}$	1
	----------(6)

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Solution	Marks
5. (a) Suppose $u(x)$ is a common factor of $f(x)-g(x)$ and $g(x)$,	
then $f(x)-g(x)=u(x)p(x)$ and $g(x)=u(x)q(x)$,	1
for some polynomials $p(x)$ and $q(x)$.	
Hence $f(x)=g(x)+u(x)p(x)$	
$=u(x)q(x)+u(x)p(x)$	
$=u(x)[p(x)+q(x)]$	
Since $p(x)+q(x)$ is a polynomial,	
$\therefore$ $u(x)$ is a factor of $f(x)$.	1
Conversely, if $u(x)$ is a common factor of $f(x)$ and $g(x)$,	
then $f(x)=u(x)r(x)$ and $g(x)=u(x)s(x)$	
for some polynomials $r(x)$ and $s(x)$.	
Hence $f(x)-g(x)=u(x)r(x)-u(x)s(x)=u(x)[r(x)-s(x)]$	
Since $r(x)-s(x)$ is a polynomial,	
$\therefore$ $u(x)$ is a factor of $f(x)-g(x)$.	1
3 marks if both parts correct; 2 marks if one part correct; 1 mark for formulating the definition of factor.	
(b) $f(x)-g(x)=x^3-2x^2+2x-1=(x-1)(x^2-x+1)$	1A
By (a), every common factor of $f(x)-g(x)$ and $g(x)$ is a common factor of $f(x)$ and $g(x)$.	
By division, $(x-1)$ is not a factor of $g(x)$ while (x^2-x+1) is a factor of $g(x)$.	
$\therefore$ x^2-x+1 is a common factor of $f(x)$ and $g(x)$.	1M
$\because$ $f(x)=(x^2-x+1)(x^2-2x+3)$ and $g(x)=(x^2-x+1)(x^2-3x+4)$	1A
and the H. C. F. of (x^2-2x+3) and (x^2-3x+4) is 1,	
$\therefore$ the H.C.F. of $f(x)$ and $g(x)$ is (x^2-x+1).	1A

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 -1+1 | 1-3+6-5+3 | 1-4+8-7+4 | 1
      | 1-2 +2 -1 | 1-3+6-5+3 |
      |  -1+4-4+3 |  -1+2-2 +1| -1+1
      |  -1+2 -2+1|  -1+1 -1  |
      | 2|  2 -2+2 |    1 -1+1 |
      |     1 -1 +1|    1 -1+1 |
      |            |         0 |
```

Solution	Marks
(division above)	1M+2A
$\therefore$ the H.C.F. of $f(x)$ and $g(x)$ is (x^2-x+1).	1A
	----------(7)

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Solution	Marks
6. (a) $z_1z_2z_3 = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$	
$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$	
$= \cos 2\pi + i\sin 2\pi$	
$= 1$	1A
(b) $z_k = \cos\theta_k + i\sin\theta_k$ and $\frac{1}{z_k} = \cos\theta_k - i\sin\theta_k$	
$\therefore \quad \cos\theta_k = \frac{1}{2}(z_k + \frac{1}{z_k})$	1
$z_k^2 = \cos 2\theta_k + i\sin 2\theta_k$ and $\frac{1}{z_k^2} = \cos 2\theta_k - i\sin 2\theta_k$	
$\therefore \quad \cos 2\theta_k = \frac{1}{2}(z_k^2 + \frac{1}{z_k^2})$	1
$4\cos\theta_1\cos\theta_2\cos\theta_3$	
$= 4\left[\frac{1}{2}(z_1+\frac{1}{z_1})\right]\left[\frac{1}{2}(z_2+\frac{1}{z_2})\right]\left[\frac{1}{2}(z_3+\frac{1}{z_3})\right]$	
$= \frac{1}{2}\left(z_1z_2z_3 + \frac{z_1}{z_2z_3} + \frac{z_2}{z_1z_3} + \frac{z_3}{z_1z_2} + \frac{z_1z_2}{z_3} + \frac{z_2z_3}{z_1} + \frac{z_1z_3}{z_2} + \frac{1}{z_1z_2z_3}\right)$	1A
$= \frac{1}{2}\left(1 + z_1^2 + z_2^2 + z_3^2 + \frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2} + 1\right) \quad (\because \quad z_1z_2z_3 = 1)$	
$= 1 + \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2 + \frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2}\right)$	1A
$\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3$	
$= \frac{1}{2}\left(z_1^2 + \frac{1}{z_1^2}\right) + \frac{1}{2}\left(z_2^2 + \frac{1}{z_2^2}\right) + \frac{1}{2}\left(z_3^2 + \frac{1}{z_3^2}\right)$	
$= \frac{1}{2}\left(z_1^2 + z_2^2 + z_3^2 + \frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2}\right)$	
$= 4\cos\theta_1\cos\theta_2\cos\theta_3 - 1$	1
$\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3$	
$= 2\cos(\theta_1+\theta_2)\cos(\theta_1-\theta_2) + (2\cos^2\theta_3 - 1)$	1A
$= 2\cos\theta_3\cos(\theta_1-\theta_2) + (2\cos^2\theta_3 - 1)$	
$= 2\cos\theta_3[\cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2)] - 1$	1A
$= 2\cos\theta_3[2\cos\theta_1\cos\theta_2] - 1$	
$= 4\cos\theta_1\cos\theta_2\cos\theta_3 - 1$	1
	---------(6)

Solution	Marks
7. (a) $\dfrac{m(m-1)\cdots(m-k+1)}{m^k} = \dfrac{m-1}{m}\cdot\dfrac{m-2}{m}\cdots\dfrac{m-k+1}{m}$	
$= (1-\dfrac{1}{m})(1-\dfrac{2}{m})\cdots(1-\dfrac{k-1}{m})$	
$< (1-\dfrac{1}{m+1})(1-\dfrac{2}{m+1})\cdots(1-\dfrac{k-1}{m+1})$	1M
$= \dfrac{m}{m+1}\cdot\dfrac{m-1}{m+1}\cdots\dfrac{m-k+2}{m+1}$	
$= \dfrac{(m+1)m(m-1)\cdots(m-k+2)}{(m+1)^k}$	1
When $k=1$,	
L.S. of the inequality is $\dfrac{m}{m}=1$,	
R.S. of the inequality is $\dfrac{m+1}{m+1}=1$.	
The inequality does not hold when $k=1$.	1
(b) $(1+\dfrac{1}{m})^m = 1+\sum_{k=1}^{m} C_k^m\cdot\dfrac{1}{m^k}$	1A
$= 1+\sum_{k=1}^{m}\dfrac{1}{k!}\cdot\dfrac{m(m-1)\cdots(m-k+1)}{m^k}$	
$\le 1+\sum_{k=1}^{m}\dfrac{1}{k!}\cdot\dfrac{(m+1)m\cdots(m-k+2)}{(m+1)^k}$ (by (a))	
$< 1+\left\{\sum_{k=1}^{m}\dfrac{(m+1)m\cdots(m-k+2)}{k!}\cdot\dfrac{1}{(m+1)^k}\right\}+\dfrac{1}{(m+1)^{m+1}}$	1M
$= (1+\dfrac{1}{m+1})^{m+1}$	1
When $m=1$, L.S. $= (1+\dfrac{1}{1})^1 = 2$	
R.S. $= (1+\dfrac{1}{2})^2 = 2.25$	
$\therefore$ L.S. $<$ R.S.	
When $m \ge 2$,	
$(1+\dfrac{1}{m})^m = 2+\sum_{k=2}^{m} C_k^m\cdot\dfrac{1}{m^k}$	1A
$= 2+\sum_{k=2}^{m}\dfrac{1}{k!}\cdot\dfrac{m(m-1)\cdots(m-k+1)}{m^k}$	
$< 2+\sum_{k=2}^{m}\dfrac{1}{k!}\cdot\dfrac{(m+1)m\cdots(m-k+2)}{(m+1)^k}$ (by (a))	
$< 2+\left\{\sum_{k=2}^{m}\dfrac{(m+1)m\cdots(m-k+2)}{k!}\cdot\dfrac{1}{(m+1)^k}\right\}+\dfrac{1}{(m+1)^{m+1}}$	1M
$= (1+\dfrac{1}{m+1})^{m+1}$	1
	----------(6)

Solution	Marks
8. (a) (i) (S) has a unique solution if and only if	
$\Delta = \begin{vmatrix} a & -2 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & a \end{vmatrix}$	1M
$= -(a^2-1)$	
$\neq 0$	
i.e. $a^2 \neq 1$	1
The augmented matrix of (S) is	
$A = \left(\begin{array}{ccc\|c} a & -2 & 1 & 0 \\ 1 & -1 & 2 & b \\ 0 & 1 & a & b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -1 & 2 & b \\ 0 & 1 & a & b \\ a & -2 & 1 & 0 \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & -1 & 2 & b \\ 0 & 1 & a & b \\ a & -2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -1 & 2 & b \\ 0 & 1 & a & b \\ 0 & a-2 & -2a+1 & -ab \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & -1 & 2 & b \\ 0 & 1 & a & b \\ 0 & 0 & -a^2+1 & -2b(a-1) \end{array}\right)$	1M
(S) has a unique solution if and only if $-a^2+1 \neq 0$, i.e. $a^2 \neq 1$.	1
In this case,	
$x = \dfrac{\begin{vmatrix} 0 & -2 & 1 \\ b & -1 & 2 \\ b & 1 & a \end{vmatrix}}{-(a^2-1)} = \dfrac{2b(a-1)}{-(a^2-1)}$	1M for Cramer's Rule
$= -\dfrac{2b}{a+1}$	
$y = \dfrac{\begin{vmatrix} a & 0 & 1 \\ 1 & b & 2 \\ 0 & b & a \end{vmatrix}}{-(a^2-1)} = \dfrac{b(a-1)^2}{-(a^2-1)}$	
$= -\dfrac{b(a-1)}{a+1}$	
$z = \dfrac{\begin{vmatrix} a & -2 & 0 \\ 1 & -1 & b \\ 0 & 1 & b \end{vmatrix}}{-(a^2-1)} = \dfrac{-2b(a-1)}{-(a^2-1)}$	
$= \dfrac{2b}{a+1}$	1A+1A (1A for anyone, 1A for all)

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Solution	Marks
When $a^2 \neq 1$,	
$A \sim \left(\begin{array}{ccc\|c} 1 & -1 & 2 & b \\ 0 & 1 & a & b \\ 0 & 0 & 1 & \frac{2b}{a+1} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -1 & 2 & b \\ 0 & 1 & 0 & -\frac{b(a-1)}{a+1} \\ 0 & 0 & 1 & \frac{2b}{a+1} \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & 0 & 2 & \frac{2b}{a+1} \\ 0 & 1 & 0 & -\frac{b(a-1)}{a+1} \\ 0 & 0 & 1 & \frac{2b}{a+1} \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 0 & -\frac{2b}{a+1} \\ 0 & 1 & 0 & -\frac{b(a-1)}{a+1} \\ 0 & 0 & 1 & \frac{2b}{a+1} \end{array}\right)$	1M
Hence $x = -\frac{2b}{a+1}$, $y = -\frac{b(a-1)}{a+1}$, $z = \frac{2b}{a+1}$.	1A+1A (1A for anyone, 1A for all)
(ii) (1) When $a = 1$, (S) becomes	
$\begin{cases} x - 2y + z = 0 & (1) \\ x - y + 2z = b & (2) \\ y + z = b & (3) \end{cases}$	
(1) – (2) $-y - z = -b$	
i.e. $y + z = b$	
which is the same as equation (3) ,	
∴ (S) is consistent for all values of b .	1A
∴ $\begin{cases} x = 2b - 3t \\ y = b - t \\ z = t \end{cases}$ where $t \in \mathbf{R}$	1A
When $a = 1$,	
$A = \left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 1 & -1 & 2 & b \\ 0 & 1 & 1 & b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & b \\ 0 & 1 & 1 & b \end{array}\right)$	
(S) is consistent for all values of b ,	1A
and $\begin{cases} x = 2b - 3t \\ y = b - t \\ z = t \end{cases}$ where $t \in \mathbf{R}$	1A
(2) When $a = -1$, (S) becomes	
$\begin{cases} -x - 2y + z = 0 & (1) \\ x - y + 2z = b & (2) \\ y - z = b & (3) \end{cases}$	
(1) + (2) $-3y + 3z = b$	
i.e. $3y - 3z = -b$	
∴ (S) is consistent when $3b = -b$ only, i.e. $b = 0$.	1A
Hence $\begin{cases} x = -t \\ y = z = t \end{cases}$, where $t \in \mathbf{R}$	1A

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Solution	Marks
When $a=-1$,	
$A=\left(\begin{array}{ccc\|c} -1 & -2 & 1 & 0 \\ 1 & -1 & 2 & b \\ 0 & 1 & -1 & b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & b \\ 0 & 1 & -1 & b \end{array}\right)$	
$\therefore$ (S) is consistent when $b=-3b$ only, i.e. $b=0$.	1A
Hence $\begin{cases} x=-t \\ y=z=t \end{cases}$, where $t \in \mathbf{R}$	1A
	---------(9)
(b) Substitute $b=-1$ into (S), then (S) becomes the first 3 equations of (T).	
Using (a), there are two possibilities : (i) $a^2 \neq 1$ and $b=-1$; (ii) $a=1$ and $b=-1$.	1M
(i) If $a^2 \neq 1$ and $b=-1$, then (S) has a unique solution.	
(T) is consistent if the solution $\begin{cases} x=\dfrac{2}{a+1} \\ y=\dfrac{a-1}{a+1} \\ z=-\dfrac{2}{a+1} \end{cases}$ satisfies the 4th equation of (T),	
$\therefore$ $5(\dfrac{2}{a+1})-2\left(\dfrac{a-1}{a+1}\right)-\dfrac{2}{a+1}=a$	1M
$a^2+3a-10=0$	
$a=2$ or -5.	1A
When $a=2$, solution of (T) is $\begin{cases} x=\dfrac{2}{3} \\ y=\dfrac{1}{3} \\ z=-\dfrac{2}{3} \end{cases}$; When $a=-5$, solution of (T) is $\begin{cases} x=-\dfrac{1}{2} \\ y=\dfrac{3}{2} \\ z=\dfrac{1}{2} \end{cases}$	1A
(ii) If $a=1$ and $b=-1$, (T) is consistent if the solution $\begin{cases} x=-2-3t \\ y=-1-t \\ z=t \end{cases}$ satisfies the 4th equation of (T),	
$\therefore$ $5(-2-3t)-2(-1-t)+t=1$	1M
$t=-\dfrac{3}{4}$	
The solution of (T) is $\begin{cases} x=\dfrac{1}{4} \\ y=-\dfrac{1}{4} \\ z=-\dfrac{3}{4} \end{cases}$	1A
	---------(6)

Solution	Marks
9. (a) (i) Suppose that $\lambda_1\mathbf{u}+\lambda_2\mathbf{v}+\lambda_3\mathbf{w}=\mathbf{0}$ for some scalars λ_1, λ_2 and λ_3.	
Then $\mathbf{u}\cdot(\lambda_1\mathbf{u}+\lambda_2\mathbf{v}+\lambda_3\mathbf{w})=0$	1M
$\lambda_1\mathbf{u}\cdot\mathbf{u}+\lambda_2\mathbf{u}\cdot\mathbf{v}+\lambda_3\mathbf{u}\cdot\mathbf{w}=0$	
Since $\mathbf{u}\neq\mathbf{0}$ and $\mathbf{u}\cdot\mathbf{v}=0$, $\mathbf{u}\cdot\mathbf{w}=0$, $\therefore\ \lambda_1=0$	
Similarly, $\lambda_2=\lambda_3=0$.	
$\therefore$ u, v and w are linearly independent.	1
(ii) Let $\mathbf{u}=\mathbf{i}$, $\mathbf{v}=\mathbf{j}$ and $\mathbf{w}=\mathbf{j}+\mathbf{k}$.	
Suppose that $\lambda_1\mathbf{u}+\lambda_2\mathbf{v}+\lambda_3\mathbf{w}=\mathbf{0}$ for some scalars λ_1, λ_2 and λ_3.	
Then $\mathbf{u}\cdot(\lambda_1\mathbf{u}+\lambda_2\mathbf{v}+\lambda_3\mathbf{w})=0$	1M
$\lambda_1\,\mathbf{i}\cdot\mathbf{i}+\lambda_2\,\mathbf{i}\cdot\mathbf{j}+\lambda_3\,\mathbf{i}\cdot(\mathbf{j}+\mathbf{k})=0$	
$\lambda_1=0$	
Similarly, $\lambda_2=0$ and $\lambda_3=0$.	1M
$\therefore$ u, v and w are linearly independent.	1
On the other hand, $\mathbf{v}\cdot\mathbf{w}=1\neq0$.	1A
$\therefore$ u, v and w are not orthogonal.	
	----------(6)

Solution	Marks
Suppose that $\lambda_1\mathbf{u}+\lambda_2\mathbf{v}+\lambda_3\mathbf{w}=\mathbf{0}$ for some scalars λ_1, λ_2 and λ_3.	
Then $\begin{cases}\lambda_1 = 0\\ \lambda_2 = 0\\ \lambda_2+\lambda_3 = 0\end{cases}$	1M
$\therefore\ \lambda_1=0$, $\lambda_2=0$ and $\lambda_3=0$.	1M
$\therefore$ u, v and w are linearly independent.	1
On the other hand, $\mathbf{v}\cdot\mathbf{w}=1\neq0$.	1A
$\therefore$ u, v and w are not orthogonal.	

Solution	Marks

(b) (i)

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1^2+u_2^2+u_3^2 & u_1v_1+u_2v_2+u_3v_3 & u_1w_1+u_2w_2+u_3w_3 \\ u_1v_1+u_2v_2+u_3v_3 & v_1^2+v_2^2+v_3^2 & v_1w_1+v_2w_2+v_3w_3 \\ u_1w_1+u_2w_2+u_3w_3 & v_1w_1+v_2w_2+v_3w_3 & w_1^2+w_2^2+w_2^3 \end{pmatrix}$$ 1A

Since **u** , **v** and **w** are non-zero orthogonal vectors in $\mathbf{R}^3$,

$\therefore\quad u_1^2+u_2^2+u_3^2 \neq 0,\quad v_1^2+v_2^2+v_3^2 \neq 0,\quad w_1^2+w_2^2+w_3^2 \neq 0,$

$u_1v_1+u_2v_2+u_3v_3 = 0\ ,\ \cdots\ ,\ v_1w_1+v_2w_2+v_3w_3 = 0$ 1A

$$\therefore\quad \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

$$= \begin{pmatrix} {u_1}^2+{u_2}^2+{u_3}^2 & 0 & 0 \\ 0 & {v_1}^2+{v_2}^2+{v_3}^2 & 0 \\ 0 & 0 & {w_1}^2+{w_2}^2+{w_3}^2 \end{pmatrix}$$

Hence

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$ 1M for $\det(AB) = (\det A)(\det B)$

$$= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2) \neq 0\ .$$

$$\therefore\quad \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0\ .$$ 1

(ii) Consider

$$\begin{cases} u_1\lambda_1+v_1\lambda_2+w_1\lambda_3 = p_1 \\ u_2\lambda_1+v_2\lambda_2+w_2\lambda_3 = p_2 \\ u_3\lambda_1+v_3\lambda_2+w_3\lambda_3 = p_3 \end{cases} \qquad (*)$$ 1M

Using (i),

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0$$

$\therefore$ the system (*) has (unique) solution λ_1, λ_2, λ_3 .

$\therefore$ $\mathbf{p} = \lambda_1\mathbf{u}+\lambda_2\mathbf{v}+\lambda_3\mathbf{w}$ for some scalars λ_1, λ_2 and λ_3 .

$\therefore$ **p** is a linear combination of **u** , **v** and **w** . 1

----------(6)

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Solution	Marks
(c) (i) $\because$ $\mathbf{x}\cdot\mathbf{y}=0$	
$\mathbf{y}\cdot\mathbf{z}=0$	
and $\mathbf{x}\cdot\mathbf{z}=\frac{1}{2}-\frac{1}{2}=0$	
$\therefore$ $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are orthogonal vectors	1
(ii) Let $\mathbf{q}=\lambda_1\mathbf{x}+\lambda_2\mathbf{y}+\lambda_3\mathbf{z}$.	
Using (*) ,	
$\begin{cases} \frac{1}{\sqrt{2}}\lambda_1 + \frac{1}{\sqrt{2}}\lambda_3 = -1 \\ \lambda_2 = 2 \\ -\frac{1}{\sqrt{2}}\lambda_1 + \frac{1}{\sqrt{2}}\lambda_3 = 0 \end{cases}$	1M
$\therefore$ $\lambda_1=-\frac{\sqrt{2}}{2}$, $\lambda_2=2$, $\lambda_3=-\frac{\sqrt{2}}{2}$	
$\therefore$ $\mathbf{q}=-\frac{\sqrt{2}}{2}\mathbf{x}+2\mathbf{y}-\frac{\sqrt{2}}{2}\mathbf{z}$.	1A
	----------(3)

Solution	Marks
10. (a) $\because \sum_{i=1}^{n}(a_ix+b_i)^2 \geq 0$ for all $x \in \mathbf{R}$	
$\therefore \left(\sum_{i=1}^{n} a_i^2\right)x^2 + 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \sum_{i=1}^{n} b_i^2 \geq 0$ for all $x \in \mathbf{R}$	1A
The required inequality is trivial when $\sum_{i=1}^{n} a_i^2 = 0$.	1
Suppose $\sum_{i=1}^{n} a_i^2 > 0$.	
For the quadratic expression to be non-negative for all values of $x \in \mathbf{R}$,	
$4\left(\sum_{i=1}^{n} a_ib_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0$	
i.e. $\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$	1
If $\left(\sum_{i=1}^{n} a_ib_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$,	
the quadratic equation $\left(\sum_{i=1}^{n} a_i^2\right)x^2 + 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \sum_{i=1}^{n} b_i^2 = 0$	
has (equal) real roots.	
Hence $\left(\sum_{i=1}^{n} a_i^2\right)x^2 + 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \sum_{i=1}^{n} b_i^2 = 0$ for some nonzero $x \in \mathbf{R}$	
since $\sum_{i=1}^{n} b_i^2 \neq 0$.	
i.e. $\sum_{i=1}^{n}(a_ix+b_i)^2 = 0$ for some nonzero $x \in \mathbf{R}$	1
Thus $a_ix + b_i = 0$ for all $i = 1, 2, \cdots, n$.	
$\therefore \frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}\left(= -\frac{1}{x}\right)$.	1
Conversely, if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = p$ for some $p \in \mathbf{R}$, then	
$a_1 = pb_1$, $a_2 = pb_2$, …, $a_n = pb_n$	
Hence $\left(\sum_{i=1}^{n} a_ib_i\right)^2 = \left(\sum_{i=1}^{n} pb_i^2\right)^2 = p^2\left(\sum_{i=1}^{n} b_i^2\right)^2$	
and $\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} p^2b_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$	
$= p^2\left(\sum_{i=1}^{n} b_i^2\right)^2$	
$\therefore \left(\sum_{i=1}^{n} a_ib_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$	1
	----------(6)

Solution	Marks
(b) (i) Putting $a_i = x_i$ and $b_i = 1$ for all $i = 1, 2, \ldots, n$ in (a)	
$\left(\sum_{i=1}^{n} x_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\cdot n$.	1A
Dividing both sides by n^2 ,	
$\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 \le \frac{\sum_{i=1}^{n} x_i^2}{n}$.	1
Putting $a_i = x_i$ and $b_i = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$ in (a)	
$\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{n^2}\right)$	1A
i.e. $\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 \le \frac{\sum_{i=1}^{n} x_i^2}{n}$.	1
(ii) Putting $b_i = \sqrt{\lambda_i}$ and $a_i = \sqrt{\lambda_i}\, x_i$ for all $i = 1, 2, \ldots, n$ in (a),	1A
$\left(\sum_{i=1}^{n} \lambda_i x_i\right)^2 \le \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \lambda_i x_i^2\right)$.	1
The necessary and sufficient condition is	
$\frac{\sqrt{\lambda_1}\, x_1}{\sqrt{\lambda_1}} = \frac{\sqrt{\lambda_2}\, x_2}{\sqrt{\lambda_2}} = \cdots = \frac{\sqrt{\lambda_n}\, x_n}{\sqrt{\lambda_n}}$	
i.e. $x_1 = x_2 = \ldots = x_n$.	1A
(iii) Putting $\lambda_i = \frac{1}{t^i}$ and $x_i = y_i$ into the inequality in (b) (ii),	1A Put $b_i = \frac{1}{\sqrt{t^i}}$, $a_i = \frac{y_i}{\sqrt{t^i}}$ in (a)
$\left(\sum_{i=1}^{n} \frac{1}{t^i}\cdot y_i\right)^2 \le \left(\sum_{i=1}^{n} \frac{1}{t^i}\right)\left(\sum_{i=1}^{n} \frac{y_i^2}{t^i}\right)$.	1
Since $\sum_{i=1}^{n} \frac{1}{t^i} = \frac{\frac{1}{t}(1-\frac{1}{t^n})}{1-\frac{1}{t}}$	
$= \frac{1}{t-1}(1-\frac{1}{t^n})$	
and $t \ge 2$,	
$\therefore \sum_{i=1}^{n} \frac{1}{t^i} < 1$	1A
$\sum_{i=1}^{n} \frac{1}{t^i} \le \sum_{i=1}^{n} \frac{1}{2^i}$ ($\because$ $t \ge 2$)	
$= 1 - \frac{1}{2^n}$	
< 1	1A
Thus $\left(\sum_{i=1}^{n} \frac{y_i}{t^i}\right)^2 < \sum_{i=1}^{n} \frac{y_i^2}{t^i}$	1
	----------(9)

Solution	Marks
11. (a) (i) Let A and B be two sufficiently large positive numbers.	
Since $f(x) = x^3(1 - \frac{3p}{x^2} + \frac{1}{x^3})$,	
$\therefore$ $f(A) > 0$ and $f(-B) < 0$.	1M
Hence the equation $f(x) = 0$ has at least one real root.	1
The complex roots of a real polynomial equation are in conjugate pairs.	1M
The degree of the given equation is 3 , therefore it has at least one real root.	1
(ii) $f'(x) = 3x^2 - 3p$	1A
When $p \le 0$,	
$f'(x) > 0$ $\forall x \ne 0$.	
$\therefore$ $f(x)$ is strictly increasing for all x .	1
Hence $f(x) = 0$ has at most one real root.	
Combining (a) (i) , $f(x) = 0$ has one and only one real root.	1
(iii) Note that $f''(x) = 6x$.	
If $p > 0$, then	
$f'(x) = 0 \Leftrightarrow x = \pm\sqrt{p}$,	
Since $f''(-\sqrt{p}) < 0$ and $f''(\sqrt{p}) > 0$,	
the maximum value of $f(x)$ is	
$f(-\sqrt{p}) = 2p^{\frac{3}{2}} + 1$	
and the minimum value of $f(x)$ is	
$f(\sqrt{p}) = -2p^{\frac{3}{2}} + 1$.	1A both $f(\sqrt{p})$ and $f(-\sqrt{p})$
(1) For the equation $f(x) = 0$ to have exactly one real root,	
$(2p^{\frac{3}{2}} + 1)(-2p^{\frac{3}{2}} + 1) > 0$	
$0 < p < \frac{1}{\sqrt[3]{4}}$.	
(2) For the equation $f(x) = 0$ to have exactly two distinct real roots,	
$2p^{\frac{3}{2}} + 1 = 0$ or $-2p^{\frac{3}{2}} + 1 = 0$	
$p = \frac{1}{\sqrt[3]{4}}$	
(3) For the equation $f(x) = 0$ to have three distinct real roots,	1M for considering the max and min
$(2p^{\frac{3}{2}} + 1)(-2p^{\frac{3}{2}} + 1) < 0$	1A + 1A (1A for one correct answer 1A for all correct answers)
$p > \frac{1}{\sqrt[3]{4}}$	
	----------(9)

Solution	Marks
(b) (i) $g'(x) = 4(x^3 + 1)$	1A
$g''(x) = 12x^2 > 0$ for all $x \neq 0$	
Using (a)(ii), $g'(x) = 0$ has one and only one real root and it gives the minimum value of $g(x)$.	1(or $g'(x) = 0$: $x = -1$ or $\frac{1 \pm \sqrt{3}i}{2}$)
Thus $g(x) = 0$ has at most two real roots.	1
(ii) The minimum value of $g(x)$ is	
$g(-1) = 1 - 4 + a$	
$= a - 3$	1A
The equation $g(x) = 0$ has two distinct real roots if and only if	
$a - 3 < 0$	1M
i.e. $a < 3$	1
	----------(6)

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Solution	Marks
12. (a) (i) $A^3 + A^2 + A + I = 0$	
$A(-A^2 - A - I) = (-A^2 - A - I)A = I$	1A
$\therefore$ A has an inverse	1
$A(-A^2 - A - I) = I$	
$\lvert A \rvert \cdot \lvert -A^2 - A - I \rvert = 1$	1A
$\therefore$ $\lvert A \rvert \neq 0$	
$\therefore$ A has an inverse	1
$\because$ $A^3 + A^2 + A + I = 0$	
$\therefore$ $(A - I)(A^3 + A^2 + A + I) = 0$	
$A^4 - I = 0$	1A
$\therefore$ $\lvert A \rvert \neq 0$	
$\therefore$ A has an inverse	1
$A^{-1} = -(A^2 + A + I)$	1A
$A^{-1} = A^3$	1A
(ii) $\because$ $A^3 + A^2 + A + I = 0$	
$\therefore$ $(A - I)(A^3 + A^2 + A + I) = 0$	
$A^4 - I = 0$	
i.e. $A^4 = I$	1
(iii) $(A^{-1})^3(A^3 + A^2 + A + I) = 0$	
$(A^{-1}A)^3 + A^{-1}(A^{-1}A)^2 + (A^{-1})^2(A^{-1}A) + (A^{-1})^3 = 0$	
$(A^{-1})^3 + (A^{-1})^2 + A^{-1} + I = 0$	1
$(A^{-1})^3 + (A^{-1})^2 + A^{-1} + I$	
$= A^4\left((A^{-1})^3 + (A^{-1})^2 + A^{-1} + I\right)$ $\quad (\because \quad A^4 = I)$	
$= A + A^2 + A^3 + A^4$	
$= -I + I$ $\quad (\because \quad A^3 + A^2 + A = -I$ and $A^4 = I)$	
$= 0$	1
$(A^{-1})^3 + (A^{-1})^2 + A^{-1} + I$	
$= (A^3)^3 + (A^3)^2 + A^3 + I$	
$= A^9 + A^6 + A^3 + I$	
$= A + A^2 + A^3 + I$	
$= 0$	1
(iv) The identity matrix I is one such example because	
$I^3 + I^2 + I + I$	
$= 4I \neq 0$	
and $\lvert I \rvert \neq 0$.	1
	---------(6)

Solution	Marks

(b) (i)

$$X^2=\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$$

$$X^3=\begin{pmatrix}-1&-1&-1\\1&0&1\\1&1&0\end{pmatrix}$$

$\therefore \quad X^3+X^2+X+I$

$$=\begin{pmatrix}-1&-1&-1\\1&0&1\\1&1&0\end{pmatrix}+\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}+\begin{pmatrix}1&1&1\\-1&-1&0\\-1&0&-1\end{pmatrix}+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

1M for attempting to verify $X^3+X^2+X+I=0$

1A if X^2 and X^3 are correct

$= 0$

Using (a)(i)

$$X^{-1}=-(X^2+X+I)$$

$$=\begin{pmatrix}-1&-1&-1\\1&0&1\\1&1&0\end{pmatrix}$$

1A

$$|X|=\begin{vmatrix}1&1&1\\-1&-1&0\\-1&0&-1\end{vmatrix}=-1$$

1A

$\therefore \quad X^{-1}$ exists

$$X^{-1}=-\begin{pmatrix}\begin{vmatrix}-1&0\\0&-1\end{vmatrix}&-\begin{vmatrix}1&1\\0&-1\end{vmatrix}&\begin{vmatrix}1&1\\-1&0\end{vmatrix}\\-\begin{vmatrix}-1&0\\-1&-1\end{vmatrix}&\begin{vmatrix}1&1\\-1&-1\end{vmatrix}&-\begin{vmatrix}1&1\\-1&0\end{vmatrix}\\\begin{vmatrix}-1&-1\\-1&0\end{vmatrix}&-\begin{vmatrix}1&1\\-1&0\end{vmatrix}&\begin{vmatrix}1&1\\-1&-1\end{vmatrix}\end{pmatrix}$$

1M for attempting to find the adjoint matrix

$$=\begin{pmatrix}-1&-1&-1\\1&0&1\\1&1&0\end{pmatrix}$$

1A

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Solution	Marks
(ii) $\because \quad X^3 = X^{-1}$ $\therefore \quad X^4 = I$ The possible values of X^n are, (p = positive integer)	or using (a)(ii)
(1) when $n = 4p$, $X^n = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (2) when $n = 4p+1$, $X^n = X = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$ (3) when $n = 4p+2$, $X^n = X^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ (4) when $n = 4p+3$, $X^n = X^3 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	1M for considering 4 cases 1A for one correct answer 1A for all correct answers
(iii) Let $Y = -I$	1A
Then $Y^3 + Y^2 + Y + I$ $= -I + I - I + I$ $= 0$	1
$\therefore \quad Y = -I$ is one possible answer.	
Let $Z = X^3$ Then $Z^3 + Z^2 + Z + I$ $= Z^9 + Z^6 + Z^3 + I$ $= X + X^2 + X^3 + I$ $= 0$	
$\therefore \quad Z = X^3$ is one possible answer.	1 ---------(9)

Marking Scheme:
2 marks for 1 correct answer.
3 marks for 2 correct answers.

Solution	Marks
13. (a) (i) The case is trivial when $n=1$.	
Assume that	
$x_{k+2}-x_k=(-1)^k\cdot\frac{2^{k-1}}{3^k}(x_1-x_2)$ for some integer $k\ge 1$.	1
Then $x_{k+3}-x_{k+1}=(\frac{1}{3}x_{k+2}+\frac{2}{3}x_{k+1})-x_{k+1}$	1
$=\frac{1}{3}x_{k+2}-\frac{1}{3}x_{k+1}$	
$=\frac{1}{3}x_{k+2}-(x_{k+2}-\frac{2}{3}x_k)$	1
$=-\frac{2}{3}(x_{k+2}-x_k)$	
$=-\frac{2}{3}\cdot(-1)^k\cdot\frac{2^{k-1}}{3^k}(x_1-x_2)$	
$=(-1)^{k+1}\cdot\frac{2^k}{3^{k+1}}(x_1-x_2)$	1
Hence the equality holds for all integer $n\ge 1$.	
$x_{n+2}-x_n=\frac{1}{3}x_{n+1}+\frac{2}{3}x_n-x_n$	1
$=\frac{-1}{3}(x_n-x_{n+1})$	1
$=\frac{-1}{3}(x_n-\frac{1}{3}x_n-\frac{2}{3}x_{n-1})$	
$=\frac{-1}{3}(\frac{2}{3})(x_n-x_{n-1})$	
$=(-1)^2\frac{2}{3^2}(x_{n-1}-x_n)$	1
$=(-1)^3\frac{2^2}{3^3}(x_{n-2}-x_{n-1})$	
$=(-1)^n\frac{2^{n-1}}{3^n}(x_1-x_2)$	1
(ii) Since $x_1>x_2$,	
$\therefore$ $x_{n+2}-x_n<0$ if n is odd	
and $x_{n+2}-x_n>0$ if n is even.	
$\therefore$ $\{x_1,x_3,\ldots\}$ is strictly decreasing and	
$\{x_2,x_4,\ldots\}$ is strictly increasing.	1
	----------(5)
(b) (i) $x_{2n}=\frac{1}{3}x_{2n-1}+\frac{2}{3}x_{2n-2}$	
$<\frac{1}{3}x_{2n-1}+\frac{2}{3}x_{2n}$ ($\because$ $\{x_{2n}\}$ is strictly increasing)	1
$\therefore$ $\frac{1}{3}x_{2n}<\frac{1}{3}x_{2n-1}$	
i.e. $x_{2n}<x_{2n-1}$	1

Solution	Marks
(ii) By (a) and the above result,	
$\{x_1, x_3, x_5, \dots\}$ is decreasing and bounded below by x_2 ,	1
and $\{x_2, x_4, x_6, \dots\}$ is increasing and bounded above by x_1 ,	1
$\therefore$ both sequences converge .	
Let $\lim\limits_{n\to\infty} x_{2n-1} = \ell_1$ and $\lim\limits_{n\to\infty} x_{2n} = \ell_2$.	
Since $3x_{2n+2} - x_{2n+1} - 2x_{2n} = 0$	
$\therefore$ $3\ell_2 - \ell_1 - 2\ell_2 = 0$	1M
$\therefore$ $\ell_1 = \ell_2$	1
	----------(6)
(c) Using (a),	
$\sum\limits_{n=1}^{p}(x_{n+2} - x_n) = \sum\limits_{n=1}^{p}(-1)^n \cdot \frac{2^{n-1}}{3^n}(x_1 - x_2)$	
i.e. $x_{p+2} + x_{p+1} - x_2 - x_1 = (x_1 - x_2)\cdot\frac{(-\frac{1}{3})\left[1-\left(-\frac{2}{3}\right)^p\right]}{1-\left(-\frac{2}{3}\right)}$	1A+1A
$= -\frac{\left[1-\left(-\frac{2}{3}\right)^p\right](x_1 - x_2)}{5}$	
When $p \to \infty$,	
$2\ell - x_2 - x_1 = -\frac{x_1 - x_2}{5}$, where $\ell = \lim\limits_{n\to\infty} x_n$	1M
$\therefore$ $\ell = \frac{2x_1 + 3x_2}{5}$	1A
$\because$ $x_{n+2} = \frac{1}{3}x_{n+1} + \frac{2}{3}x_n$	
$\therefore$ $\sum\limits_{n=1}^{p} x_{n+2} = \frac{1}{3}\left(\sum\limits_{n=1}^{p} x_{n+1}\right) + \frac{2}{3}\left(\sum\limits_{n=1}^{p} x_n\right)$	
$\left(\sum\limits_{n=3}^{p} x_n\right) + x_{p+1} + x_{p+2} = \frac{1}{3}\left[\left(\sum\limits_{n=2}^{p} x_n\right) + x_{p+1}\right] + \frac{2}{3}\left(\sum\limits_{n=1}^{p} x_n\right)$	1A + 1A
$\left(\sum\limits_{n=3}^{p} x_n\right) + x_{p+1} + x_{p+2} = \left(\sum\limits_{n=2}^{p} x_n\right) + \frac{1}{3}x_{p+1} + \frac{2}{3}x_1$	
$x_{p+1} + x_{p+2} = x_2 + \frac{1}{3}x_{p+1} + \frac{2}{3}x_1$	
$x_{p+1} + \frac{2}{3}x_{p+2} = \frac{2}{3}x_1 + x_2$	
When $p \to \infty$,	
$\ell + \frac{2}{3}\ell = \frac{2}{3}x_1 + x_2$, where $\ell = \lim\limits_{n\to\infty} x_n$	1M
$\therefore$ $\ell = \frac{2x_1 + 3x_2}{5}$	1A
	----------(4)

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Solution	Marks
1. $\because \dfrac{3-x}{(1+x)(1+x^2)}=\dfrac{2}{x+1}+\dfrac{-2x+1}{x^2+1}$	1M+1A
$\therefore \displaystyle\int\frac{3-x}{(1+x)(1+x^2)}\,dx=\int\frac{2dx}{x+1}+\int\frac{-2x+1}{x^2+1}\,dx$	
$=2\ln\lvert x+1\rvert-\displaystyle\int\frac{d(x^2+1)}{(x^2+1)}+\int\frac{dx}{x^2+1}$	
$=2\ln\lvert x+1\rvert-\ln(x^2+1)+\tan^{-1}x+C$	1A+1A
$\therefore \displaystyle\int_0^{\infty}\frac{(3-x)}{(1+x)(1+x^2)}\,dx=\lim_{x\to\infty}\left\{\ln\left[\frac{(x+1)^2}{(x^2+1)}\right]+\tan^{-1}x\right\}-0$	1M
$=\dfrac{\pi}{2}$	1A
	-----------(6)
2. f is continuous at $x=\dfrac{\pi}{2}$	
$\Rightarrow \displaystyle\lim_{x\to\frac{\pi}{2}^-}f(x)=\lim_{x\to\frac{\pi}{2}^+}f(x)=f(\frac{\pi}{2})$	1M
$\Rightarrow \dfrac{a\pi}{2}=e^{\frac{b\pi}{2}}\sin\dfrac{\pi}{2}$	
i.e. $\dfrac{a\pi}{2}=e^{\frac{b\pi}{2}}$. .... (1)	1
$\displaystyle\lim_{x\to\frac{\pi}{2}^-}\frac{f(x)-f(\frac{\pi}{2})}{x-\frac{\pi}{2}}=\lim_{x\to\frac{\pi}{2}^-}\frac{ax-\frac{a\pi}{2}}{x-\frac{\pi}{2}}=a$	1M
$\displaystyle\lim_{x\to\frac{\pi}{2}^+}\frac{f(x)-f(\frac{\pi}{2})}{x-\frac{\pi}{2}}=\lim_{x\to\frac{\pi}{2}^+}\frac{e^{bx}\sin x-\frac{a\pi}{2}}{x-\frac{\pi}{2}}$	
$=\displaystyle\lim_{x\to\frac{\pi}{2}^+}(e^{bx}\cos x+be^{bx}\sin x)$	
$=be^{\frac{b\pi}{2}}$	
For differentiable at $x=\dfrac{\pi}{2}$,	
$a=be^{\frac{b\pi}{2}}$ .... (2)	1A
Solving (1) and (2),	
$a=\dfrac{2e}{\pi}$ $b=\dfrac{2}{\pi}$	1A
	----------(5)

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Solution	Marks
3. (a) $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	
$=\lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$	1M definition and $f(x+y)=f(x)f(y)$
$=\lim_{h\to 0}\frac{f(h)-1}{h}\cdot f(x)$	
$=f(x)$	1
(b) $\frac{d}{dx}\left(\frac{f(x)}{e^x}\right)=\frac{e^x f'(x)-e^x f(x)}{e^{2x}}$	
$=0$ (result of (a))	
$\therefore f(x)=Ce^x$ for some constant C.	1A
Sub. $x=0$, we get $C=f(0)$.	
By (ii), $f(0+0)=[f(0)]^2 \Rightarrow f(0)=0$ or $f(0)=1$	1A
By (i), $\lim_{x\to 0}(f(x)-1)=0 \Rightarrow \lim_{x\to 0}f(x)=1 \Rightarrow f(0)=1$	
Thus $C=f(0)=1$. Hence, $f(x)=e^x$.	1
	----------(5)
4. (a) The distance between any point on ℓ and O is	
$\sqrt{3t^2+3t^2+(3-\sqrt{2}t)^2}=\sqrt{8t^2-6\sqrt{2}t+9}$	
Suppose the coordinates of A and B are $(\sqrt{3}t_1, \sqrt{3}t_1, 3-\sqrt{2}t_1)$	
and $(\sqrt{3}t_2, \sqrt{3}t_2, 3-\sqrt{2}t_2)$ respectively.	
Then t_1 and t_2 are the roots of the equation	
$\sqrt{8t^2-6\sqrt{2}t+9}=r$.	1M
Hence $t_1+t_2=\frac{3\sqrt{2}}{4}$	}
and $t_1t_2=\frac{9-r^2}{8}$.	} 1A
$AB^2=(\sqrt{3}t_1-\sqrt{3}t_2)^2+(\sqrt{3}t_1-\sqrt{3}t_2)^2+[(3-\sqrt{2}t_1)-(3-\sqrt{2}t_2)]^2$	
$=8(t_1-t_2)^2$	1A
$=8[(t_1+t_2)^2-4t_1t_2]$	
$=4r^2-27$	
$AB=\sqrt{4r^2-27}$	1A
(b) If ΔOAB is equilateral, then	
$4r^2-27=r^2$	
Hence $r=3$	1A
	------------(5)

Solution	Marks
5. (a) g is even, $g(-x)=g(x)$.	
g is odd, $g(-x)=-g(x)$.	
Hence $g(x)=-g(x)$.	
Thus $g(x)=0$	1
(b) (i) $F(-x)=\frac{1}{2}[f(-x)+f(x)]=F(x)$ $\therefore$ F is even .	1
$G(-x)=\frac{1}{2}[f(-x)-f(x)]=-G(x)$ $\therefore$ G is odd .	1
(ii) $f(x)=M(x)+N(x)$ and	
$f(-x)=M(-x)+N(-x)=M(x)-N(x)$	1A
$\therefore$ $f(x)+f(-x)=2M(x)$ and $f(x)-f(-x)=2N(x)$	
$\therefore$ $M(x)=\frac{f(x)+f(-x)}{2}=F(x)$ and $N(x)=\frac{f(x)-f(-x)}{2}=G(x)$	1+1
$f(x)=M(x)+N(x)=F(x)+G(x)$	1A
$M(x)-F(x)=G(x)-N(x)$ (*)	
$\because$ M and F are even ,	
$M(-x)-F(-x)=M(x)-F(x)$	
$\therefore$ $M-F$ is even .	
$\because$ G and N are odd ,	1
$G(-x)-N(-x)=-G(x)+N(x)=-[G(x)-N(x)]$	
$\therefore$ $G-N$ is odd .	
Using the results of (a) and (*) ,	
$M(x)-F(x)=G(x)-N(x)=0$	
i.e. $M(x)=F(x)$ and $N(x)=G(x)$	1
	----------(6)
6. (a) Let $y=\left(\frac{1}{x}\right)^{\sin x}$.	
$\ln y=-\sin x\ln x$ $(x>0)$	
$\lim_{x\to 0^+}\ln y=-\lim_{x\to 0^+}\sin x\ln x$	1A
$=-\lim_{x\to 0^+}\frac{\ln x}{\operatorname{cosec} x}$	
$=+\lim_{x\to 0^+}\frac{\frac{1}{x}}{\cot x\operatorname{cosec} x}$	1A
$=\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\tan x$	
$=0$	
$\lim_{x\to 0^+}y=e^0=1$	1A

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Solution	Marks
(b) $\because$ $f(t)$ is continuous for all t,	
$\therefore$ $\dfrac{d}{dx}\displaystyle\int_0^x f(t)dt = f(x)$ by the Fundamental Theorem of Calculus	
$\displaystyle\lim_{x\to 0}\frac{\int_0^x f(t)dt - x}{x^3} = \lim_{x\to 0}\frac{f(x)-1}{3x^2}$	1A
$\displaystyle = \lim_{x\to 0}\frac{f'(x)}{6x}$	
$\displaystyle = \lim_{x\to 0}\frac{x\cos x - \sin x}{6x^3}$	1A
$\displaystyle = \lim_{x\to 0}\frac{(\cos x - x\sin x) - \cos x}{18x^2}$	
$\displaystyle = -\lim_{x\to 0}\frac{\sin x}{18x}$	
$\displaystyle = -\frac{1}{18}$	1A
$\displaystyle\lim_{x\to 0}\frac{\int_0^x f(t)dt - x}{x^3} = \lim_{x\to 0}\frac{f(x)-1}{3x^2}$	1A
$\displaystyle = \lim_{x\to 0}\frac{\frac{\sin x}{x}-1}{3x^2}$	
$\displaystyle = \lim_{x\to 0}\frac{\sin x - x}{3x^3}$	
$\displaystyle = \lim_{x\to 0}\frac{\cos x - 1}{9x^2}$	1A
$\displaystyle = \lim_{x\to 0}\frac{-\sin x}{18x}$	
$\displaystyle = -\frac{1}{18}$	1A
	----------(6)
7. (a) The greatest value of r is 2.	1A
$1-\cos 4\theta = 2$	
$\cos 4\theta = -1$	
$4\theta = (2n+1)\pi$	
$\theta = \dfrac{(2n+1)\pi}{4}$	
For $0 \le \theta \le 2\pi$,	
$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$	1A
The points are	
$(2, \frac{\pi}{4})$, $(2, \frac{3\pi}{4})$, $(2, \frac{5\pi}{4})$, $(2, \frac{7\pi}{4})$.	

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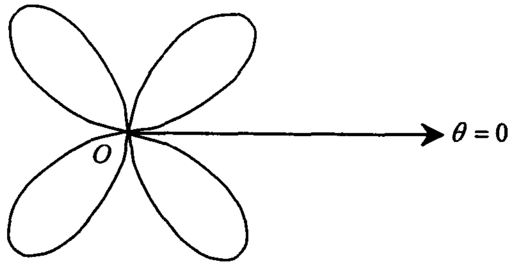
Solution	Marks
(b) $r = 0$ when $\cos 4\theta = 1$	
i.e. $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.	
	1A for 4 loops + 1A for each loop in the right quadrant
(c) The area of the shaded region is	
$\frac{1}{2}\int_0^{2\pi}(1-\cos 4\theta)^2 \, d\theta$	1A
$= \frac{1}{2}\int_0^{2\pi}(1-2\cos 4\theta + \cos^2 4\theta) \, d\theta$	
$= \frac{1}{2}\int_0^{2\pi}(\frac{3}{2} - 2\cos 4\theta + \frac{1}{2}\cos 8\theta) \, d\theta$	1M for $\cos^2 4\theta = \frac{1+\cos 8\theta}{2}$
$= \frac{1}{2}\left[\frac{3}{2}\theta - \frac{1}{2}\sin 4\theta + \frac{1}{16}\sin 8\theta\right]_0^{2\pi}$	
$= \frac{3\pi}{2}$	1A
The area of the shaded region is	
$4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 4\theta)^2 \, d\theta\right)$	1A
$= 2\int_0^{\frac{\pi}{2}}(1-2\cos 4\theta + \cos^2 4\theta) \, d\theta$	
$= 2\int_0^{\frac{\pi}{2}}(1-2\cos 4\theta + \frac{1}{2} + \frac{1}{2}\cos 8\theta) \, d\theta$	1M for $\cos^2 4\theta = \frac{1+\cos 8\theta}{2}$
$= 2\int_0^{\frac{\pi}{2}}\left(\frac{3}{2} - 2\cos 4\theta + \frac{1}{2}\cos 8\theta\right) d\theta$	
$= 2\left[\frac{3}{2}\theta - \frac{1}{2}\sin 4\theta + \frac{1}{16}\sin 8\theta\right]_0^{\frac{\pi}{2}}$	
$= \frac{3\pi}{2}$	1A
	----------(7)

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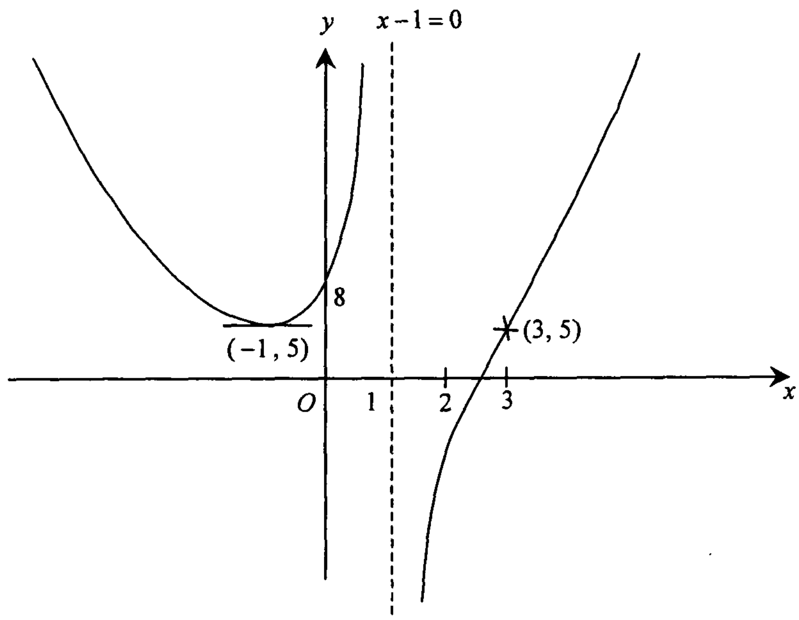
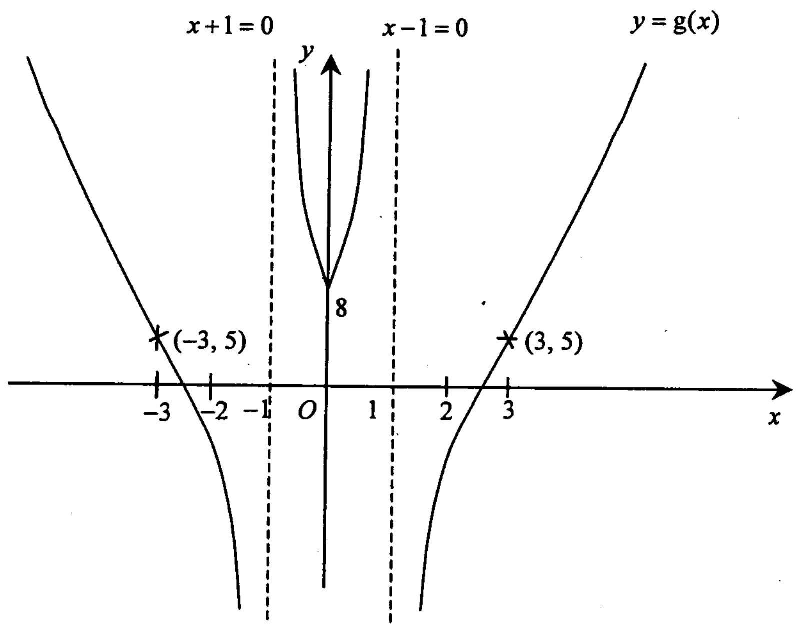
Solution	Marks
8. (a) $f(x) = x^2 - \frac{8}{x-1}$.	
$f'(x) = 2x + \frac{8}{(x-1)^2}$.	1A
$f''(x) = 2 - \frac{16}{(x-1)^3}$	1A
$= 2 + \frac{16}{(1-x)^3}$	
$f(x) = \frac{x^3 - x^2 - 8}{x-1}$	
$f'(x) = \frac{(3x^2-2x)(x-1)-(x^3-x^2-8)}{(x-1)^2}$	
$= \frac{2x^3-4x^2+2x+8}{(x-1)^2}$	1A
$f''(x) = \frac{(6x^2-8x+2)(x-1)^2-(2x^3-4x^2+2x+8)[2(x-1)]}{(x-1)^4}$	
$= \frac{2x^3-6x^2+6x-18}{(x-1)^3}$	1A
$= \frac{2(x^3-3x^2+3x-1)-16}{(x-1)^3}$	
$= \frac{2(x-1)^3-16}{(x-1)^3}$	
$= 2 - \frac{16}{(x-1)^3}$	
	----------(2)
(b) $f'(x) = 2x + \frac{8}{(x-1)^2}$	
$= \frac{2(x+1)(x^2-3x+4)}{(x-1)^2}$	
$= \frac{2(x+1)\left[(x-\frac{3}{2})^2+\frac{7}{4}\right]}{(x-1)^2}$	
(i) $f'(x) > 0$ when $x > -1$ and $x \neq 1$.	(Accept if $x \neq 1$ is omitted)
(ii) $f'(x) < 0$ when $x < -1$.	1A for (i) and (ii)
(iii) $\because$ $f''(x) = \frac{2(x-3)(x^2+3)}{(x-1)^3}$	
$\therefore$ $f''(x) > 0$ when $x < 1$ or $x > 3$.	1A
(iv) $f''(x) < 0$ when $1 < x < 3$.	1A
	----------(3)

Solution	Marks
(c) $\because$ $f'(-1)=0$ and $f''(-1)=4>0$	
$\therefore$ $f'(x)$ is minimum at $x=-1$ and the minimum value is 5 .	
$\therefore$ the minimum point is $(-1, 5)$	1A
$f''(x)=0$ when $x=3$,	
$\because$ $f''(x)<0$ when $1<x<3$	
and $f''(x)>0$ when $x>3$,	
$\therefore$ there is a point of inflection at $x=3$.	
The point of inflection is $(3, 5)$.	1A
	----------(2)
(d) $x-1=0$ is a vertical asymptote.	1A
$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\left(\frac{x^2-\frac{8}{x-1}}{x}\right)=\lim_{x\to\infty}\left[x-\frac{8}{x(x-1)}\right]$ The limit does not exist, therefore there is no oblique asymptotes.	can be omitted
	----------(1)
(e)	
	1A extreme point, point of inflexion and asymptote 1A Shape of the curve
	----------(2)

Solution	Marks
(f) (i) $g(x) = f(\|x\|) = \begin{cases} x^2 - \dfrac{8}{x-1} & \text{if } x \ge 0 \text{ and } x \ne 1 \\ x^2 + \dfrac{8}{x+1} & \text{if } x < 0 \text{ and } x \ne -1 \end{cases}$	
$\lim_{x \to 0^-} \dfrac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^-} \left[\dfrac{\dfrac{x^3 + x^2 + 8}{x+1} - 8}{x - 0} \right] = \lim_{x \to 0^-} \dfrac{x^2 + x - 8}{x+1} = -8$ $\lim_{x \to 0^+} \dfrac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^+} \left[\dfrac{\dfrac{x^3 - x^2 - 8}{x-1} - 8}{x - 0} \right] = \lim_{x \to 0^+} \dfrac{x^2 - x - 8}{x-1} = 8$	1M+1A
$g'(0)$ does not exist.	1A
(ii) $\because$ $g(-x) = f(\|-x\|) = f(\|x\|) = g(x)$ $\therefore$ $g(x)$ is an even function. $\therefore$ the graph of $g(x)$ is symmetrical about the y-axis.	
	1M for symmetry 1A
	----------(5)

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Solution	Marks
9. (a) $\int e^{-x}\sin x\,dx = \int -\sin x\,d(e^{-x}) = -e^{-x}\sin x + \int e^{-x}\cos x\,dx$	1M
$= -e^{-x}\sin x - \int \cos x\,d(e^{-x})$	
$= -e^{-x}\sin x - e^{-x}\cos x + \int e^{-x}(-\sin)\,dx$	1A
$\int e^{-x}\sin x\,dx = -\dfrac{e^{-x}}{2}[\sin x + \cos x] + C$	1A
$\int e^{-x}\sin x\,dx = -\int e^{-x}\,d(\cos x) = -e^{-x}\cos x - \int e^{-x}\cos x\,dx$	1M
$= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx$	1A
$\int e^{-x}\sin x\,dx = -\dfrac{e^{-x}}{2}[\sin x + \cos x] + C$	1A
	----------(3)
(b) (i) Let $x = y + kT$, $\int_{a+kT}^{b+kT} e^{-x}f(x)\,dx = \int_a^b e^{-(y+kT)}f(y+kT)\,dy$	1M
$= e^{-kT}\int_a^b e^{-y}f(y)\,dy$	
$= e^{-kT}\int_a^b e^{-x}f(x)\,dx$	1
(ii) $I_n = \int_0^T e^{-x}f(x)\,dx + \int_{0+T}^{T+T} e^{-x}f(x)\,dx + \ldots + \int_{0+(n-1)T}^{T+(n-1)T} e^{-x}f(x)\,dx$	1M
$= I_1 + e^{-T}I_1 + \ldots + e^{-(n-1)T}I_1$	1A
$= \dfrac{1-e^{-nT}}{1-e^{-T}}I_1$	1
When $n=1$, L.S. $= I_1 =$ R.S.	
$\therefore$ the statement is true for $n=1$.	
Assume that $I_k = \dfrac{1-e^{-kT}}{1-e^{-T}}I_1$ for some positive integer k.	
Then, $I_{k+1} = \int_0^{(k+1)T} e^{-x}f(x)\,dx$	
$= \int_0^{kT} e^{-x}f(x)\,dx + \int_{kT}^{(k+1)T} e^{-x}f(x)\,dx$	1M
$= I_k + e^{-kT}\int_0^T e^{-x}f(x)\,dx$ (by (b)(i))	1A
$= \dfrac{1-e^{-kT}}{1-e^{-T}}I_1 + e^{-kT}I_1$	
$= \dfrac{1-e^{-(k+1)T}}{1-e^{-T}}I_1$	1
Thus, by induction, $I_n = \dfrac{1-e^{-nT}}{1-e^{-T}}I_1$ for any positive integer n.	

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Solution	Marks
(iii) Since $f(x)\ge 0$, $\therefore e^{-x}f(x)\ge 0$.	
$I_n=\int_0^{nT}e^{-x}f(x)\,dx\le\int_0^{\ell}e^{-x}f(x)\,dx\le\int_0^{(n+1)T}e^{-x}f(x)\,dx=I_{n+1}$	1M
$\frac{1-e^{-nT}}{1-e^{-T}}I_1\le\int_0^{\ell}e^{-x}f(x)\,dx\le\frac{1-e^{-(n+1)T}}{1-e^{-T}}I_1$	1
When $\ell\to\infty$, $n\to\infty$ and	
$\lim_{n\to\infty}\frac{1-e^{-(n+1)T}}{1-e^{-T}}I_1=\lim_{n\to\infty}\left(\frac{1-e^{-nT}}{1-e^{-T}}\right)I_1=\frac{I_1}{1-e^{-T}}$	1A
$\therefore\int_0^{\infty}e^{-x}f(x)\,dx=\lim_{\ell\to\infty}\int_0^{\ell}e^{-x}f(x)\,dx=\frac{I_1}{1-e^{-T}}$.	1A
	----------(9)
(c) $f(x)=\lvert\sin x\rvert$, $f(x+\pi)=\lvert\sin(x+\pi)\rvert=\lvert-\sin x\rvert=f(x)$	
$\therefore$ f is periodic with period π.	1A
By (b)(iii), $\int_0^{\infty}e^{-x}\lvert\sin x\rvert\,dx=\frac{1}{1-e^{-\pi}}\int_0^{\pi}e^{-x}\sin x\,dx$	
$=\frac{1}{1-e^{-\pi}}\left[-\frac{e^{-x}}{2}(\sin x+\cos x)\right]_0^{\pi}$	1M
$=\frac{e^{-\pi}+1}{2(1-e^{-\pi})}$	1A
$=\frac{e^{\pi}+1}{2(e^{\pi}-1)}$	
$f(x)=\lvert\sin x\rvert$, $f(x+2\pi)=\lvert\sin(x+2\pi)\rvert=\lvert\sin x\rvert=f(x)$	
$\therefore$ f is periodic with period 2π.	1A
By (b)(iii),	
$\int_0^{\infty}e^{-x}\lvert\sin x\rvert\,dx=\frac{1}{1-e^{-2\pi}}\int_0^{2\pi}e^{-x}\lvert\sin x\rvert\,dx$	
$=\frac{1}{1-e^{-2\pi}}\left(\int_0^{\pi}e^{-x}\sin x\,dx-\int_{\pi}^{2\pi}e^{-x}\sin x\,dx\right)$	
$=\frac{1}{1-e^{-2\pi}}\left\{\left[\frac{-e^{-x}}{2}(\sin x+\cos x)\right]_0^{\pi}+\left[\frac{e^{-x}}{2}(\sin x+\cos x)\right]_{\pi}^{2\pi}\right\}$	1M
$=\frac{1}{1-e^{-2\pi}}\left(\frac{e^{-2\pi}+2e^{-\pi}+1}{2}\right)$	1A
$=\frac{(e^{-\pi}+1)^2}{2(1-e^{-2\pi})}$	
$=\frac{e^{-\pi}+1}{2(1-e^{-\pi})}$	
$=\frac{e^{\pi}+1}{2(e^{\pi}-1)}$	
	----------(3)

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Solution	Marks
10. (a) (i) $G(x) = \int_0^x g(t)\,dt$	
$\le \int_0^x 1\cdot dt \qquad (\because \quad 0 \le g(t) \le 1)$	
$= x$	1
$\phi'(x) = f(G(x))\cdot G'(x) - f(x)g(x)$	1M
$= f(G(x))\cdot g(x) - f(x)g(x)$	1A
Since f is decreasing and $G(x) \le x$, therefore $f(G(x)) \ge f(x)$.	
Hence $\phi'(x) \ge f(x)\cdot g(x) - f(x)g(x) = 0 \ (\because \ 0 \le g(x) \le 1)$	1
(ii) $\because \quad G(0) = \int_0^0 g(t)\,dt = 0$	
$\therefore \quad \phi(0) = \int_0^{G(0)} f(t)\,dt - \int_0^0 f(t)g(t)\,dt$	
$= \int_0^0 f(t)\,dt$	
$= 0$.	1A
Since $\phi'(x) \ge 0$ for all $x \in (0,1)$ and ϕ is continuous on $[0, 1]$,	
we have $\phi(1) \ge \phi(0) = 0$.	1A
$\therefore \quad \phi(1) = \int_0^{G(1)} f(t)\,dt - \int_0^1 f(t)g(t)\,dt \ge 0$	
$\therefore \quad \int_0^1 f(t)g(t)\,dt \le \int_0^{G(1)} f(t)\,dt$	1
	-----------(7)
(b) (i) $G(1) + H(1) = \int_0^1 g(t)\,dt + \int_0^1 [1 - g(t)]\,dt$	
$= \int_0^1 1\cdot dt$	
$= 1$	1
(ii) $\because \quad 0 \le g(t) \le 1$	
$\therefore \quad 0 \le 1 - g(t) \le 1$	1M
Using (a)(ii), (replace $g(t)$ by $1 - g(t)$)	
$\int_0^1 f(t)[1 - g(t)]\,dt \le \int_0^{H(1)} f(t)\,dt$	1A
$\int_0^1 f(t)[1 - g(t)]\,dt \le \int_0^{1-G(1)} f(t)\,dt$	1A
$\therefore \quad \int_0^1 f(t)\,dt - \int_0^{1-G(1)} f(t)\,dt \le \int_0^1 f(t)g(t)\,dt$	
i.e. $\int_{1-G(1)}^1 f(t)\,dt \le \int_0^1 f(t)g(t)\,dt$.	1
	-----------(5)

Solution	Marks
(c) Combining the results of (a)(ii) and (b)(ii), we have $$\int_{1-G(1)}^{1} f(t)\,dt \le \int_0^1 f(t)g(t)\,dt \le \int_0^{G(1)} f(t)\,dt .$$	
Put $g(x) = x^n$. Then $0 \le g(x) \le 1$ for all $x \in [0, 1]$ and $G(1) = \int_0^1 t^n\,dt = \frac{1}{n+1}$.	1A
Hence $$\int_{1-\frac{1}{n+1}}^{1} f(t)\,dt \le \int_0^1 f(t)\cdot t^n\,dt \le \int_0^{\frac{1}{n+1}} f(t)\,dt ,$$ i.e. $$\int_{\frac{n}{n+1}}^{1} f(t)\,dt \le \int_0^1 f(t)\cdot t^n\,dt \le \int_0^{\frac{1}{n+1}} f(t)\,dt .$$	1
$\because \lim_{n\to\infty} \frac{n}{n+1} = 1$ and $\lim_{n\to\infty} \frac{1}{n+1} = 0$ $\therefore \lim_{n\to\infty} \int_{\frac{n}{n+1}}^{1} f(t)\,dt = \lim_{n\to\infty} \int_0^{\frac{1}{n+1}} f(t)\,dt = 0$	
Thus, $\lim_{n\to\infty} \int_0^1 f(t)\cdot t^n\,dt = 0$.	1 -----------(3)

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Solution	Marks
11. (a) For C_2, $\frac{dy}{dx}=\frac{2}{y}$.	
The equation of PS is	
$y-2s=\frac{2}{2s}(x-s^2)$	1M+1A
$x-sy+s^2=0$	
Similarly the equation of PT is	
$x-ty+t^2=0$	
P is on the tangents,	
$(p^2-1)-2ps+s^2=0$	
$(p^2-1)-2pt+t^2=0$.	
$\therefore$ s, t are the roots of the equation	
$(p^2-1)-2px+x^2=0$.	1M
Hence $s+t=2p$	
and $st=p^2-1$	1
The equation of PS is	
$\frac{y-2s}{x-s^2}=\frac{2p-2s}{p^2-1-s^2}$	
$(2p-2s)x+(s^2+1-p^2)y+(2sp^2-2ps^2-2s)=0$	
Similarly the equation of PT is	1A
$(2p-2t)x+(t^2+1-p^2)y+(2tp^2-2pt^2-2t)=0$	
Note that $2y\frac{dy}{dx}=4 \Rightarrow \frac{dy}{dx}=\frac{2}{y}$	
$\therefore$ $\left.\frac{dy}{dx}\right\|_S=\frac{2}{2s}=\frac{1}{s}$ and $\left.\frac{dy}{dx}\right\|_T=\frac{2}{2t}=\frac{1}{t}$	
$\therefore$ $\frac{2p-2s}{p^2-1-s^2}=\frac{1}{s}$ and $\frac{2p-2t}{p^2-1-t^2}=\frac{1}{t}$	1M
$\therefore$ $s^2-2ps+(p^2-1)=0$ and $t^2-2pt+(p^2-1)=0$	
$\therefore$ s and t are the roots of $x^2-2px+(p^2-1)=0$	1M
Hence $s+t=2p$ and $st=p^2-1$	1
	----------(4)

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Solution	Marks
(b) $\begin{vmatrix} q^2 & 2q & 1 \\ s^2 & 2s & 1 \\ t^2 & 2t & 1 \end{vmatrix} = 2\left\{q^2\begin{vmatrix} s & 1 \\ t & 1 \end{vmatrix} - q\begin{vmatrix} s^2 & 1 \\ t^2 & 1 \end{vmatrix} + \begin{vmatrix} s^2 & s \\ t^2 & t \end{vmatrix}\right\}$	
$= 2\{(s-t)q^2 - (s^2-t^2)q + st(s-t)\}$	
$= 2(s-t)[q^2 - (s+t)q + st]$	
Area of $\Delta SQT = \lvert s-t\rvert \cdot \lvert q^2 - (s+t)q + st\rvert$	1M+1A
Using the results of (a),	
$\lvert s-t\rvert = \sqrt{(s+t)^2 - 4st}$	
$= \sqrt{(2p)^2 - 4(p^2-1)}$	1M
$= 2$	
Note that area of $\Delta SQT = 2\lvert q^2 - 2pq + p^2 - 1\rvert$	
$= 2\lvert (q-p)^2 - 1\rvert$.	1A
$\because$ $p = \frac{s+t}{2}$, $(q-p)^2 \le 1$ and $\lvert s-t\rvert = 2$.	
$\therefore$ the area is maximum if and only if $q = p$.	2
The area of ΔSQT is a maximum if and only if the tangent to C_2 at Q is parallel to ST.	2A
The equation of ST is	
$y - 2s = \frac{2(t-s)}{(t^2-s^2)}(x - s^2)$	1M
$(t+s)(y-2s) = 2(x-s^2)$	
$(s+t)y = 2x + 2st$	
Using the results of (a),	
$py = x + (p^2-1)$ ....(1)	1M
The equation of the tangent to C_2 is	
$x - qy + q^2 = 0$ ....(2)	1A
Comparing the equations (1) and (2), we get $q = p$.	1
	----------(6)
(c) The equation of ST is	
$y - 2s = \frac{2(t-s)}{(t^2-s^2)}(x - s^2)$	
$(s+t)(y-2s) = 2(x-s^2)$	
$(s+t)y = 2x + 2st$	1A
Using the results of (a) ,	
$py = x + (p^2-1)$ ....(1)	1M
The equation of PQ is	
$y = 2p$ ....(2)	1A
Eliminating p from (1) and (2) ,	1M
$\left(\frac{y}{2}\right)y = x + \left(\frac{y}{2}\right)^2 - 1$	
$y^2 = 4(x-1)$	1
	------------(5)

Solution	Marks
12. (a) By Mean Value Theorem ,	
$\frac{g(b)-g(a)}{b-a} = g'(c)$ for some $c \in (a, b)$.	1M
Since $g(a) = g(b) = 0$, hence $g'(c) = 0$.	1A
Since $g'(x)$ is decreasing on (a, b) ,	
$g'(x) \geq 0$ for all $x \in (a, c)$ and $g'(x) \leq 0$ for all $x \in (c, b)$	
$\therefore$ g is increasing on (a, c) and decreasing on (c, b)	1
$\therefore$ $g(x) \geq g(a)$ for all $x \in (a, c)$ and $g(x) \geq g(b)$ for all $x \in (c, b)$	
$\therefore$ $g(x) \geq 0$ for all $x \in (a, c)$ and $g(x) \geq 0$ for all $x \in (c, b)$	1M
By the continuity of g ,	
$g(x) \geq 0$ for all $x \in [a, b]$	1
By Mean Value Theorem ,	
$\frac{g(b)-g(a)}{b-a} = g'(c)$ for some $c \in (a, b)$.	1M
Since $g(a) = g(b) = 0$, hence $g'(c) = 0$.	1A
Let x_1 , $x_2 \in (a, c)$ with $x_1 < x_2$.	
Using Mean Value Theorem ,	
$\frac{g(x_2)-g(x_1)}{x_2 - x_1} = g'(\xi)$ for some $\xi \in (x_1, x_2)$.	
Since $a < x_1 < \xi < x_2 < c$ and $g'(x)$ is decreasing, therefore	
$g'(\xi) \geq g'(c) = 0$.	
Hence $g(x_2) \geq g(x_1)$, i.e. $g(x)$ is increasing on (a, c)	
Let x_3 , $x_4 \in (c, b)$ with $x_3 < x_4$.	
Using Mean Value Theorem,	
$\frac{g(x_4)-g(x_3)}{x_4 - x_3} = g'(\eta)$ for some $\eta \in (x_3, x_4)$.	
Since $c < x_3 < \eta < x_4 < b$ and $g'(x)$ is decreasing, therefore	
$g'(\eta) \leq g'(c) = 0$.	
Hence $g(x_4) \leq g(x_3)$, i.e. $g(x)$ is decreasing on (c, b) .	1
Since $g(x)$ is increasing on (a, c) ,	
$\therefore$ $g(x) \geq g(a) = 0$ for all $a < x \leq c$.	
Similarly, $\because$ $g(x)$ is decreasing on (c, b) ,	1M
$\therefore$ $g(x) \geq g(b) = 0$ for all $c \leq x < b$.	
Hence $g(x) \geq 0$ for all $a \leq x \leq b$.	1
	-----------(5)

Solution	Marks
(b) $g(x) = (b-x)f(a) + (x-a)f(b) - (b-a)f(x)$	
$g'(x) = -f(a) + f(b) - (b-a)f'(x)$	
$g''(x) = -(b-a)f''(x) \le 0 \quad (\because f''(x) \ge 0)$	
Furthermore $g(a) = g(b) = 0$,	
using the result of (a), $g(x) \ge 0$ for all $x \in \mathbf{I}$.	
Hence $(b-x)f(a) + (x-a)f(b) - (b-a)f(x) \ge 0$.	1M for checking the conditions
Thus $f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$.	1
Without loss of generality, we may assume that $x_1 < x_2$ and $\lambda_1, \lambda_2 > 0$.	
Let $x = \lambda_1 x_1 + \lambda_2 x_2 = \lambda_1 x_1 + (1-\lambda_1)x_2$	
Then $x_1 < x < x_2$	1M for checking $x \in \mathbf{I}$ +1A
Sub. into $f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$,	
$f(\lambda_1 x_1 + \lambda_2 x_2)$	
$= f(x)$	
$\le \frac{x_2 - [\lambda_1 x_1 + (1-\lambda_1)x_2]}{x_2 - x_1} f(x_1) + \frac{[\lambda_1 x_1 + (1-\lambda_1)x_2] - x_1}{x_2 - x_1} f(x_2)$	
$= \frac{-\lambda_1 x_1 + \lambda_1 x_2}{x_2 - x_1} f(x_1) + \frac{(\lambda_1 - 1)x_1 + (1-\lambda_1)x_2}{x_2 - x_1} f(x_2)$	
$= \lambda_1 f(x_1) + \lambda_2 f(x_2)$	1
	----------(5)
(c) Let $f(x) = -\ln x$	1A
$f'(x) = -\frac{1}{x}$	
$f''(x) = \frac{1}{x^2} > 0$ for all $x > 0$.	1A for checking condition
(i) When $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 + \lambda_2 = 1$,	
using the result of (b)	
$-\ln(\lambda_1 x_1 + \lambda_2 x_2) \le -\lambda_1 \ln x_1 - \lambda_2 \ln x_2 = -\ln(x_1^{\lambda_1} x_2^{\lambda_2})$.	
Thus $\lambda_1 x_1 + \lambda_2 x_2 \ge e^{\ln x_1^{\lambda_1} x_2^{\lambda_2}} = x_1^{\lambda_1} x_2^{\lambda_2}$.	1
(ii) Putting $\lambda_k = \frac{\beta_k}{\beta_1 + \beta_2}$, where $\beta_k > 0$ and $k = 1, 2$ in (i),	1A
then $\lambda_k > 0$ and $\lambda_1 + \lambda_2 = 1$.	
$\therefore \frac{\beta_1 x_1 + \beta_2 x_2}{\beta_1 + \beta_2} \ge \left(x_1^{\beta_1} x_2^{\beta_2}\right)^{\frac{1}{\beta_1 + \beta_2}}$	
i.e. $\left(\frac{\beta_1 x_1 + \beta_2 x_2}{\beta_1 + \beta_2}\right)^{\beta_1 + \beta_2} \ge x_1^{\beta_1} x_2^{\beta_2}$	1
	----------(5)

Solution	Marks
13. (a) (i) $I_{2n}(\theta)=\int_0^{\theta}\tan^{2n-2}u(\sec^2 u-1)\,du$	
$=\int_0^{\theta}\tan^{2n-2}u\sec^2 u\,du-\int_0^{\theta}\tan^{2n-2}u\,du$	1M
$=\left[\dfrac{\tan^{2n-1}u}{2n-1}\right]_0^{\theta}-I_{2n-2}(\theta)$	
$=\dfrac{\tan^{2n-1}\theta}{2n-1}-I_{2n-2}(\theta)$	1
(ii) $\int_0^x\dfrac{t^{2n}}{1+t^2}dt=\int_0^{\tan^{-1}x}\dfrac{\tan^{2n}u}{1+\tan^2 u}\cdot\sec^2 u\,du=\int_0^{\tan^{-1}x}\tan^{2n}u\,du$	1A
$=\dfrac{\tan^{2n-1}(\tan^{-1}x)}{2n-1}-I_{2n-2}(\tan^{-1}x)$	
$=\dfrac{x^{2n-1}}{2n-1}-I_{2n-2}(\tan^{-1}x)$	
$=\dfrac{x^{2n-1}}{2n-1}-\left(\dfrac{x^{2n-3}}{2n-3}-I_{2n-4}(\tan^{-1}x)\right)$	1A
$=\dfrac{x^{2n-1}}{2n-1}-\dfrac{x^{2n-3}}{2n-3}+\cdots+(-1)^{n-1}\dfrac{x}{1}+(-1)^n I_0$	
$\because\ I_0=\int_0^{\tan^{-1}x}du=\tan^{-1}x$	
$\therefore\ \int_0^x\dfrac{t^{2n}}{1+t^2}dt=\dfrac{x^{2n-1}}{2n-1}-\dfrac{x^{2n-3}}{2n-3}+\cdots+(-1)^{n-1}\dfrac{x}{1}+(-1)^n\tan^{-1}x$	1
	----------(5)
(b) (i) For $0\le t\le x$,	
$\dfrac{t^{2n}}{1+x^2}\le\dfrac{t^{2n}}{1+t^2}\le t^{2n}$	
$\therefore\ \int_0^x\dfrac{t^{2n}}{1+x^2}dt\le\int_0^x\dfrac{t^{2n}}{1+t^2}dt\le\int_0^x t^{2n}dt$	
$\dfrac{1}{1+x^2}\cdot\left[\dfrac{t^{2n+1}}{2n+1}\right]_0^x\le\int_0^x\dfrac{t^{2n}}{1+t^2}dt\le\left[\dfrac{t^{2n+1}}{2n+1}\right]_0^x$	
$\dfrac{x^{2n+1}}{(2n+1)(1+x^2)}\le\int_0^x\dfrac{t^{2n}}{1+t^2}dt\le\dfrac{x^{2n+1}}{2n+1}$	1+1
(ii) Using (a)(ii), we have	
$\tan^{-1}x+(-1)^n\left(\dfrac{x^{2n-1}}{2n-1}-\dfrac{x^{2n-3}}{2n-3}+\cdots+(-1)^{n-1}\dfrac{x}{1}\right)=(-1)^n\int_0^x\dfrac{t^{2n}}{1+t^2}dt$	1A
$\tan^{-1}x-\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\cdots+\dfrac{(-1)^{n-1}x^{2n-1}}{2n-1}\right)=(-1)^n\int_0^x\dfrac{t^{2n}}{1+t^2}dt$	
Hence, for $x\ge 0$	
$\left\lvert\tan^{-1}x-\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\cdots+\dfrac{(-1)^{n-1}x^{2n-1}}{2n-1}\right)\right\rvert=\int_0^x\dfrac{t^{2n}}{1+t^2}dt$	
Using the result of (b)(i) and putting $x=1$,	
$\dfrac{1}{2(2n+1)}\le\left\lvert\dfrac{\pi}{4}-\left(1-\dfrac{1}{3}+\dfrac{1}{5}-\cdots+\dfrac{(-1)^{n-1}}{2n-1}\right)\right\rvert\le\dfrac{1}{2n+1}$	1

Solution	Marks

(iii) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{5}{12}$ and $\tan 4\alpha = \dfrac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \dfrac{120}{119}$ — 1A+1A

$$\tan\left(4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}\right)$$

$$= \frac{\tan\left(4\tan^{-1}\frac{1}{5}\right) - \tan\left(\tan^{-1}\frac{1}{239}\right)}{1+\tan\left(4\tan^{-1}\frac{1}{5}\right)\tan\left(\tan^{-1}\frac{1}{239}\right)}$$

$$= \frac{\frac{120}{119} - \frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}} = 1 \quad \text{and} \quad 0 < 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) < \pi$$

$$\therefore \frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$ — 1

Using

$$\left|\tan^{-1}x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1}\right)\right| = \int_0^x \frac{t^{2n}}{1+t^2}\,dt \quad \text{for all } x \ge 0,$$

$$\left|\tan^{-1}\frac{1}{5} - \sum_{p=1}^{n}\frac{(-1)^{p-1}}{2p-1}\cdot\left(\frac{1}{5}\right)^{2p-1}\right| = \int_0^{\frac{1}{5}} \frac{t^{2n}}{1+t^2}\,dt$$

$$\le \frac{\left(\frac{1}{5}\right)^{2n+1}}{2n+1} = \frac{1}{(2n+1)\cdot 5^{2n+1}}$$

Similarly,

$$\left|\tan^{-1}\frac{1}{239} - \sum_{p=1}^{n}\frac{(-1)^{p-1}}{2p-1}\cdot\left(\frac{1}{239}\right)^{2p-1}\right| \le \frac{1}{(2n+1)\cdot 239^{2n+1}}$$

— 1A (either one)

Since $\dfrac{\pi}{4} = 4\tan^{-1}\left(\dfrac{1}{5}\right) - \tan^{-1}\left(\dfrac{1}{239}\right)$,

$$\therefore \left|\frac{\pi}{4} - \sum_{p=1}^{n}\frac{(-1)^{p-1}}{2p-1}\cdot\left(\frac{4}{5^{2p-1}} - \frac{1}{239^{2p-1}}\right)\right|$$

$$= \left|4\left(\tan^{-1}\frac{1}{5} - \sum_{p=1}^{n}\frac{(-1)^{p-1}}{2p-1}\cdot\frac{1}{5^{2p-1}}\right) - \left(\tan^{-1}\frac{1}{239} - \sum_{p=1}^{n}\frac{(-1)^{p-1}}{2p-1}\cdot\frac{1}{239^{2p-1}}\right)\right|$$

$$\le 4\left|\tan^{-1}\frac{1}{5} - \sum_{p=1}^{n}\frac{(-1)^{p-1}}{2p-1}\cdot\frac{1}{5^{2p-1}}\right| + \left|\tan^{-1}\frac{1}{239} - \sum_{p=1}^{n}\frac{(-1)^{p-1}}{2p-1}\cdot\frac{1}{239^{2p-1}}\right|$$ — 1M

$$\le 4\cdot\frac{1}{(2n+1)\cdot 5^{2n+1}} + \frac{1}{(2n+1)\cdot 239^{2n+1}}$$

$$= \frac{1}{(2n+1)}\left(\frac{4}{5^{2n+1}} + \frac{1}{239^{2n+1}}\right)$$ — 1

----------(10)