Advanced Level Pure Mathematics

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Makers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. In the marking scheme, steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles.

For Section A:

7. Marks may be deducted for poor presentation (*pp*). The symbol *pp-1* should be used to denote 1 mark deducted for *pp*.
   a. At most deduct 1 mark for *pp* in each question, up to a maximum of 2 marks for section A.
   b. For similar *pp*, deduct 1 mark for the first time that it occurs. Do not penalize candidates twice in the paper for the same *pp*.
   c. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

8. Marks entered in the Page Total Box should be the NET total scored on that page.

For Section B:

9. Markers should devise a detailed marking scheme for his/her question after reading a number of live scripts and trail mark 30 scripts based on this scheme before attending the markers' meeting.

10. The Chief Examiner would checkmark these 30 scripts and finalize the marking scheme with the marker.

11. Each marker must hand in a marker's report together with a copy of the revised marking scheme for his/her question. Any changes to the marking scheme must be highlighted and notes should be added to make it clear what each mark is for and the treatment of any special considerations.

Solution	Marks
1. (a) Let $\frac{8}{x(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x}+\frac{C}{x+2}$, then	1M can be skipped
$Ax(x+2)+B(x-2)(x+2)+Cx(x-2)=8$	
$\Rightarrow$ $A+B+C=0$, $2A-2C=0$ and $-4B=8$	
$\Rightarrow$ $B=-2$ and $A=C=1$	
$\therefore$ $\frac{8}{x(x-2)(x+2)}=\frac{1}{x-2}-\frac{2}{x}+\frac{1}{x+2}$	1A
(b) $\sum_{r=3}^{2001}\frac{8}{r(r-2)(r+2)}$	
$=\sum_{r=3}^{2001}\left(\frac{1}{r-2}-\frac{2}{r}+\frac{1}{r+2}\right)$	
$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1999}\right)-2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{2001}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots+\frac{1}{2003}\right)$	1M can be skipped
$=1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{2000}-\frac{1}{2001}+\frac{1}{2002}+\frac{1}{2003}$	1A
$<1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}$	1
$=\frac{11}{12}$	
	----------(5)

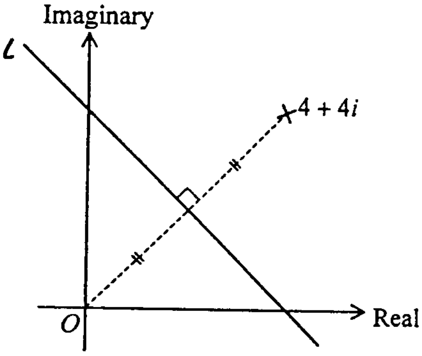
Solution	Marks
2. (a) Clearly, $\sum_{r=1}^{1} r^2 = \frac{1}{6}(1)(1+1)(2\times1+1)$.	
Assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$, then	
$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$	1A
$= \frac{1}{6}(k+1)(2k^2+7k+6)$	
$= \frac{1}{6}(k+1)(k+2)(2k+3)$	1A
By the principle of mathematical induction, the result follows.	
(b) Summing up the equations	
$\begin{array}{llllll} a_1 & = & & & & 6 \\ a_2 & = a_1 & +3\times1^2 & +9\times1 & & +6 \\ a_3 & = a_2 & +3\times2^2 & +9\times2 & & +6 \\ a_4 & = a_3 & +3\times3^2 & +9\times3 & & +6 \\ & \vdots & & & & \\ a_n & = a_{n-1} & +3(n-1)^2 & +9(n-1) & & +6 \end{array}$	
we have $a_n = 3\sum_{r=1}^{n-1} r^2 + 9\sum_{r=1}^{n-1} r + 6n$	1M
$= \frac{3}{6}(n-1)n(2n-1) + \frac{9n(n-1)}{2} + 6n$	1A+1A 1A for any 2 terms
$= \frac{1}{2}n(n-1)(2n-1+9)+6n$	
$= n(n+1)(n+2)$ $\boxed{n^3+3n^2+2n}$	
	----------(5)
3. (a) For $x>0$, let $f(x) = x^\lambda - \lambda x - 1 + \lambda$ where $0<\lambda<1$.	
$f'(x) = \lambda x^{\lambda-1} - \lambda = \lambda\left(\frac{1}{x^{1-\lambda}}-1\right)$	1A
$\begin{cases} >0 & \text{when } 0<x<1 \\ <0 & \text{when } x>1 \end{cases}$	1M
$\therefore$ $f(x) \le f(1)$ for $x>0$.	
$x^\lambda - \lambda x - 1 + \lambda \le 0$	
$x^\lambda \le (1-\lambda) + \lambda x$	1 no deduction for strict inequality
(b) Let $x = \frac{a}{b}$ and $\lambda = \frac{1}{p}$, then $x>0$, $0<\lambda<1$ and $\frac{1}{q} = 1-\lambda$.	1A for defining x and λ
From (a), $\left(\frac{a}{b}\right)^{\frac{1}{p}} \le \frac{1}{q} + \frac{1}{p}\cdot\frac{a}{b}$	
$a^{\frac{1}{p}} b^{1-\frac{1}{p}} \le \frac{b}{q} + \frac{a}{p}$	
$a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{a}{p} + \frac{b}{q}$	1 no deduction for strict inequality
	----------(5)

Solution	Marks
4. (a) $\overrightarrow{AB}=-a\mathbf{i}+b\mathbf{j}$ and $\overrightarrow{AC}=-a\mathbf{i}+c\mathbf{k}$	1A
$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a&b&0\\-a&0&c\end{vmatrix}=bc\mathbf{i}+ac\mathbf{j}+ab\mathbf{k}$	1A
(b) $S_{\Delta ABC}{}^2=\left\|\frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}\right\|^2$	1A can be skipped
$=\frac{1}{4}[(bc)^2+(ac)^2+(ab)^2]$	1M
$=\left(\frac{ab}{2}\right)^2+\left(\frac{bc}{2}\right)^2+\left(\frac{ca}{2}\right)^2$	
$=S_{\Delta OAB}{}^2+S_{\Delta OBC}{}^2+S_{\Delta OCA}{}^2$	1
	----------(5)
5. (a) $\frac{T_{k+1}}{T_k}=\frac{C_k^{2n}x^k}{C_{k-1}^{2n}x^{k-1}}$	
$=\frac{(2n)!}{k!(2n-k)!}\cdot\frac{(k-1)!(2n-k+1)!}{(2n)!}x$	1M
$=\frac{2n-k+1}{k}x$	1A
$T_{k+1}-T_k=\frac{(2n)!}{k!(2n-k)!}x^k-\frac{(2n)!}{(k-1)!(2n-k+1)!}x^{k-1}$	$\boxed{\text{1M}}$
$=\frac{(2n)!x^{k-1}}{(2n-k)!(k-1)!}\left(\frac{x}{k}-\frac{1}{2n-k+1}\right)$	$\boxed{\text{1A}}$
For $x=\frac{1}{3}$ and $k=1,2,\cdots,2n+1$,	
$T_{k+1}\ge T_k$ iff $\frac{2n-k+1}{3k}\ge 1$ $\boxed{\frac{1}{3k}\ge\frac{1}{2n-k+1}}$	
$k\le\frac{2n+1}{4}$	1A
(b) For $x=\frac{1}{3}$, $n=15$ and $k=1,2,\cdots,31$,	
$T_{k+1}\ge T_k$ iff $k\le\frac{2\times15+1}{4}=7.75$	1M
Since $T_8=C_7^{30}\left(\frac{1}{3}\right)^7\approx930.86$ and $T_7=C_6^{30}\left(\frac{1}{3}\right)^6\approx814.51$,	
$\therefore$ The greatest term is T_8 $C_7^{30}\left(\frac{1}{3}\right)^7$	1A
	----------(5)

Solution	Marks
6. If $f[f(x)]=[f(x)]^2$, then	
$a(ax^2+bx+c)^2+b(ax^2+bx+c)+c=(ax^2+bx+c)^2$	1A
$(a-1)(ax^2+bx+c)^2+b(ax^2+bx+c)+c=0$ ....(*)	
Since coefficient of $x^4=a^2(a-1)=0$ and $a\neq 0$, $\therefore$ $a=1$.	1M
Hence (*) becomes $bx^2+b^2x+bc+c=0$.	1M
Comparing the coefficients, we have $b=c=0$.	1
$a[f(x)]^2+b\,f(x)+c=[f(x)]^2$	
$f(x)[(a-1)f(x)+b]=-c$	$\boxed{1A}$
Since $\deg(f(x))>0$, $\therefore$ $(a-1)f(x)+b=0$ and hence $c=0$.	$\boxed{1M}$
Again since $(a-1)f(x)=-b$, we have $a-1=0$ and $b=0$.	$\boxed{1M}$
Hence $a=1$, $b=0$ and $c=0$.	$\boxed{1}$
$\therefore$ $f(x)=x^2$	
	----------(4)
7. (a) $M=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$	1A+1A 1A for each column
(b) Since $\lambda^2=\lvert M\rvert=2$, we have $\lambda=\sqrt{2}$.	1M
Hence $M=\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}$.	
Thus the geometric meaning of T is "a rotation which rotates any vector anticlockwise through $\frac{\pi}{4}$ about the origin, followed by an enlargement with factor $\sqrt{2}$ ".	① 1A rotation with details 1A enlargement with details ② $\boxed{1A}$ rotation and enlargement $\boxed{1A}$ details of rotation and enlargement (Choose the one which would give higher marks) -----------(5)

Solution — Marks

8. (a)



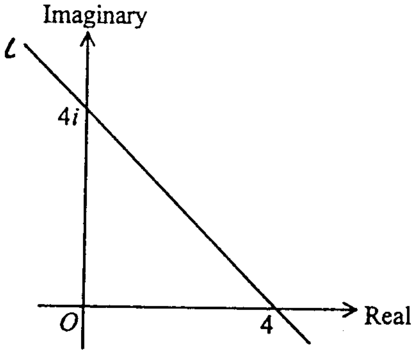
1M+1A — 1M for perpendicular bisector

Let $z = x + yi$, then $\lvert z-(4+4i)\rvert = \lvert z\rvert$ becomes

$$(x-4)^2+(y-4)^2=x^2+y^2$$

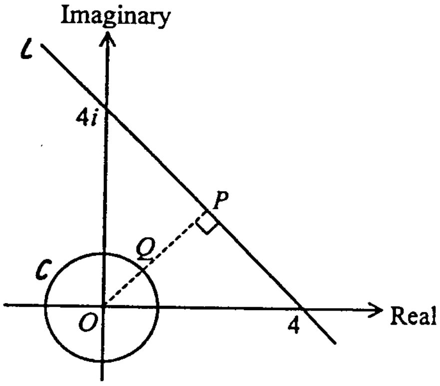
or $x+y=4$

$\therefore$ The graph of ℓ is



$\boxed{1M}+\boxed{1A}$ — 1M for correct equation or graph based on eqtn.

(b) Add C to the figure in (a),



1M

The complex number representing P is $2+2i$. — 1A

$PQ = OP - OQ = \sqrt{2^2+2^2}-1 = 2\sqrt{2}-1$. — 1A

The complex number representing Q is $\dfrac{2+2i}{2\sqrt{2}} = \dfrac{1+i}{\sqrt{2}}$. $\boxed{\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i}$ — 1A — Deduct 1 mark if P is denoted by Q and vice versa

----------(6)

Solution | Marks

9. (a) Let Δ be the determinant of the coefficients of (S).

$$\Delta = \begin{vmatrix} 1 & \lambda & 1 \\ \lambda & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= -2+3\lambda+\lambda+3-1-2\lambda^2$$

$$= -2\lambda(\lambda-2)$$ 1M

(S) has a unique solution iff $\Delta \neq 0$

iff $\lambda \neq 0$ and $\lambda \neq 2$ 1

---------(2)

(b) (i) When $\lambda \neq 0$ and $\lambda \neq 2$, (S) is consistent for all values of k, 1A

$$x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} k & \lambda & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{vmatrix}}{-2\lambda(\lambda-2)} = -\frac{-2k-\lambda+1-1-k-2\lambda}{-2\lambda(\lambda-2)} = \frac{3(\lambda+k)}{2\lambda(\lambda-2)}$$ 1M

$$y = \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} 1 & k & 1 \\ \lambda & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix}}{-2\lambda(\lambda-2)} = \frac{2+3k-\lambda-3+1-2\lambda k}{-2\lambda(\lambda-2)} = \frac{2\lambda k+\lambda-3k}{2\lambda(\lambda-2)}$$

$$z = \frac{\Delta_z}{\Delta} = \frac{\begin{vmatrix} 1 & \lambda & k \\ \lambda & -1 & 1 \\ 3 & 1 & -1 \end{vmatrix}}{-2\lambda(\lambda-2)} = \frac{1+3\lambda+\lambda k+3k-1+\lambda^2}{-2\lambda(\lambda-2)} = \frac{(\lambda+3)(\lambda+k)}{-2\lambda(\lambda-2)}$$

1A+1A 1A for anyone
1A for the rest

Alternatively,
The augmented matrix of (S) is

$$\left(\begin{array}{ccc|c} 1 & \lambda & 1 & k \\ \lambda & -1 & 1 & 1 \\ 3 & 1 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \lambda & 1 & k \\ \lambda-1 & -\lambda-1 & 0 & -k+1 \\ 1 & -2\lambda+1 & 0 & -2k-1 \end{array}\right)$$

$R_2 \leftarrow R_2 - R_1$
$R_3 \leftarrow R_3 - 3R_1$

$$\sim \left(\begin{array}{ccc|c} 1 & \lambda & 1 & k \\ \lambda & -3\lambda & 0 & -3k \\ 0 & -2\lambda(\lambda-2) & 0 & -2\lambda k-\lambda+3k \end{array}\right)$$

$R_2 \leftarrow R_2 + R_3$
$R_3 \leftarrow \lambda R_3 - R_2$

$\boxed{1M}$

$$\therefore \quad y = \frac{2\lambda k+\lambda-3k}{2\lambda(\lambda-2)}$$

$$x = \frac{1}{\lambda}\left[-3k+3\lambda\left(\frac{2\lambda k+\lambda-3k}{2\lambda(\lambda-2)}\right)\right] = \frac{3(\lambda+k)}{2\lambda(\lambda-2)}$$

$$z = k-\left(\frac{3(\lambda+k)}{2\lambda(\lambda-2)}\right)-\lambda\left(\frac{2\lambda k+\lambda-3k}{2\lambda(\lambda-2)}\right) = \frac{(\lambda+3)(\lambda+k)}{-2\lambda(\lambda-2)}$$

$\boxed{1A}$+$\boxed{1A}$ 1A for anyone
1A for the rest

Solution | Marks

(ii) When $\lambda = 0$, the augmented matrix of (S) becomes

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & k \\ 0 & -1 & 1 & 1 \\ 3 & 1 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 0 & 1 & k \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -3k \end{array}\right)$$

$R_3 \leftarrow R_3 - 3R_1 + R_2$

$\therefore$ (S) is consistent iff $k = 0$. 1A

When $\lambda = 0$ and $k = 0$, $\begin{cases} x + z = 0 \\ -y + z = 1 \end{cases}$.

S.S. of $(S) = \{(t, -(t+1), -t) : t \in \mathbf{R}\}$ 1A

$\boxed{\{(-t, t-1, t) : t \in \mathbf{R}\}}$

(iii) When $\lambda = 2$, the augmented matrix of (S) becomes

$$\left(\begin{array}{ccc:c} 1 & 2 & 1 & k \\ 2 & -1 & 1 & 1 \\ 3 & 1 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 2 & 1 & k \\ 0 & 5 & 1 & 2k-1 \\ 0 & 0 & 0 & -k-2 \end{array}\right)$$

$R_3 \leftarrow R_3 - R_1 - R_2$

$R_2 \leftarrow -(R_2 - 2R_1)$

$\therefore$ (S) is consistent iff $k = -2$. 1A

When $\lambda = 2$ and $k = -2$, $\begin{cases} x + 2y + z = -2 \\ 5y + z = -5 \end{cases}$.

S.S. of $(S) = \{(3(1+t), t, -5(1+t)) : t \in \mathbf{R}\}$ 1A

$\boxed{\left\{\left(-\frac{3t}{5}, -\frac{t+5}{5}, t\right) : t \in \mathbf{R}\right\}}$ $\boxed{\left\{\left(t, \frac{t-3}{3}, -\frac{5t}{3}\right) : t \in \mathbf{R}\right\}}$

----------(8)

(c) The system of equations is (S) when $\lambda = 0$ and $k = 0$.

If some solution $(t, -(t+1), -t)$ satisfies $(x-p)^2 + y^2 + z^2 = 1$, then

$$(t-p)^2 + (t+1)^2 + t^2 = 1$$

$$3t^2 - 2(p-1)t + p^2 = 0$$

1M+1A

If some solution $(-t, t-1, t)$ satisfies $(x-p)^2 + y^2 + z^2 = 1$, then

$$(t+p)^2 + (t-1)^2 + t^2 = 1$$

$$3t^2 + 2(p-1)t + p^2 = 0$$

$\boxed{\text{1M}}+\boxed{\text{1A}}$

For t to have real solution,

$$4(p-1)^2 - 12p^2 \geq 0$$

1M

$$2p^2 + 2p - 1 \leq 0$$

$$\frac{-1-\sqrt{3}}{2} \leq p \leq \frac{-1+\sqrt{3}}{2}$$

$\boxed{-1.366 \leq p \leq 0.366}$

1A+1A 1A for either side

----------(5)

Solution | Marks

10. (a) (i) Clearly, $a_1, b_1 > 0$ and $a_1^2 - 2b_1^2 = (-1)^1$.

Assume $a_k, b_k > 0$ and $a_k^2 - 2b_k^2 = (-1)^k$ where $k \in \mathbf{Z}^+$. } 1A

Then $a_{k+1} = a_k + 2b_k > 0$, $b_{k+1} = a_k + b_k > 0$ and

$$\begin{aligned} a_{k+1}^2 - 2b_{k+1}^2 &= (a_k + 2b_k)^2 - 2(a_k + b_k)^2 \\ &= -\left(a_k^2 - 2b_k^2\right) \\ &= (-1)^{k+1} \end{aligned}$$ 1A

By the principle of mathematical induction, the result follows.

(ii) Clearly, $\left(1+\sqrt{2}\right)^1 = a_1 + b_1\sqrt{2}$.

Assume $\left(1+\sqrt{2}\right)^k = a_k + b_k\sqrt{2}$ where k is a positive integer. } 1A

$$\begin{aligned} \left(1+\sqrt{2}\right)^{k+1} &= \left(1+\sqrt{2}\right)\left(a_k + b_k\sqrt{2}\right) \\ &= \left(a_k + 2b_k\right) + \left(a_k + b_k\right)\sqrt{2} \\ &= a_{k+1} + b_{k+1}\sqrt{2} \end{aligned}$$ 1A

By the principle of mathematical induction, the result follows.

----------(4)

(b) Let $n = 1, 2, 3, \ldots$.

(i) $$u_{n+1} = \frac{a_{n+1}}{b_{n+1}} = \frac{a_n + 2b_n}{a_n + b_n} = \frac{\frac{a_n}{b_n} + 2}{\frac{a_n}{b_n} + 1} = \frac{u_n + 2}{u_n + 1}$$ 1

(ii) From (a)(i), $a_n^2 - 2b_n^2 = (-1)^n$.

$\therefore$ $$\left(\frac{a_n}{b_n}\right)^2 = 2 + \frac{(-1)^n}{b_n^2}$$ 1A

$$u_n = \sqrt{2 + \frac{(-1)^n}{b_n^2}}$$ [since $u_n = \frac{a_n}{b_n} > 0$]

Hence $u_{2n-1} < \sqrt{2}$ and $u_{2n} > \sqrt{2}$. [since $b_n > 0$] 1M+1

Alternatively,

Since $a_{2n-1}^2 - 2b_{2n-1}^2 = (-1)^{2n-1} = -1 < 0$ $\boxed{\text{1M}}$

$\therefore$ $$\left(\frac{a_{2n-1}}{b_{2n-1}}\right)^2 < 2$$

$u_{2n-1} < \sqrt{2}$ [since $u_n = \frac{a_n}{b_n} > 0$] $\boxed{1}$

Similarly, $a_{2n}^2 - 2b_{2n}^2 = (-1)^{2n} = 1 > 0$

$\therefore$ $$\left(\frac{a_{2n}}{b_{2n}}\right)^2 > 2$$

$u_{2n} > \sqrt{2}$ [since $u_n = \frac{a_n}{b_n} > 0$] $\boxed{1}$

Solution	Marks
(iii) Using (b)(i), $u_{n+2}=\frac{u_{n+1}+2}{u_{n+1}+1}=\frac{\frac{u_n+2}{u_n+1}+2}{\frac{u_n+2}{u_n+1}+1}=\frac{3u_n+4}{2u_n+3}$	1
Hence $u_{n+2}-u_n=\frac{3u_n+4}{2u_n+3}-u_n$ $=\frac{-2u_n{}^2+4}{2u_n+3}$ $=\frac{-2\left(u_n-\sqrt{2}\right)\left(u_n+\sqrt{2}\right)}{2u_n+3}$	1A
$\Rightarrow$ $u_{2n+1}-u_{2n-1}=\frac{-2\left(u_{2n-1}-\sqrt{2}\right)\left(u_{2n-1}+\sqrt{2}\right)}{2u_{2n-1}+3}>0$ [since $u_{2n-1}<\sqrt{2}$]	1
and $u_{2n+2}-u_{2n}=\frac{-2\left(u_{2n}-\sqrt{2}\right)\left(u_{2n}+\sqrt{2}\right)}{2u_{2n}+3}<0$ [since $u_{2n}>\sqrt{2}$]	1
i.e. the sequence $\{u_1,u_3,u_5,\ldots\}$ is strictly increasing and the sequence $\{u_2,u_4,u_6,\ldots\}$ is strictly decreasing.	
(iv) Since $\{u_1,u_3,u_5,\ldots\}$ is strictly increasing and bounded above by $\sqrt{2}$ and $\{u_2,u_4,u_6,\ldots\}$ is strictly decreasing and bounded below by $\sqrt{2}$, therefore both the sequences converge.	1
Let $\lim\limits_{n\to\infty}u_{2n-1}=a$ and $\lim\limits_{n\to\infty}u_{2n}=b$.	
From (b)(iii), $a=\frac{3a+4}{2a+3}$ and $b=\frac{3b+4}{2b+3}$ $\therefore$ $a^2=b^2=2$ $\Rightarrow$ $a=b=\sqrt{2}$ [since $u_n>0$]	1+1A
From (b)(i), $a=\frac{b+2}{b+1}$ and $b=\frac{a+2}{a+1}$ $\therefore$ $ab=-a+b+2$ and $ab=a-b+2$ $-a+b+2=a-b+2$ $a=b$	$\boxed{1}$
Putting $b=a$ into $ab=-a+b+2$, we have $a^2=2$ $\therefore$ $a=\sqrt{2}$ [since $u_n>0$]	$\boxed{1A}$
	----------(11)

Solution	Marks
11. (a) $\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = \frac{1+\left(2\cos^2\frac{\theta}{2}-1\right)+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{1-\left(1-2\sin^2\frac{\theta}{2}\right)-i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}$	1M
$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{-2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}$	1M
$= i\cot\frac{\theta}{2}$	1
	----------(3)
(b) Clearly, $z \neq -1$. (*) can be rewritten as $\left(\frac{z-1}{z+1}\right)^n = -1$.	1A
Using De Moivre's theorem, $\frac{z-1}{z+1} = \cos\frac{(2k+1)\pi}{n} + i\sin\frac{(2k+1)\pi}{n}$ where $k = 0, 1, 2, \ldots, n-1$.	1A
For $k = 0, 1, 2, \ldots, n-1$, let $\theta_k = \frac{(2k+1)\pi}{n}$.	
$z - 1 = z\cos\theta_k + iz\sin\theta_k + \cos\theta_k + i\sin\theta_k$	
$z = \frac{1+\cos\theta_k + i\sin\theta_k}{1-\cos\theta_k - i\sin\theta_k}$	1A
$= i\cot\frac{\theta_k}{2}$	
$= i\cot\frac{(2k+1)\pi}{2n}$	
Putting $\alpha_k = \cot\frac{(2k+1)\pi}{2n}$, the result follows.	1
Alternatively,	
Clearly, $z \neq 1$. (*) can be rewritten as $\left(\frac{z+1}{z-1}\right)^n = -1$.	$\boxed{1A}$
$\frac{z+1}{z-1} = \cos\frac{(2k+1)\pi}{n} + i\sin\frac{(2k+1)\pi}{n}$ where $k = 0, 1, 2, \ldots, n-1$.	$\boxed{1A}$
For $k = 0, 1, 2, \ldots, n-1$, let $\theta_k = \frac{(2k+1)\pi}{n}$.	
$z + 1 = z\cos\theta_k + iz\sin\theta_k - \cos\theta_k - i\sin\theta_k$	
$z = -\frac{1+\cos\theta_k + i\sin\theta_k}{1-\cos\theta_k - i\sin\theta_k}$	$\boxed{1A}$
$= -i\cot\frac{\theta_k}{2}$	
$= -i\cot\frac{(2k+1)\pi}{2n}$	
Putting $\alpha_k = -\cot\frac{(2k+1)\pi}{2n}$, the result follows.	$\boxed{1}$
	----------(4)

Solution	Marks
(c) Expanding the LHS, (*) becomes	
$[z^n + C_1^n z^{n-1} + C_2^n z^{n-2} + \cdots + 1] + [z^n - C_1^n z^{n-1} + C_2^n z^{n-2} - \cdots + (-1)^n] = 0$	
$z^n + C_2^n z^{n-2} + C_4^n z^{n-4} + \cdots = 0$	1A
Using the relations between the roots and coefficients, we have	
$\sum_{k=0}^{n-1} i\alpha_k = 0$	1M
and $\sum_{\substack{j,k=0 \\ j\neq k}}^{n-1} (i\alpha_j)(i\alpha_k) = C_2^n$	1M
Hence $\sum_{k=0}^{n-1} (i\alpha_k)^2 = \left(\sum_{k=0}^{n-1} i\alpha_k\right)^2 - 2\sum_{\substack{j,k=0 \\ j\neq k}}^{n-1} (i\alpha_j)(i\alpha_k)$	1A
$= 0 - 2\left(\frac{n(n-1)}{2}\right)$	
$\sum_{k=0}^{n-1} \alpha_k{}^2 = n(n-1)$	1
	----------(5)
(d) $\sum_{k=0}^{n-1} d_k{}^2 = \sum_{k=0}^{n-1} \lvert i\alpha_k - r(\cos\beta + i\sin\beta)\rvert^2$	1M for $d_k{}^2 = \cdots$
$= \sum_{k=0}^{n-1} \left[(r\cos\beta)^2 + (\alpha_k - r\sin\beta)^2\right]$	
$= r^2 \sum_{k=0}^{n-1} (\cos^2\beta + \sin^2\beta) + \sum_{k=0}^{n-1} \alpha_k{}^2 - 2r\sin\beta \sum_{k=0}^{n-1} \alpha_k$	1A
$= nr^2 + n(n-1)$ $\boxed{n(r^2 + n - 1)}$	
which is independent of β.	1
	----------(3)

Solution	Marks
12. (a) Let x, y and z be real numbers such that $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$.	1M
The equation is equivalent to $\begin{pmatrix}7&8&3\\2&-7&13\\17&-6&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.	
Since $\begin{vmatrix}7&8&3\\2&-7&13\\17&-6&3\end{vmatrix}=-147+1768-36+357+546-48=2440\neq0$,	1A
$\therefore$ The system has the trivial solution only, i.e. $x=y=z=0$.	
Hence $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.	1
<u>Alternatively</u>,	
$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\begin{vmatrix}7&8&3\\2&-7&13\\17&-6&3\end{vmatrix}=-147+1768-36+357+546-48=2440\neq0$	$\boxed{1M}+\boxed{1A}$
Hence $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.	$\boxed{1}$
	----------(3)
(b) Let $\mathbf{v}=\lambda\,\mathbf{a}+\mu\,\mathbf{b}=\alpha\,\mathbf{c}+\beta\,\mathbf{d}$.	
Then $(\lambda\,\mathbf{a}+\mu\,\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\alpha\,\mathbf{c}+\beta\,\mathbf{d})\cdot(\mathbf{c}\times\mathbf{d})=0$	1M+1A
$\lambda\,\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})+\mu\,\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})=0$	
$\lambda:\mu=-\mathbf{b}\cdot\mathbf{c}\times\mathbf{d}:\mathbf{a}\cdot\mathbf{c}\times\mathbf{d}$	1
Since $\mathbf{c}\times\mathbf{d}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\17&-6&3\\2r&-4r&-5r\end{vmatrix}$	1M
$=r(42\mathbf{i}+91\mathbf{j}-56\mathbf{k})$	
$=7r(6\mathbf{i}+13\mathbf{j}-8\mathbf{k})$	
$\therefore$ $\lambda:\mu=-7r(2\times6-7\times13-13\times8):7r(7\times6+8\times13-3\times8)$	
$=183r:122r$	
$=3:2$	1A
	----------(5)
(c) (i) If A, B, C and D are coplanar, then	
$(\mathbf{d}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=0$	1M
$\begin{vmatrix}2r-7&-4r-8&-5r-3\\2-7&-7-8&13-3\\17-7&-6-8&3-3\end{vmatrix}=0$	1A
$\begin{vmatrix}2r-7&-4r-8&-5r-3\\-1&-3&2\\5&-7&0\end{vmatrix}=0$	
$7(2r-7)-5(4r+8)-11(5r+3)=0$	
$r=-2$	1A
(ii) Let $\mathbf{e}$ be the position vector of E and $BE:EA=\lambda_1:\mu_1$.	
Since E lies on CD, $\mathbf{e}$ is also a linear combination of $\mathbf{c}$ and $\mathbf{d}$.	1M
Using (b), $\lambda_1:\mu_1=3:2$.	1M
$\mathbf{e}=\dfrac{\lambda_1\mathbf{a}+\mu_1\mathbf{b}}{\lambda_1+\mu_1}=\dfrac{3}{5}(7\mathbf{i}+8\mathbf{j}+3\mathbf{k})+\dfrac{2}{5}(2\mathbf{i}-7\mathbf{j}+13\mathbf{k})=5\mathbf{i}+2\mathbf{j}+7\mathbf{k}$	1M+1A
	----------(7)

Solution	Marks
13. (a) (i) $P(\overline{\alpha}) = (\overline{\alpha})^4 + a(\overline{\alpha})^3 + b(\overline{\alpha})^2 + c\overline{\alpha} + d$	1M
$= \overline{\alpha^4} + a\overline{\alpha^3} + b\overline{\alpha^2} + c\overline{\alpha} + d$	
$= \overline{P(\alpha)}$	
$= 0$	1
∴ if α is a complex root of $P(x) = 0$, $\overline{\alpha}$ is also a root of $P(x) = 0$.	
(ii) $(x-\alpha)(x-\overline{\alpha}) = x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha}$	
$= x^2 - 2\,\mathrm{Re}(\alpha)x + \lvert\alpha\rvert^2$	1A
∴ $(x-\alpha)(x-\overline{\alpha})$ is a quadratic polynomial in x with real coefficients.	
Since $P(x)$ is a polynomial of degree 4, therefore the equation $P(x) = 0$ has 4 roots.	
(I) If all the 4 roots are real, then $P(x)$ can be factorized into 4 linear factors with real coefficients. Hence	
$P(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$ for some $a_1, a_2, a_3, a_4 \in R$	
$= [x^2 - (a_1 + a_2)x + a_1a_2][x^2 - (a_3 + a_4)x + a_3a_4]$	1
(II) If there is a complex root α, then $\overline{\alpha}$ is also a root. Hence	
$P(x) = (x-\alpha)(x-\overline{\alpha})Q(x)$ for some polynomial $Q(x)$	
$= [x^2 - 2\,\mathrm{Re}(\alpha)x + \lvert\alpha\rvert^2]Q(x)$	1 for one pair
Since $P(x)$ is a polynomial with real coefficients of degree 4 and $[x^2 - 2\,\mathrm{Re}(\alpha)x + \lvert\alpha\rvert^2]$ is a quadratic polynomial with real coefficients,	
∴ $Q(x)$ is also a quadratic polynomial with real coefficients.	1 for two pairs
Hence $P(x)$ is a product of two quadratic polynomials with real coefficients.	
	----------(6)

Solution	Marks
(b) (i) $g(x)=(x+k)^4+8(x+k)^3+23(x+k)^2+26(x+k)+7$	
Coefficient of x^3 = $4k+8=0$	
$k=-2$	1A
Hence $g(x)=(x-2)^4+8(x-2)^3+23(x-2)^2+26(x-2)+7$	
$=(x^4-8x^3+24x^2-32x+16)+8(x^3-6x^2+12x-8)$	
$+23(x^2-4x+4)+26(x-2)+7$	
$=x^4-x^2-2x-1$	1A
(ii) Suppose $g(x)=(x^2+px+q)(x^2+rx+s)$.	
By comparing coefficients, $\begin{cases} p+r=0\\ q+s+pr=-1\\ ps+qr=-2\\ qs=-1\end{cases}$	1M
Clearly $p\neq 0$, otherwise $r=0$ and hence $ps+qr=0\neq -2$.	
Putting $r=-p$, $\begin{cases}-p^2+q+s=-1\\ ps-pq=-2\end{cases}$	
$\Rightarrow \begin{cases} q+s=p^2-1\\ q-s=\dfrac{2}{p}\end{cases}$	
$\Rightarrow \begin{cases} 2q=p^2-1+\dfrac{2}{p}\\ 2s=p^2-1-\dfrac{2}{p}\end{cases}$	1M
Since $qs=-1$, therefore	
$\left(p^2-1+\frac{2}{p}\right)\left(p^2-1-\frac{2}{p}\right)=-4$	
$(p^3-p+2)(p^3-p-2)=-4p^2$	
$p^6-2p^4+5p^2-4=0$	1
$(p^2-1)(p^4-p^2+4)=0$	1A
$p=\pm 1$	
Hence $p=1,\ q=1,\ r=-1,\ s=-1$	
or $p=-1,\ q=-1,\ r=1,\ s=1$	1A for any set
(iii) From (b)(ii), $g(x)=(x^2-x-1)(x^2+x+1)$.	
$\therefore$ $g(x)=0$ if $x=\dfrac{1\pm\sqrt{5}}{2}$ or $\dfrac{-1\pm\sqrt{3}i}{2}$.	1A
Since $f(x)=g(x+2)$,	
$\therefore$ $f(x)=0$ if $x=\dfrac{-3\pm\sqrt{5}}{2}$ or $\dfrac{-5\pm\sqrt{3}i}{2}$.	1A
	----------(9)

Solution	Marks
14. (a) If $a \le 1 \le b$, then	
$a+b-ab-1 = (a-1)(1-b)$ ....(1)	1A
≥ 0	
$\therefore \quad a+b \ge ab+1$	1
From (1), the equality holds iff $a-1=0$ or $1-b=0$	
iff $a=1$ or $b=1$.	1
	----------(3)
(b) If x_1 , x_2 are positive real numbers such that $x_1x_2=1$,	
without loss of generality, we may assume that $x_1 \le 1 \le x_2$.	
From (a), $x_1+x_2 \ge x_1x_2+1=2$ and	1A
the equality holds iff $x_1=1$ or $x_2=1$	
iff $x_1=x_2=1$ (since $x_1x_2=1$)	1
$x_1+x_2=x_1+\dfrac{1}{x_1}=\left(\sqrt{x_1}-\dfrac{1}{\sqrt{x_1}}\right)^2+2\ge 2$	$\boxed{\text{1A}}$
and the equality holds iff $\sqrt{x_1}=\dfrac{1}{\sqrt{x_1}}$ iff $x_1=x_2=1$.	$\boxed{1}$
$\therefore$ The statement holds for $n=2$.	
Assume the statement holds for $n=k$ where $k \in \mathrm{N}$ and $k \ge 2$.	
Let x_1 , x_2 , ..., x_{k+1} be positive real numbers such that $x_1x_2\cdots x_{k+1}=1$.	
Again there must be two numbers, say, x_1 and x_2 such that $x_1 \le 1 \le x_2$.	1M
Hence using the induction hypothesis,	
$x_1x_2+x_3+x_4+\cdots+x_{k+1} \ge k$	1A
and the equality holds iff $x_1x_2=x_3=x_4=\cdots=x_{k+1}=1$.	
Using (a) again, $x_1+x_2 \ge x_1x_2+1$ and the equality holds iff $x_1=x_2=1$.	
$\therefore \quad x_1+x_2+x_3+\cdots+x_{k+1} \ge x_1x_2+x_3+x_4+\cdots+x_{k+1}+1$	
$\ge k+1$	1
and the equality holds	
iff $(x_1=1$ or $x_2=1)$ and $x_1x_2=x_3=x_4=\cdots=x_{k+1}=1$	
iff $x_1=x_2=x_3=\cdots=x_{k+1}=1$	1
	----------(6)
(c) If a_1 , a_2 , ..., a_n are n positive real numbers with $n \ge 2$, let	
$x_i=\dfrac{a_i}{\sqrt[n]{a_1a_2\cdots a_n}}$ where $i=1,2,\ldots,n$.	1A
Then x_i 's satisfy the conditions in (b) and hence	
$\dfrac{a_1+a_2+\cdots+a_n}{\sqrt[n]{a_1a_2\cdots a_n}} \ge n$ or $\dfrac{a_1+a_2+\cdots+a_n}{n} \ge \sqrt[n]{a_1a_2\cdots a_n}$	1
and the equality holds iff $\dfrac{a_1}{\sqrt[n]{a_1a_2\cdots a_n}}=\dfrac{a_2}{\sqrt[n]{a_1a_2\cdots a_n}}=\cdots=\dfrac{a_n}{\sqrt[n]{a_1a_2\cdots a_n}}$	
iff $a_1=a_2=\cdots=a_n$	1
	----------(3)

Solution	Marks
(d) For $u \ge 0$, $(1+u)^k > 0$ for any non-negative integer k .	
For $n = 2, 3, 4, \dots$,	
$(1+u)^n - 1 = u[(1+u)^{n-1} + (1+u)^{n-2} + \cdots + 1]$	
$\frac{(1+u)^n - 1}{n} = u\frac{(1+u)^{n-1} + (1+u)^{n-2} + \cdots + 1}{n}$	
Using (c), $\frac{(1+u)^n - 1}{n} \ge u\sqrt[n]{(1+u)^{(n-1)+(n-2)+\cdots+1}}$	1M
$= u(1+u)^{\frac{1}{n}\cdot\frac{(n-1)(n-1+1)}{2}}$	1A
$= u(1+u)^{\frac{n-1}{2}}$	
$\therefore \quad (1+u)^n \ge 1 + nu(1+u)^{\frac{n-1}{2}}$	
The equality holds iff $(1+u)^{n-1} = (1+u)^{n-2} = \cdots = 1$ by (c)	
iff $u = 0$	1 ----------(3)

香港考試局
HONG KONG EXAMINATIONS AUTHORITY

2001年香港高級程度會考
HONG KONG ADVANCED LEVEL EXAMINATION 2001

純粹數學　高級程度　試卷二
PURE MATHEMATICS　A-LEVEL　PAPER 2

本評卷參考乃考試局專爲今年本科考試而編寫，供閱卷員參考之用。閱卷員在完成閱卷工作後，若將本評卷參考提供其任教會考班的本科同事參閱，本局不表反對，但須切記，在任何情況下均不得容許本評卷參考落入學生手中。學生若索閱或求取此等文件，閱卷員/教師應嚴詞拒絕，因學生極可能將評卷參考視爲標準答案，以致但知硬背死記，活剝生吞。這種落伍的學習態度，既不符現代教育原則，亦有違考試着重理解能力與運用技巧之旨。因此，本局籲請各閱卷員/教師通力合作，堅守上述原則。

This marking scheme has been prepared by the Hong Kong Examinations Authority for markers' reference. The Examinations Authority has no objection to markers sharing it, after the completion of marking, with colleagues who are teaching the subject. However, under no circumstances should it be given to students because they are likely to regard it as a set of model answers. Markers/teachers should therefore firmly resist students' requests for access to this document. Our examinations emphasise the testing of understanding, the practical application of knowledge and the use of processing skills. Hence the use of model answers, or anything else which encourages rote memorisation, should be considered outmoded and pedagogically unsound. The Examinations Authority is counting on the co-operation of markers/teachers in this regard.

考試結束後，各科評卷參考將存放於教師中心，供教師參閱。
After the examinations, marking schemes will be available for reference at the teachers' centre.

Advanced Level Pure Mathematics

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Makers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. Use of notation different from those in the marking scheme should not be penalized.

5. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

6. In the marking scheme, steps which can be skipped are ▨shaded▨ whereas alternative answers are enclosed with $\boxed{\text{rectangles}}$.

For Section A:

7. Marks may be deducted for poor presentation (*pp*). The symbol (pp-1) should be used to denote 1 mark deducted for *pp*.
   a. At most deduct 1 mark for *pp* in each question, up to a maximum of 2 marks for section A.
   b. For similar *pp*, deduct 1 mark for the first time that it occurs. Do not penalize candidates twice in the paper for the same *pp*.
   c. In any case, do not deduct any marks for *pp* in those steps where candidates could not score any marks.

8. Marks entered in the Page Total Box should be the NET total scored on that page.

For Section B:

9. Markers should devise a detailed marking scheme for his/her question after reading a number of live scripts and trail mark 30 scripts based on this scheme before attending the markers' meeting.

10. The Chief Examiner would checkmark these 30 scripts and finalize the marking scheme with the marker.

11. Each marker must hand in a marker's report together with a copy of the revised marking scheme for his/her question. Any changes to the marking scheme must be highlighted and notes should be added to make it clear what each mark is for and the treatment of any special considerations.

Solution	Marks
1. (a) $\lim\limits_{x\to 0}\frac{(e^x-e^{-x})^2}{1-\cos 2x} = \lim\limits_{x\to 0}\frac{2(e^x-e^{-x})(e^x+e^{-x})}{2\sin 2x}$ [By L'Hospital rule]	1M+1A
$= \lim\limits_{x\to 0}\frac{e^{2x}-e^{-2x}}{\sin 2x}$	
$= \lim\limits_{x\to 0}\frac{2e^{2x}+2e^{-2x}}{2\cos 2x}$ [By L'Hospital rule]	1A
$= 2$	1A
(b) Since $\left\lvert\cos\frac{1}{x}\right\rvert \le 1$, $\therefore\ -x^2 \le x^2\cos\frac{1}{x} \le x^2$	1M for considering bounded property of $\cos\frac{1}{x}$.
As $\lim\limits_{x\to 0}x^2 = 0$, $\lim\limits_{x\to 0}x^2\cos\frac{1}{x} = 0$ by sandwich rule.	1
	----------(6)
2. (a) $\int\frac{x^3}{1+x^2}\mathrm{d}x = \int\left(x-\frac{x}{1+x^2}\right)\mathrm{d}x$	1A
$= \frac{x^2}{2}-\frac{1}{2}\ln(1+x^2)+c$	1A
(b) $\int x^2\tan^{-1}x\,\mathrm{d}x = \int\tan^{-1}x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{3}\right)\mathrm{d}x$	
$= \frac{x^3}{3}\tan^{-1}x-\int\frac{x^3}{3}\cdot\frac{1}{1+x^2}\mathrm{d}x$	1M+1A
$= \frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\ln(1+x^2)+c$	1A
	----------(5)
3. Clearly, $1 \le a_1 < 2$ as $a_1 = 1$. Assume $1 \le a_k < 2$ where k is a positive integer.	1A
Then $a_{k+1} = \sqrt{\frac{4+{a_k}^2}{2}} \ge 1$ as $a_k \ge 1$ and	
$a_{k+1} < \sqrt{\frac{4+2^2}{2}} = 2$ as $a_k < 2$.	1A
By the principle of mathematical induction, $1 \le a_n < 2$ for all $n \in \mathrm{N}$.	
On the other hand, since $a_n < 2$,	
$a_{n+1} = \sqrt{\frac{4+{a_n}^2}{2}} > \sqrt{\frac{{a_n}^2+{a_n}^2}{2}} = a_n$ for all $n \in \mathrm{N}$.	1 for proving $\{a_n\}$ is increasing
$\therefore$ $\{a_n\}$ is strictly increasing and bounded above by 2 . Hence $\{a_n\}$ is convergent.	1
Let $\lim\limits_{n\to\infty}a_n = \ell$, then $2\ell^2 = 4+\ell^2$	1M
$\ell = 2$	1A
	----------(6)

Solution	Marks
4. (a) Slope of $AB=\dfrac{2a(t_1-t_2)}{a(t_1{}^2-t_2{}^2)}=\dfrac{2}{t_1+t_2}$ [since $t_1 \neq t_2$]	1M for giving steps to calculate the equation of the chord
Equation of AB is	
$\dfrac{y-2at_1}{x-at_1{}^2}=\dfrac{2}{t_1+t_2}$	
$2x-(t_1+t_2)y+2a[t_1(t_1+t_2)-t_1{}^2]=0$	
$2x-(t_1+t_2)y+2at_1t_2=0$	1A
(*Note*: the equation is also valid for the case when $t_1=-t_2$.)	
(b) If AB passes through $(a, 0)$, then $2a+2at_1t_2=0$,	1M
$t_1t_2=-1$.	
Let (x, y) be the mid-point of AB, then	
$x=\dfrac{a(t_1{}^2+t_2{}^2)}{2}$ and $y=a(t_1+t_2)$.	
Hence $\left(\dfrac{y}{a}\right)^2=(t_1+t_2)^2=(t_1{}^2+t_2{}^2)+2t_1t_2=\dfrac{2x}{a}-2$	1M for attempting the elimination process
$y^2=2ax-2a^2$	1A
	----------(5)
5. (a) For $x \in (0, 1)$,	
$F'(x)=2f(x)-2f(x)f'(x)$	1M+1A
$=2f(x)[1-f'(x)]$	
Since $0<f'(x)<1$, we have $f(x)>f(0)=0$ and $1-f'(x)>0$.	1
Hence $F'(x)>0$.	
(b) Using (a) and the fact that F is continuous on [0, 1] ,	
$F(x)>F(0)=2\int_0^0 f(t)dt-[f(0)]^2=0$ for $x \in (0, 1]$.	1A for calculating F(1) or F(0)
In particular, $F(1)=2\int_0^1 f(t)dt-[f(1)]^2>0$,	1M
$\int_0^1 f(t)dt>\dfrac{1}{2}[f(1)]^2=\dfrac{1}{2}\left(\dfrac{1}{2}\right)^2=\dfrac{1}{8}$.	1
	----------(6)

Solution	Marks
6. (a) For $x>2$, $f'(x)=3x^2-12$. $\therefore\ \lim_{x\to 2^+} f'(x)=3(2)^2-12=0$.	1
For $x<2$, $f'(x)=\frac{1}{e^2}+\frac{1-x}{e^x}$. $\therefore\ \lim_{x\to 2^-} f'(x)=\frac{1}{e^2}+\frac{1-2}{e^2}=0$.	1
(b) $f(2)=2^3-12(2)=-16$	
$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\left(\frac{x}{e^2}+\frac{x}{e^x}\right)=\frac{4}{e^2}$	1A
$\therefore$ f is not continuous at $x=2$ as $\lim_{x\to 2^-} f(x)\neq f(2)$.	1M
Since $\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{x^3-12x+16}{x-2}=\lim_{x\to 2^+}\frac{3x^2-12}{1}=0$,	$\boxed{\text{1M}}$
$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{\frac{x}{e^2}+\frac{x}{e^x}+16}{x-2}=-\infty$	$\boxed{\text{1A}}$
$\therefore\ \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}\neq\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$	
It follows that f cannot be differentiable at $x=2$.	1
	----------(5)
7. (a) $\frac{dx}{dt}=-3\sin t\cos^2 t$ and $\frac{dy}{dt}=3\sin^2 t\cos t$.	1A
Length of $\Gamma = 4\int_0^{\frac{\pi}{2}}\sqrt{(-3\sin t\cos^2 t)^2+(3\sin^2 t\cos t)^2}\,dt$	1M irrespective of limits of integration
$=12\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt$	
$=6\left[\sin^2 t\right]_0^{\frac{\pi}{2}}$	
$=6$	1A
(b) Area enclosed by $\Gamma = 4\int_0^1 y\,dx$	
$=4\int_{\frac{\pi}{2}}^0(\sin^3 t)(-3\sin t\cos^2 t)\,dt$	1A
$=12\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t\,dt$	
$=\frac{12}{8}\int_0^{\frac{\pi}{2}}(1-\cos 2t)\sin^2 2t\,dt$	1M for using double angle formula
$=\frac{3}{2}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos 4t)\,dt-\int_0^{\frac{\pi}{2}}\sin^2 2t\cos 2t\,dt\right)$	
$=\frac{3}{2}\left(\frac{1}{2}\left[t-\frac{\sin 4t}{4}\right]_0^{\frac{\pi}{2}}-\frac{1}{2}\left[\frac{\sin^3 2t}{3}\right]_0^{\frac{\pi}{2}}\right)$	1A
$=\frac{3}{8}\pi$	1A
	----------(7)

Solution	Marks
8. (a) (i) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(6-x)^{\frac{1}{3}} - \frac{1}{3}x^{\frac{2}{3}}(6-x)^{-\frac{2}{3}}$	
$= \frac{1}{3}x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}[2(6-x)-x]$	
$= x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}(4-x)$ for $x \neq 0, 6$	1A
(ii) Since $f_-'(0) = \lim_{h\to 0^-}\frac{h^{\frac{2}{3}}(6-h)^{\frac{1}{3}}}{h} = \lim_{h\to 0^-}\left(\frac{6}{h}-1\right)^{\frac{1}{3}} = -\infty$,	1M for using the 1st principle
$f_+'(0) = \lim_{h\to 0^+}\left(\frac{6}{h}-1\right)^{\frac{1}{3}} = \infty$,	
$f_-'(6) = \lim_{h\to 0^-}\frac{(6+h)^{\frac{2}{3}}(-h)^{\frac{1}{3}}}{h} = \lim_{h\to 0^-}-\left(\frac{6}{h}+1\right)^{\frac{2}{3}} = \infty$, and	
$f_+'(6) = \lim_{h\to 0^+}-\left(\frac{6}{h}+1\right)^{\frac{2}{3}} = -\infty$,	
Since both $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\left(\frac{6}{h}-1\right)^{\frac{1}{3}}$ and	$\boxed{\text{1M}}$
$\lim_{h\to 0}\frac{f(6+h)-f(6)}{h} = \lim_{h\to 0}-\left(\frac{6}{h}+1\right)^{\frac{2}{3}}$ are divergent,	
therefore both $f'(0)$ and $f'(6)$ do not exist.	1A
(iii) $f''(x) = -\frac{1}{3}x^{-\frac{4}{3}}(6-x)^{-\frac{2}{3}}(4-x) + \frac{2}{3}x^{-\frac{1}{3}}(6-x)^{-\frac{5}{3}}(4-x) - x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}$	1
$= \frac{1}{3}x^{-\frac{4}{3}}(6-x)^{-\frac{5}{3}}[-(6-x)(4-x)+2x(4-x)-3x(6-x)]$	
$= \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$ for $x \neq 0, 6$	
	----------(4)
(b) (i) $f'(x) > 0 \Leftrightarrow 0 < x < 4$ $\Leftrightarrow x \in (0,4)$	1A
(ii) $f'(x) < 0 \Leftrightarrow x < 0$ or $4 < x < 6$ or $x > 6$ $\Leftrightarrow x \in (-\infty,0)\cup(4,6)\cup(6,\infty)$	1A
(iii) $f''(x) > 0 \Leftrightarrow x > 6$ $\Leftrightarrow x \in (6,\infty)$	
(iv) $f''(x) < 0 \Leftrightarrow x < 0$ or $0 < x < 6$ $\Leftrightarrow x \in (-\infty,0)\cup(0,6)$	1A for (iii) and (iv)
	----------(3)

Solution　　Marks

(c)

x	$(-\infty, 0)$	0	(0,4)	4	(4,6)	6	$(6,\infty)$
f(x)	↓	0	↑	$2^{\frac{5}{3}}$	↓	0	↓
f ′(x)	–	Undefined	+	0	–	Undefined	–
f ″(x)	–	Undefined	–	–	–	Undefined	+

One relative minimum point: (0, 0) .　　1A

One relative maximum point: $(4, 2^{\frac{5}{3}})$.　　1A (2, 3.1748)

One point of inflexion: (6, 0) .　　1A

----------(3)

(d) Since $\lim\limits_{x\to\infty}\frac{f(x)}{x} = \lim\limits_{x\to\infty}\left(\frac{6}{x}-1\right)^{\frac{1}{3}} = -1$, and　　1M

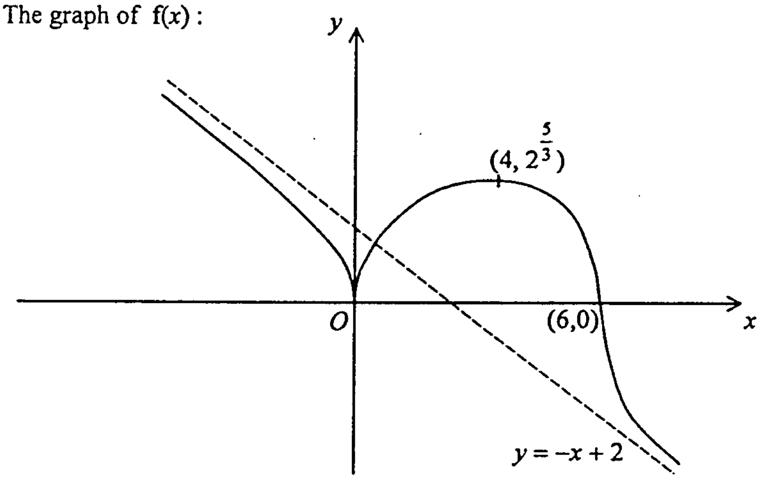
$$\lim_{x\to\infty}[f(x)+x] = \lim_{x\to\infty}\left[x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}+x\right] = \lim_{x\to\infty}\frac{\left(\frac{6}{x}-1\right)^{\frac{1}{3}}+1}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} 2\left(\frac{6}{x}-1\right)^{-\frac{2}{3}} = 2 ,$$

therefore $y=-x+2$ is an oblique asymptote of the graph of f(x) .　　1A

----------(2)

(e) The graph of f(x) :



1A Extrema and pts. of inflexion

1A Asymptote

1A Shape of the curve

----------(3)

Solution	Marks
9. (a) $P_1=(0,3)$ and $P_2=(0,-5)$ are the two points on $\mathcal{C}$ such that P_1F and P_2F are vertical.	1M
$\therefore$ The two horizontal tangents to the ellipse are L_{P_1} and L_{P_2} . i.e. $y=\pm 2$.	1A
Putting $y=1$ into $x^2+(y+1)^2=4^2$, we have $x=\pm 2\sqrt{3}$. $P_3=\left(2\sqrt{3},1\right)$ and $P_4=\left(-2\sqrt{3},1\right)$ are the two points on $\mathcal{C}$ such that P_3F and P_4F are horizontal.	1M
$\therefore$ The two vertical tangents to the ellipse are L_{P_3} and L_{P_4} . i.e. $x=\pm\sqrt{3}$.	1A
Since the circle is symmetric about the y-axis and F also lies on this line of symmetry, the ellipse must also be symmetric about the y-axis. Thus one of the axes of symmetry of the ellipse is vertical. Since $y=\pm 2$ are the horizontal tangents, the end-points of this vertical axis of the ellipse must be $(0,\pm 2)$. This implies that the centre of the ellipse is the origin and the other axes of symmetry is horizontal and passes through the origin. Since $x=\pm\sqrt{3}$ are the vertical tangents, the end-points of this horizontal axis must be $(\pm\sqrt{3},0)$.	
The equation of the ellipse is $\left(\dfrac{x}{\sqrt{3}}\right)^2+\left(\dfrac{y}{2}\right)^2=1$ $4x^2+3y^2=12$	1A
	----------(5)
(b) (i) (I) Since P lies on $\mathcal{C}$, $\therefore$ $p^2+q^2+2q-15=0$.	1M
Putting M into the equation of $\mathcal{E}$, we have $\text{LHS}=\dfrac{36p^2}{(7-q)^2}+\dfrac{48(q-1)^2}{(7-q)^2}$ $=\dfrac{12}{(7-q)^2}\left[3(-q^2-2q+15)+4(q^2-2q+1)\right]$ $=\dfrac{12}{(7-q)^2}\left[q^2-14q+49\right]$ $=12=\text{RHS}$	1
$\therefore$ M lies on $\mathcal{E}$.	
(II) The equation of the tangent, T_M , at M to $\mathcal{E}$ is $4mx+3ny=12$ or $px+(q-1)y+(q-7)=0$.	1A
Slope of $T_M=m_{T_M}=-\dfrac{4m}{3n}=\dfrac{-12p}{12(q-1)}=-\dfrac{p}{q-1}$. Slope of $FP=m_{FP}=\dfrac{q-1}{p}$.	
$\therefore$ $m_{TM}\cdot m_{FP}=-1$.	1

Solution	Marks
Putting the mid-point of FP, $\left(\frac{p}{2}, \frac{q+1}{2}\right)$, into the equation of T_M : $\text{LHS} = \frac{p^2}{2}+\frac{(q-1)(q+1)}{2}+(q-7)$ $= \frac{1}{2}\left[(-q^2-2q+15)+(q^2-1)+2(q-7)\right]$ $= 0 = \text{RHS}$.	1M+1
$MP^2 = \left(p-\frac{3p}{7-q}\right)^2+\left(q-\frac{4(q-1)}{7-q}\right)^2 = \frac{[p^2+(q+1)^2](q-4)^2}{(7-q)^2}$ $= \frac{16(q-4)^2}{(7-q)^2}$ $MF^2 = \left(\frac{3p}{7-q}\right)^2+\left(\frac{4(q-1)}{7-q}-1\right)^2 = \frac{9(15-q^2-2q)+(5q-11)^2}{(7-q)^2}$ $= \frac{16q^2-128q+256}{(7-q)^2} = MP^2$	$\boxed{1M}+\boxed{1}$
∴ T_M is the perpendicular bisector of FP .	
(ii) Let $M(m, n)$ be any point on $\mathcal{E}$. Solving $m = \frac{3p}{7-q}$ and $n = \frac{4(q-1)}{7-q}$ we have $p = \frac{8m}{n+4}$ and $q = \frac{7n+4}{n+4}$.	1A
Since M lies on $\mathcal{E}$, $4m^2+3n^2=12$. Let P be the point (p, q) and put (p, q) into $\mathcal{C}$, $\text{LHS} = \frac{64m^2}{(n+4)^2}+\frac{64(n+1)^2}{(n+4)^2}$ $= \frac{16(-3n^2+12+4n^2+8n+4)}{(n+4)^2}$ $= 16 = \text{RHS}$.	1
∴ P lies on $\mathcal{C}$. Mid-point of $FP = \left(\frac{4m}{n+4}, \frac{4(n+1)}{n+4}\right)$. Slope of $FP = \frac{3n}{4m}$. Equation of the perpendicular bisector of FP is $\frac{(n+4)y-4(n+1)}{(n+4)x-4m} = -\frac{4m}{3n}$	1A
$3n(n+4)y-12n(n+1) = -4m(n+4)x+16m^2$ $4m(n+4)x+3n(n+4)y = 12n(n+1)+4(12-3n^2)$ $4m(n+4)x+3n(n+4)y = 12(n^2+n+4-n^2)$ $4mx+3ny=12$ which is the same as the equation of the tangent to $\mathcal{E}$ at M.	1
	---------(10)

Solution	Marks
10. (a) For $n=0$, $\lim_{x\to 0^+} x(\ln x)^0 = 0$.	1
Assume $\lim_{x\to 0^+} x(\ln x)^k = 0$ where k is a non-negative integer.	
Then $\lim_{x\to 0^+} x(\ln x)^{k+1} = \lim_{x\to 0^+} \frac{(\ln x)^{k+1}}{x^{-1}}$	
$= \lim_{x\to 0^+} -\frac{(k+1)(\ln x)^k}{x^{-1}}$ [by L'Hospital rule]	1M
$= 0$ [by induction hypothesis]	1
By the principle of mathematical induction, the result follows.	
	----------(3)
(b) (i) Using the method of integration by parts,	
$\int(\ln x)^n \mathrm{d}x = x(\ln x)^n - \int x\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)^n \mathrm{d}x$	
$= x(\ln x)^n - n\int(\ln x)^{n-1}\mathrm{d}x$ for any positive integers n .	1
(ii) For $0<h<1$, $\int_h^1 \ln x\,\mathrm{d}x = [x\ln x - x]_h^1 = -1-h\ln h+h$	1M
Since $\lim_{h\to 0^+} h\ln h = 0$ by (a), $\therefore$ $\lim_{h\to 0^+}\int_h^1 \ln x\,\mathrm{d}x$ exists and	1A will be awarded for getting the right answer without taking the limit properly
$\int_0^1 \ln x\,\mathrm{d}x = \lim_{h\to 0^+}\int_h^1 \ln x\,\mathrm{d}x = -1$	1A
For $0<h<1$ and any positive integer n ,	
$\int_h^1 (\ln x)^n\mathrm{d}x = [x(\ln x)^n]_h^1 - n\int_h^1 (\ln x)^{n-1}\mathrm{d}x$	
$= -h(\ln h)^n - n\left\{[x(\ln x)^{n-1}]_h^1 - (n-1)\int_h^1(\ln x)^{n-2}\mathrm{d}x\right\}$	1M+1A
$= -h(\ln h)^n + nh(\ln h)^{n-1} + n(n-1)\int_h^1(\ln x)^{n-2}\mathrm{d}x$	
$\vdots$	
$= -h(\ln h)^n + nh(\ln h)^{n-1} - \cdots + (-1)^n(n!)h(\ln h) + (-1)^n(n!)\int_h^1 \mathrm{d}x$	1A
Since $\lim_{h\to 0^+} h(\ln h)^k = 0$ $(k=0, 1, 2, \ldots)$ by (a),	
$\therefore$ $\lim_{h\to 0^+}\int_h^1(\ln x)^n\,\mathrm{d}x$ exists and	1M
$\int_0^1(\ln x)^n\,\mathrm{d}x = \lim_{h\to 0^+}\int_h^1(\ln x)^n\,\mathrm{d}x = (-1)^n(n!)$	1A

Solution	Marks
11. (a) (i) $h(a) = f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] = f(a)g(b) - f(b)g(a)$ and	
$h(b) = f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)] = f(a)g(b) - f(b)g(a)$	
$\therefore \quad h(a) = h(b)$	1A
Since h is continuous on $[a, b]$ and differentiable in (a, b),	
there is a $c \in (a, b)$ such that	
$h'(c) = 0$ [Mean Value Theorem]	1M
i.e. $f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$	1
(ii) Since $g'(x) > 0$ for all $x \in (a, b)$,	
$\therefore$ $g(x)$ is strictly increasing on (a, b)	1
By the continuity of g at $x = a$, $g(x) > g(a)$ for any $x \in (a, b)$.	1
i.e. $g(x) - g(a) > 0$ for any $x \in (a, b)$.	
For any $x \in (a, b)$,	
$P'(x) = \dfrac{[g(x) - g(a)]f'(x) - [f(x) - f(a)]g'(x)}{[g(x) - g(a)]^2}$.	1A
Using the result of (a) and for $x \in (a, b)$,	
$P(x) = \dfrac{f(x) - f(a)}{g(x) - g(a)} = \dfrac{f'(c)}{g'(c)}$ for some $c \in (a, x)$	1A
Since $\dfrac{f'(x)}{g'(x)}$ is increasing, $\therefore$ $\dfrac{f'(c)}{g'(c)} \le \dfrac{f'(x)}{g'(x)}$.	1A
Hence $\dfrac{f(x) - f(a)}{g(x) - g(a)} \le \dfrac{f'(x)}{g'(x)}$	
$\Rightarrow \quad [g(x) - g(a)]f'(x) - [f(x) - f(a)]g'(x) \ge 0$	
$\Rightarrow \quad P'(x) \ge 0$	
i.e. $P(x)$ is also increasing on (a, b).	1
	----------(9)

Solution	Marks
(b) Using L'Hospital rule,	
$\lim\limits_{x\to 0^+} \mathrm{Q}(x) = \lim\limits_{x\to 0^+} \frac{e^x\cos x - 1}{\sin x + \cos x - 1} = \lim\limits_{x\to 0^+} \frac{e^x(\cos x - \sin x)}{\cos x - \sin x} = 1 = \mathrm{Q}(0)$	1M+1
$\therefore$ Q is continuous at $x = 0$.	
Now, Q is continuous on $\left[0, \frac{\pi}{4}\right]$ and differentiable in $\left(0, \frac{\pi}{4}\right)$.	
Putting $\mathrm{f}(x) = e^x\cos x$ and $\mathrm{g}(x) = \sin x + \cos x$, then for $x \in \left(0, \frac{\pi}{4}\right)$,	1A
$\begin{cases} \mathrm{g}'(x) = \cos x - \sin x > 0 \text{ and} \\ \frac{\mathrm{f}'(x)}{\mathrm{g}'(x)} = \frac{e^x(\cos x - \sin x)}{\cos x - \sin x} = e^x \text{ which is increasing on } \left(0, \frac{\pi}{4}\right). \end{cases}$	1 for checking $\mathrm{g}'(x) > 0$ and $\frac{\mathrm{f}'(x)}{\mathrm{g}'(x)}$ is increasing
Now for $x \in \left(0, \frac{\pi}{4}\right)$, $\mathrm{Q}(x) = \frac{e^x\cos x - 1}{\sin x + \cos x - 1} = \frac{\mathrm{f}(x) - \mathrm{f}(0)}{\mathrm{g}(x) - \mathrm{g}(0)}$.	
Using the result in (a)(ii) and the fact that Q is continuous on $\left[0, \frac{\pi}{4}\right]$,	
Q is increasing on $\left[0, \frac{\pi}{4}\right]$.	1A
Hence $\mathrm{Q}(x) \ge \mathrm{Q}(0) = 1$ for $x \in \left[0, \frac{\pi}{4}\right]$,	
$\Rightarrow \int_0^x \mathrm{Q}(t)\,\mathrm{d}t \ge \int_0^x \mathrm{d}t = x$ for $x \in \left[0, \frac{\pi}{4}\right]$.	1
	----------(6)

Solution	Marks
12. (a) $\int\frac{1}{x^2-x+1}\,dx = \int\frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\,dx$	1A
$=\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$	
$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$	1A
Let $\frac{x^2+1}{(x^2-x+1)(x^2+x+1)}\equiv\frac{ax+b}{x^2-x+1}+\frac{cx+d}{x^2+x+1}$ for some $a, b, c, d\in\mathbf{R}$	1M
Then $x^2+1\equiv(ax+b)(x^2+x+1)+(cx+d)(x^2-x+1)$	
$\Rightarrow\ a=c=0,\ b=d=\frac{1}{2}$.	1A
Hence $\int\frac{x^2+1}{(x^2-x+1)(x^2+x+1)}\,dx$	
$=\frac{1}{2}\int\frac{1}{x^2-x+1}\,dx+\frac{1}{2}\int\frac{1}{x^2+x+1}\,dx$	
$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$	1M+1A
	----------(6)
(b) For $n=1, 2, 3, \ldots$ and any fixed $x\in[0, 1)$,	
(i) $g_{n+1}(x)-g_n(x)=\frac{x^{6(n+1)-5}}{6(n+1)-5}\geq 0$ and $h_{n+1}(x)-h_n(x)=\frac{x^{6(n+1)-1}}{6(n+1)-1}\geq 0$.	
$\therefore$ $\{g_n(x)\}$ and $\{h_n(x)\}$ are increasing.	1
(ii) $g_n(x)=\sum_{r=1}^{n}\frac{x^{6r-5}}{6r-5}\leq\sum_{r=1}^{n}x^{6r-5}=\frac{x(1-x^{6n})}{1-x^6}\leq\frac{x}{1-x^6}$	1M
$h_n(x)=\sum_{r=1}^{n}\frac{x^{6r-1}}{6r-1}\leq\sum_{r=1}^{n}x^{6r-1}=\frac{x^5(1-x^{6n})}{1-x^6}\leq\frac{x^5}{1-x^6}$	
$\therefore$ Both $\{g_n(x)\}$ and $\{h_n(x)\}$ are bounded above.	
Using (i), $\lim_{n\to\infty}g_n(x)$ and $\lim_{n\to\infty}h_n(x)$ exist.	1
	----------(3)

Solution	Marks
(c) Let $n = 1, 2, 3, \ldots$.	
(i) For any fixed $x \in (0, 1)$,	
$f_n'(x) = \sum_{r=1}^{n}\left(x^{6r-6} - x^{6r-2}\right)$	
$= \left(1+x^6+x^{12}+\cdots+x^{6(n-1)}\right) - x^4\left(1+x^6+x^{12}+\cdots+x^{6(n-1)}\right)$	1A
$= \dfrac{(1-x^4)(1-x^{6n})}{1-x^6}$	1M
Since $0 < x < 1$, $x^{6n} \to 0$ as $n \to \infty$.	
$\therefore \lim_{n\to\infty} f_n'(x)$ exists.	1
(ii) For $x \in (0, 1)$, $f'(x) = \lim_{n\to\infty} f_n'(x)$	
$= \dfrac{1-x^4}{1-x^6}$	
$= \dfrac{1+x^2}{1+x^2+x^4}$	1
Hence $f(x) = \displaystyle\int \frac{1+x^2}{1+x^2+x^4}\,dx$	1
$= \displaystyle\int \frac{x^2+1}{(x^2+1)^2-x^2}\,dx$	
$= \displaystyle\int \frac{x^2+1}{(x^2-x+1)(x^2+x+1)}\,dx$	
$= \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right) + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right) + c$	
for $x \in (0, 1)$	
Noting that $f(0) = 0$ since $f_n(0) = 0$ for all n and that f is continuous on $[0, 1)$, we have $c = 0$.	
i.e. $f(x) = \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right) + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$	1A
	----------(6)

Solution	Marks
13. (a) Using the given condition, $f(1+x)=\dfrac{f(x)}{1+f(x)}$.	
Assume $f(k+x)=\dfrac{f(x)}{1+k\,f(x)}$ where $k\in\mathbf{Z}^+$. Then	
$f[(k+1)+x] = \dfrac{f(k+x)}{1+f(k+x)}$	1
$= \dfrac{\frac{f(x)}{1+k\,f(x)}}{1+\frac{f(x)}{1+k\,f(x)}}$	1
$= \dfrac{f(x)}{1+(k+1)f(x)}$.	
By the principle of mathematical induction, the result follows.	1 ----------(3)
(b) $f(x_n) = f\left[1+\dfrac{f(1)}{1+n\,f(1)}\right]$	
$= \dfrac{f\left[\frac{f(1)}{1+n\,f(1)}\right]}{1+f\left[\frac{f(1)}{1+n\,f(1)}\right]}$	1A
$= \dfrac{f[f(n+1)]}{1+f[f(n+1)]}$	
$= \dfrac{n+1}{n+2}$	1
By the continuity of f ,	
$f(1) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\dfrac{n+1}{n+2} = 1$	1
For $n>1$, $f(n) = f[(n-1)+1]$	
$= \dfrac{f(1)}{1+(n-1)\,f(1)}$ (by (a))	
$= \dfrac{1}{n}$	1
For $n\in N$, $f\left(\dfrac{1}{n}\right) = f[f(n)] = n$	1
	----------(5)

Solution	Marks
(c) (i) From (b), $f\left(\frac{1}{2}\right)=2$. $\therefore$ S(2) is true.	1
(ii) Assume that S(h) is true for $2 \le h \le q$ and $h \in \mathbf{N}$. For $0 < p < q+1$, there are $\alpha, \beta \in \mathrm{N}$ with $0 \le \beta < p$ such that $q+1 = \alpha p + \beta$.	1M
$\therefore \quad f\left(\frac{q+1}{p}\right) = f\left(\alpha+\frac{\beta}{p}\right)$ $= \frac{f\left(\frac{\beta}{p}\right)}{1+\alpha f\left(\frac{\beta}{p}\right)}$ [from (a)] $= \frac{\frac{p}{\beta}}{1+\alpha\left(\frac{p}{\beta}\right)}$ [by induction hypothesis] $= \frac{p}{\alpha p+\beta}$ $= \frac{p}{q+1}$	1
Hence $f\left[f\left(\frac{q+1}{p}\right)\right] = f\left(\frac{p}{q+1}\right)$, $f\left(\frac{p}{q+1}\right) = \frac{q+1}{p}$ i.e. S(q + 1) is true.	1
(iii) From (i), S(2) is true. From (ii), assuming that S(h) is true for $2 \le h \le q$, it is proved that S(q + 1) is true. By the principle of mathematical induction, S(q) is true for $q = 2, 3, 4, ..$	
For any positive rational number x, (I) If x is an integer, then $f(x)=\frac{1}{x}$ by (b).	1
(II) If x is not an integer, then $x = n + r$ for some non-negative integer n and $0 < r < 1$.	1
Let $r = \frac{p}{q}$ where $p, q \in \mathrm{N}$ and $0 < p < q$. Then $f(x) = f(n+\frac{p}{q})$ $= \frac{f\left(\frac{p}{q}\right)}{1+n f\left(\frac{p}{q}\right)}$ [by (a)] $= \frac{\frac{q}{p}}{1+n\cdot\frac{q}{p}}$ [Since S(q) is true] $= \frac{1}{n+\frac{p}{q}}$ $= \frac{1}{x}$	1
	----------(7)