2002-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2002

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A and any FOUR questions in Section B.
3. You are provided with one AL(E) answer book and four AL(D) answer books.

   Section A : Write your answers in the AL(E) answer book.

   Section B : Use a separate AL(D) answer book for each question and put the question number on the front cover of each answer book.
4. The four AL(D) answer books should be tied together with the green tag provided. The AL(E) answer book and the four AL(D) answer books must be handed in separately at the end of the examination.
5. Unless otherwise specified, all working must be clearly shown.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. A sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 3$ and $a_{n+2} = 2a_{n+1} + a_n$ for $n = 1, 2, 3, \cdots$. Prove by mathematical induction that

$$a_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} \quad \text{for} \quad n = 1, 2, 3, \cdots .$$

(5 marks)

2. (a) Express $\left|\frac{z-1}{z-4}\right| = 2$ in the form of $|z-c| = r$, where c and r are constants.

(b) Shade the region represented by $\left\{z \in \mathbf{C} : \left|\frac{z-1}{z-4}\right| < 2\right\}$ in the Argand plane.

(5 marks)

3. (a) Write down the matrix A representing the rotation in the Cartesian plane anticlockwise about the origin by $45°$.

(b) Write down the matrix B representing the enlargement in the Cartesian plane with scale factor $\sqrt{2}$.

(c) Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $V = BAX$, where A and B are the matrices defined in (a) and (b). If $V^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V = -4$, express y in terms of x .

(5 marks)

4. Let $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ and $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$, $\mathbf{c} = \mathbf{j} + \mathbf{k}$.

(a) Prove that $\mathbf{a}$ is not perpendicular to $\mathbf{b} \times \mathbf{c}$.

(b) Find all unit vectors which are perpendicular to both $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$.

(c) If $\theta \in [0, \pi]$ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, prove that $\frac{\pi}{4} < \theta < \frac{\pi}{3}$.

(6 marks)

5. (a) Let $\mathrm{f}(x)$ and $\mathrm{g}(x)$ be polynomials.

Prove that a non-zero polynomial $\mathrm{u}(x)$ is a common factor of $\mathrm{f}(x)$ and $\mathrm{g}(x)$ if and only if $\mathrm{u}(x)$ is a common factor of $\mathrm{f}(x) - \mathrm{g}(x)$ and $\mathrm{g}(x)$.

(b) Let $\mathrm{f}(x) = x^4 - 3x^3 + 6x^2 - 5x + 3$ and $\mathrm{g}(x) = x^4 - 4x^3 + 8x^2 - 7x + 4$.
Using (a) or otherwise, find the H.C.F. of $\mathrm{f}(x)$ and $\mathrm{g}(x)$.

(7 marks)

6. For $k = 1, 2, 3$, let $z_k = \cos\theta_k + i\sin\theta_k$ be complex numbers, where $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

(a) Evaluate $z_1 z_2 z_3$.

(b) Prove that $\cos\theta_k = \frac{1}{2}(z_k + \frac{1}{z_k})$ and $\cos 2\theta_k = \frac{1}{2}(z_k{}^2 + \frac{1}{z_k{}^2})$.

Hence or otherwise, prove that
$\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3 = 4\cos\theta_1 \cos\theta_2 \cos\theta_3 - 1$.

(6 marks)

Go on to the next page

7. (a) Let m and k be positive integers and $k \le m$.

Prove that $\dfrac{m(m-1)\cdots(m-k+1)}{m^k} < \dfrac{(m+1)m\cdots(m-k+2)}{(m+1)^k}$ for $k > 1$.

Prove that the above inequality does not hold when $k = 1$.

(b) Let m be a positive integer.

Using (a) or otherwise, prove that $(1+\dfrac{1}{m})^m < (1+\dfrac{1}{m+1})^{m+1}$.

(6 marks)

Go on to the next page

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

8. (a) Consider the system of linear equations in x , y , z

$$(S):\begin{cases} ax - 2y + z = 0 \\ x - y + 2z = b \\ y + az = b \end{cases}, \text{ where } a,\ b \in \mathbf{R} .$$

(i) Show that (S) has a unique solution if and only if $a^2 \ne 1$. Solve (S) in this case.

(ii) For each of the following cases, determine the value(s) of b for which (S) is consistent, and solve (S) for such value(s) of b .

(1) $a = 1$,

(2) $a = -1$.

(9 marks)

(b) Consider the system of linear equations in x , y , z

$$(T):\begin{cases} ax - 2y + z = 0 \\ x - y + 2z = -1 \\ y + az = -1 \\ 5x - 2y + z = a \end{cases}, \text{ where } a \in \mathbf{R} .$$

Find all the values of a for which (T) is consistent.
Solve (T) for each of these values of a .

(6 marks)

9. Vectors **u** , **v** and **w** in $\mathbf{R}^3$ are said to be orthogonal if and only if $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{u}=0$.

(a) (i) Show that if **u** , **v** and **w** are non-zero orthogonal vectors, then **u** , **v** and **w** are linearly independent.

(ii) Give a counter example to show that the converse of the statement in (i) is not true.

(6 marks)

(b) Let $\mathbf{u}=(u_1,\ u_2,\ u_3)$, $\mathbf{v}=(v_1,\ v_2,\ v_3)$ and $\mathbf{w}=(w_1,\ w_2,\ w_3)$ be three non-zero orthogonal vectors in $\mathbf{R}^3$.

(i) By computing the product

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix},$$

show that

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}\neq 0 .$$

(ii) Let $\mathbf{p}=(p_1\ ,\ p_2\ ,\ p_3)$ be a vector in $\mathbf{R}^3$. Show that **p** is a linear combination of **u** , **v** and **w**.

(6 marks)

(c) Let $\mathbf{x}=(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}})$, $\mathbf{y}=(0,1,0)$, $\mathbf{z}=(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$ and $\mathbf{q}=(-1,2,0)$ be vectors in $\mathbf{R}^3$.

(i) Show that **x** , **y** and **z** are orthogonal.

(ii) Express **q** as a linear combination of **x** , **y** and **z** .

(3 marks)

Go on to the next page

10. (a) Let $a_1, a_2, \ldots, a_n$ be real numbers and $b_1, b_2, \ldots, b_n$ be non-zero real numbers .

By considering $\sum_{i=1}^{n}(a_i x+b_i)^2$, or otherwise, prove Schwarz's inequality $\left(\sum_{i=1}^{n}a_ib_i\right)^2\leq\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)$, and that the equality holds if and only if $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}$.

(6 marks)

(b) (i) Prove that $\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)^2\leq\frac{\sum_{i=1}^{n}x_i^2}{n}$, where $x_1, x_2, \ldots, x_n$ are real numbers.

(ii) Prove that $\left(\sum_{i=1}^{n}\lambda_i x_i\right)^2\leq\left(\sum_{i=1}^{n}\lambda_i\right)\left(\sum_{i=1}^{n}\lambda_i x_i^2\right)$, where $x_1, x_2, \ldots, x_n$ are real numbers and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive numbers.

Find a necessary and sufficient condition for the equality to hold.

(iii) Using (b)(ii) or otherwise, prove that

$\left(\frac{y_1}{t}+\frac{y_2}{t^2}+\cdots+\frac{y_n}{t^n}\right)^2<\frac{y_1^{\ 2}}{t}+\frac{y_2^{\ 2}}{t^2}+\cdots+\frac{y_n^{\ 2}}{t^n}$, where $y_1, y_2, \ldots, y_n$ are real numbers, not all zero, and $t\geq 2$.

(9 marks)

11. (a) Let $f(x) = x^3 - 3px + 1$, where $p \in \mathbf{R}$.

(i) Show that the equation $f(x) = 0$ has at least one real root.

(ii) Using differentiation or otherwise, show that if $p \le 0$, then the equation $f(x) = 0$ has one and only one real root.

(iii) If $p > 0$, find the range of values of p for each of the following cases:

(1) the equation $f(x) = 0$ has exactly one real root,

(2) the equation $f(x) = 0$ has exactly two distinct real roots,

(3) the equation $f(x) = 0$ has three distinct real roots.

(9 marks)

(b) Let $g(x) = x^4 + 4x + a$, where $a \in \mathbf{R}$.

(i) Prove that the equation $g(x) = 0$ has at most two real roots.

(ii) Prove that the equation $g(x) = 0$ has two distinct real roots if and only if $a < 3$.

(6 marks)

Go on to the next page

12. (a) Let A be a 3×3 matrix such that

$$A^3 + A^2 + A + I = 0 ,$$

where I is the 3×3 identity matrix.

(i) Prove that A has an inverse, and find A^{-1} in terms of A .

(ii) Prove that $A^4 = I$.

(iii) Prove that $(A^{-1})^3 + (A^{-1})^2 + A^{-1} + I = 0$.

(iv) Find a 3×3 invertible matrix B such that $B^3 + B^2 + B + I \neq 0$.

(6 marks)

(b) Let $X = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$.

(i) Using (a)(i) or otherwise, find X^{-1} .

(ii) Let n be a positive integer. Find X^n .

(iii) Find two 3×3 matrices Y and Z , other than X , such that $Y^3 + Y^2 + Y + I = 0$, $Z^3 + Z^2 + Z + I = 0$.

(9 marks)

13. Let $\{x_n\}$ be a sequence of real numbers such that $x_1 > x_2$ and $3x_{n+2} - x_{n+1} - 2x_n = 0$ for $n = 1, 2, 3 \ldots$.

(a) (i) Show that for $n \geq 1$, $x_{n+2} - x_n = (-1)^n \cdot \dfrac{2^{n-1}}{3^n}(x_1 - x_2)$.

(ii) Show that the sequence $\{x_1, x_3, x_5, \ldots\}$ is strictly decreasing and that the sequence $\{x_2, x_4, x_6, \ldots\}$ is strictly increasing.

(5 marks)

(b) (i) For any positive integer n , show that $x_{2n} < x_{2n-1}$.

(ii) Show that the sequences $\{x_1, x_3, x_5, \ldots\}$ and $\{x_2, x_4, x_6, \ldots\}$ converge to the same limit.

(6 marks)

(c) By considering $\sum_{n=1}^{p}(x_{n+2} - x_n)$ or otherwise, find $\lim\limits_{n\to\infty} x_n$ in terms of x_1 and x_2 .

[You may use the fact, without proof, that from (b)(ii) , $\lim\limits_{n\to\infty} x_n$ exists.]

(4 marks)

END OF PAPER

2002-AL
P MATH
PAPER 2

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2002

PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A and any FOUR questions in Section B.
3. You are provided with one AL(E) answer book and four AL(D) answer books.

   Section A : Write your answers in the AL(E) answer book.

   Section B : Use a separate AL(D) answer book for each question and put the question number on the front cover of each answer book.
4. The four AL(D) answer books should be tied together with the green tag provided. The AL(E) answer book and the four AL(D) answer books must be handed in separately at the end of the examination.
5. Unless otherwise specified, all working must be clearly shown.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Find the indefinite integral $\int \frac{3-x}{(1+x)(1+x^2)} \mathrm{d}x$.

   Hence evaluate the improper integral $\int_0^{\infty} \frac{3-x}{(1+x)(1+x^2)} \mathrm{d}x$.

   (6 marks)

2. Let $\mathrm{f}(x) = \begin{cases} ax & \text{if } x \le \frac{\pi}{2} , \\ e^{bx} \sin x & \text{if } x > \frac{\pi}{2} . \end{cases}$

   If f is continuous at $\frac{\pi}{2}$, show that $\frac{a\pi}{2} = e^{\frac{b\pi}{2}}$.

   Furthermore, if f is differentiable at $\frac{\pi}{2}$, find the values of a and b .

   (5 marks)

3. Let $\mathrm{f} : \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying the following conditions:

   (i) $\lim_{x \to 0} \frac{\mathrm{f}(x) - 1}{x} = 1$;

   (ii) $\mathrm{f}(x+y) = \mathrm{f}(x)\,\mathrm{f}(y)$ for all x and y .

   (a) Prove that $\mathrm{f}'(x)$ exists and $\mathrm{f}'(x) = \mathrm{f}(x)$ for every $x \in \mathbf{R}$.

   (b) By considering the derivative of $\frac{\mathrm{f}(x)}{e^x}$, show that $\mathrm{f}(x) = e^x$.

   (5 marks)

Go on to the next page

4. The equation of a straight line ℓ in $\mathbf{R}^3$ is

$$\begin{cases} x = t\sqrt{3} \\ y = t\sqrt{3} \\ z = 3-t\sqrt{2} \end{cases}, \quad t \in \mathbf{R}.$$

Let A and B be two points on ℓ with $OA = OB = r$, where O is the origin.

(a) Express the distance between A and B in terms of r.

(b) If ΔOAB is an equilateral triangle, find the value of r.

(5 marks)

5. (a) If the function $\mathrm{g}: \mathbf{R} \to \mathbf{R}$ is both even and odd, show that $\mathrm{g}(x) = 0$ for all $x \in \mathbf{R}$.

(b) For any function $\mathrm{f}: \mathbf{R} \to \mathbf{R}$, define

$\mathrm{F}(x) = \frac{1}{2}[\mathrm{f}(x) + \mathrm{f}(-x)]$ and $\mathrm{G}(x) = \frac{1}{2}[\mathrm{f}(x) - \mathrm{f}(-x)]$.

(i) Show that F is an even function and G is an odd function.

(ii) If $\mathrm{f}(x) = \mathrm{M}(x) + \mathrm{N}(x)$ for all $x \in \mathbf{R}$, where M is even and N is odd, show that $\mathrm{M}(x) = \mathrm{F}(x)$ and $\mathrm{N}(x) = \mathrm{G}(x)$ for all $x \in \mathbf{R}$.

(6 marks)

6. (a) Find $\lim_{x \to 0^+} \left(\frac{1}{x}\right)^{\sin x}$.

(b) Let $\mathrm{f}(t) = \begin{cases} \frac{\sin t}{t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$

Find $\lim_{x \to 0} \frac{\int_0^x \mathrm{f}(t)\,\mathrm{d}t - x}{x^3}$.

(6 marks)

7. Let Γ be the curve with polar equation $r = 1 - \cos 4\theta$, $0 \le \theta \le 2\pi$.

(a) Find the polar coordinates of all the points on Γ that are farthest from the pole O.

(b) Sketch the curve Γ.

(c) Find the area enclosed by Γ.

(7 marks)

Go on to the next page

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

8. Let $f(x) = x^2 - \dfrac{8}{x-1}$ $(x \neq 1)$.

(a) Find $f'(x)$ and $f''(x)$.

(2 marks)

(b) Determine the range of values of x for each of the following cases:

(i) $f'(x) > 0$,

(ii) $f'(x) < 0$,

(iii) $f''(x) > 0$,

(iv) $f''(x) < 0$.

(3 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of $f(x)$.

(2 marks)

(d) Find the asymptote(s) of the graph of $f(x)$.

(1 mark)

(e) Sketch the graph of $f(x)$.

(2 marks)

(f) Let $g(x) = f(|x|)$ $(|x| \neq 1)$.

(i) Is $g(x)$ differentiable at $x = 0$? Why?

(ii) Sketch the graph of $g(x)$.

(5 marks)

9. (a) Find $\int e^{-x} \sin x \, dx$.

(3 marks)

(b) Let $f : \mathbf{R} \to [0, \infty)$ be a periodic function with period T .

(i) Prove that $\displaystyle\int_{a+kT}^{b+kT} e^{-x} f(x) dx = e^{-kT} \int_a^b e^{-x} f(x) dx$ for any positive integer k .

(ii) Let $I_n = \displaystyle\int_0^{nT} e^{-x} f(x) dx$.

Prove that $I_n = \dfrac{1 - e^{-nT}}{1 - e^{-T}} I_1$ for any positive integer n .

(iii) If ℓ is a positive number and n is a positive integer such that $nT \leq \ell \leq (n+1)T$, prove that

$$\frac{1 - e^{-nT}}{1 - e^{-T}} I_1 \leq \int_0^{\ell} e^{-x} f(x) dx \leq \frac{1 - e^{-(n+1)T}}{1 - e^{-T}} I_1 \ .$$

Hence find the improper integral $\displaystyle\int_0^{\infty} e^{-x} f(x) dx$ in terms of I_1 and T .

(9 marks)

(c) Using the results of (a) and (b)(iii), evaluate $\displaystyle\int_0^{\infty} e^{-x} |\sin x| dx$.

(3 marks)

Go on to the next page

10. Let f and g be continuous functions defined on $[0,1]$ such that f is decreasing and $0 \le \mathrm{g}(x) \le 1$ for all $x \in [0,1]$.

For $x \in [0,1]$, define $\mathrm{G}(x) = \int_0^x \mathrm{g}(t)\mathrm{d}t$ and

$\phi(x) = \int_0^{\mathrm{G}(x)} \mathrm{f}(t)\mathrm{d}t - \int_0^x \mathrm{f}(t)\mathrm{g}(t)\mathrm{d}t$.

(a) (i) Prove that $\mathrm{G}(x) \le x$. Hence prove that $\phi'(x) \ge 0$ for all $x \in (0,1)$.

(ii) Evaluate $\phi(0)$ and hence prove that $\int_0^1 \mathrm{f}(t)\mathrm{g}(t)\mathrm{d}t \le \int_0^{\mathrm{G}(1)} \mathrm{f}(t)\mathrm{d}t$.

(7 marks)

(b) Let $\mathrm{H}(x) = \int_0^x [1-\mathrm{g}(t)]\mathrm{d}t$ for all $x \in [0,1]$.

(i) Prove that $\mathrm{G}(1) + \mathrm{H}(1) = 1$.

(ii) Using (a)(ii), prove that $\int_{1-\mathrm{G}(1)}^1 \mathrm{f}(t)\mathrm{d}t \le \int_0^1 \mathrm{f}(t)\mathrm{g}(t)\mathrm{d}t$.

(5 marks)

(c) Using the results of (a)(ii) and (b)(ii), prove that

$\int_{\frac{n}{n+1}}^1 \mathrm{f}(t)\mathrm{d}t \le \int_0^1 \mathrm{f}(t)\cdot t^n \mathrm{d}t \le \int_0^{\frac{1}{n+1}} \mathrm{f}(t)\mathrm{d}t$, where n is a positive integer.

Hence show that $\lim_{n\to\infty} \int_0^1 \mathrm{f}(t)\cdot t^n \mathrm{d}t = 0$.

(3 marks)

11. Consider the parabolas $C_1 : y^2 = 4(x+1)$ and $C_2 : y^2 = 4x$.
Let $P(p^2 - 1, 2p)$ be a point on C_1 . The two tangents drawn from P to C_2 touch C_2 at the points $S(s^2, 2s)$ and $T(t^2, 2t)$.

(a) Find the equations of PS and PT and hence show that $s + t = 2p$, $st = p^2 - 1$.

(4 marks)

(b) $Q(q^2, 2q)$ is a point on the arc ST of C_2 . Prove that the area of ΔSQT is a maximum if and only if $q = p$.

(6 marks)

(c) Let Q be the point in (b) where the area of ΔSQT is a maximum. If the straight line PQ cuts the chord ST at M, find the equation of the locus of M as P moves along C_1 .

(5 marks)

Go on to the next page

12. (a) Let $g(x)$ be a function continuous on $[a, b]$, differentiable in (a, b), with $g'(x)$ decreasing on (a, b) and $g(a) = g(b) = 0$. Using Mean Value Theorem, show that there exists $c \in (a, b)$ such that g is increasing on (a, c) and decreasing on (c, b).
Hence show that $g(x) \geq 0$ for all $x \in [a, b]$.

(5 marks)

(b) Let f be a twice differentiable function and $f''(x) \geq 0$ on an open interval $\mathbf{I}$.
Suppose $a, b, x \in \mathbf{I}$ with $a < x < b$.
By considering the function
$g(x) = (b-x)f(a) + (x-a)f(b) - (b-a)f(x)$ or otherwise, show that

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) .$$

Hence, or otherwise, prove that $f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$ for all $x_1, x_2 \in \mathbf{I}$, where $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$.

(5 marks)

(c) Let x_1 and x_2 be positive numbers.

(i) If $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$, prove that

$$\lambda_1 x_1 + \lambda_2 x_2 \geq x_1^{\lambda_1} x_2^{\lambda_2} .$$

(ii) If β_1, β_2 are positive numbers, prove that

$$\left(\frac{\beta_1 x_1 + \beta_2 x_2}{\beta_1 + \beta_2} \right)^{\beta_1 + \beta_2} \geq x_1^{\beta_1} x_2^{\beta_2} .$$

(5 marks)

13. (a) (i) Let $I_n(\theta) = \int_0^\theta \tan^n u \, du$, where n is a non-negative integer and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Show that $I_{2n}(\theta) = \frac{\tan^{2n-1}\theta}{2n-1} - I_{2n-2}(\theta)$ for all $n \geq 1$.

(ii) Using the substitution $t = \tan u$, or otherwise, show that

$$\int_0^x \frac{t^{2n}}{1+t^2} dt = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3} + \cdots + (-1)^{n-1} \frac{x}{1} + (-1)^n \tan^{-1} x$$

for any positive integer n.

(5 marks)

(b) (i) Let $x \geq 0$ and n be a positive integer. Prove that

$$\frac{x^{2n+1}}{(2n+1)(1+x^2)} \leq \int_0^x \frac{t^{2n}}{1+t^2} dt \leq \frac{x^{2n+1}}{2n+1} .$$

(ii) Using (a) or otherwise, show that

$$\frac{1}{2(2n+1)} \leq \left| \frac{\pi}{4} - \sum_{p=1}^{n} \frac{(-1)^{p-1}}{2p-1} \right| \leq \frac{1}{2n+1} .$$

(iii) Suppose that $\tan \alpha = \frac{1}{5}$. Evaluate $\tan 2\alpha$ and $\tan 4\alpha$, and show that $\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$.

Hence prove that

$$\left| \frac{\pi}{4} - \sum_{p=1}^{n} \frac{(-1)^{p-1}}{2p-1} \left(\frac{4}{5^{2p-1}} - \frac{1}{239^{2p-1}} \right) \right| \leq \frac{1}{(2n+1)} \left(\frac{4}{5^{2n+1}} + \frac{1}{239^{2n+1}} \right) .$$

(10 marks)

END OF PAPER