2001-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2001

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A and any FOUR questions in Section B.
3. You are provided with one AL(E) answer book and four AL(D) answer books.
   Section A : Write your answers in the AL(E) answer book.
   Section B : Use a separate AL(D) answer book for each question and put the question number on the front cover of each answer book.
4. The four AL(D) answer books should be tied together with the green tag provided. The AL(E) answer book and the four AL(D) answer books must be handed in separately at the end of the examination.
5. Unless otherwise specified, all working must be clearly shown.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Resolve $\frac{8}{x(x-2)(x+2)}$ into partial fractions.

   (b) Show that $\sum_{r=3}^{2001} \frac{8}{r(r-2)(r+2)} < \frac{11}{12}$.

(5 marks)

2. (a) Show that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

   (b) A sequence $\{a_n\}$ is defined as follows:

   $a_1 = 6$,

   $a_{k+1} = a_k + 3k^2 + 9k + 6$ for $k = 1, 2, 3, \ldots$.

   Find a_n in terms of n .

(5 marks)

3. (a) Let $0 < \lambda < 1$. Show that $x^{\lambda} \le (1-\lambda) + \lambda x$ for all $x > 0$.

   (b) Let a , b , p and q be positive numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{a}{p} + \frac{b}{q}$.

(5 marks)

Go on to the next page

4. A , B , C are the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ respectively and O is the origin.

   (a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

   (b) Let $S_{\Delta XYZ}$ denote the area of the triangle with vertices X , Y and Z . Prove that ${S_{\Delta ABC}}^2 = {S_{\Delta OAB}}^2 + {S_{\Delta OBC}}^2 + {S_{\Delta OCA}}^2$.

(5 marks)

5. Let the kth term in the binomial expansion of $(1+x)^{2n}$ in ascending powers of x be denoted by T_k , i.e. $T_k = C_{k-1}^{2n} x^{k-1}$.

   (a) If $x = \frac{1}{3}$, find the range of values of k such that $T_{k+1} \ge T_k$.

   (b) Find the greatest term in the expansion if $x = \frac{1}{3}$ and $n = 15$.

(5 marks)

6. Let $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \ne 0$. Show that if $f[f(x)] = [f(x)]^2$ for all x , then $f(x) = x^2$.

(4 marks)

7. A 2×2 matrix M is the matrix representation of a transformation T in $\mathbf{R}^2$. T transforms $(1, 0)$ and $(0, 1)$ to $(1, 1)$ and $(-1, 1)$ respectively.

(a) Find M.

(b) Find $\lambda > 0$ such that M can be decomposed as $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$.

Hence describe the geometric meaning of T.

(5 marks)

8. Let L be the straight line $|z-(4+4i)| = |z|$ and C be the circle $|z| = 1$.

(a) Sketch L on an Argand diagram.

(b) Let P, Q be points on L and C respectively such that PQ is equal to the shortest distance between L and C. Find PQ and the complex numbers representing P and Q.

(6 marks)

Go on to the next page

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

9. Consider the system of linear equations

$$(S): \begin{cases} x + \lambda y + z = k \\ \lambda x - y + z = 1 \\ 3x + y + 2z = -1 \end{cases} \quad \text{where } \lambda, k \in \mathbf{R}.$$

(a) Show that (S) has a unique solution if and only if $\lambda \neq 0$ and $\lambda \neq 2$.

(2 marks)

(b) For each of the following cases, determine the value(s) of k for which (S) is consistent. Solve (S) in each case.

(i) $\lambda \neq 0$ and $\lambda \neq 2$,

(ii) $\lambda = 0$,

(iii) $\lambda = 2$.

(8 marks)

(c) If some solution (x, y, z) of $\begin{cases} x + z = 0 \\ -y + z = 1 \\ 3x + y + 2z = -1 \end{cases}$

satisfies $(x-p)^2 + y^2 + z^2 = 1$, find the range of values of p.

(5 marks)

10. Let $\begin{cases} a_1 = 1 \\ b_1 = 1 \end{cases}$ and $\begin{cases} a_n = a_{n-1} + 2b_{n-1} \\ b_n = a_{n-1} + b_{n-1} \end{cases}$, $n = 2, 3, 4, \ldots$.

(a) Show that for any positive integer n ,

(i) $a_n, b_n > 0$ and $a_n{}^2 - 2b_n{}^2 = (-1)^n$;

(ii) $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$.

(4 marks)

(b) For $n = 1, 2, 3, \ldots$, define $u_n = \dfrac{a_n}{b_n}$.

(i) Show that $u_{n+1} = \dfrac{u_n + 2}{u_n + 1}$.

(ii) Show that $u_{2n-1} < \sqrt{2}$ and $u_{2n} > \sqrt{2}$.

(iii) Show that $u_{n+2} = \dfrac{3u_n + 4}{2u_n + 3}$.

Hence show that the sequence $\{u_1, u_3, u_5, \ldots\}$ is strictly increasing and the sequence $\{u_2, u_4, u_6, \ldots\}$ is strictly decreasing.

(iv) Show that the sequences $\{u_1, u_3, u_5, \ldots\}$ and $\{u_2, u_4, u_6, \ldots\}$ converge to the same limit.

Find this limit.

(11 marks)

Go on to the next page

11. (a) Show that $\dfrac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = i\cot\dfrac{\theta}{2}$.

(3 marks)

(b) Let n be a positive integer. Show that all the roots of the equation

$$(z-1)^n + (z+1)^n = 0 \qquad \ldots\ldots\ldots(*)$$

can be written as $i\alpha_k$, where $\alpha_k \in \mathbf{R}$, $k = 0, 1, \ldots, n-1$.

(4 marks)

(c) If $i\alpha_k$ $(k = 0, 1, \ldots, n-1)$ are the roots of (*) in (b), using the relations between the roots and coefficients, show that

$\sum_{k=0}^{n-1} \alpha_k{}^2 = n(n-1)$.

(5 marks)

(d) Let P_0 , P_1 , …, P_{n-1} be the n points in an Argand plane representing the roots of (*) in (b), and O be the origin. Q is the point representing $r(\cos\beta + i\sin\beta)$ where $r \geq 0$ and $\beta \in \mathbf{R}$. If d_k is the distance between P_k and Q , show that $\sum_{k=0}^{n-1} d_k{}^2$ is independent of β .

(3 marks)

12. The position vectors of four points A, B, C and D are $\mathbf{a} = 7\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 7\mathbf{j} + 13\mathbf{k}$, $\mathbf{c} = 17\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{d} = r(2\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})$ respectively, where r is a non-zero real number.

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.

(3 marks)

(b) If $\mathbf{v} = \lambda\mathbf{a} + \mu\mathbf{b}$, where $\lambda, \mu \in \mathbf{R}$ and $\lambda, \mu \neq 0$, and $\mathbf{v}$ is also a linear combination of vectors $\mathbf{c}$ and $\mathbf{d}$, show that $\lambda : \mu = (-\mathbf{b} \cdot \mathbf{c} \times \mathbf{d}) : (\mathbf{a} \cdot \mathbf{c} \times \mathbf{d})$.

Hence evaluate $\lambda : \mu$.

(5 marks)

(c) Suppose the four points A, B, C and D are coplanar.

(i) Find r.

(ii) Let E be the intersection of AB and CD. Using (b) or otherwise, find $BE : EA$ and the position vector of E.

(7 marks)

Go on to the next page

13. (a) Let $\mathrm{P}(x) = x^4 + ax^3 + bx^2 + cx + d$ where $a, b, c, d \in \mathbf{R}$.

(i) Show that if α is a complex root of $\mathrm{P}(x) = 0$, then $\overline{\alpha}$ is also a root of $\mathrm{P}(x) = 0$.

(ii) For any $\alpha \in \mathbf{C}$, show that $(x - \alpha)(x - \overline{\alpha})$ is a quadratic polynomial in x with real coefficients.

Hence show that $\mathrm{P}(x)$ can be factorized as a product of two quadratic polynomials with real coefficients.

(6 marks)

(b) Let $\mathrm{f}(x) = x^4 + 8x^3 + 23x^2 + 26x + 7$ and $\mathrm{g}(x) = \mathrm{f}(x + k)$ where $k \in \mathbf{R}$ and the coefficient of x^3 in $\mathrm{g}(x)$ is zero.

(i) Find k and the coefficients of $\mathrm{g}(x)$.

(ii) Suppose $\mathrm{g}(x) = (x^2 + px + q)(x^2 + rx + s)$ where $p, q, r, s \in \mathbf{R}$. By comparing coefficients or otherwise, show that $p^6 - 2p^4 + 5p^2 - 4 = 0$.

Hence find p, q, r and s.

(iii) Find all roots of $\mathrm{f}(x) = 0$.

(9 marks)

14. (a) If a, b are two real numbers such that $a \le 1 \le b$, show that $a+b \ge ab+1$ and the equality holds if and only if $a=1$ or $b=1$.
(3 marks)

(b) Show by induction that if x_1, x_2, …, x_n are n $(n \ge 2)$ positive real numbers such that $x_1 x_2 \cdots x_n = 1$, then $x_1 + x_2 + \cdots + x_n \ge n$ and the equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$.
(6 marks)

(c) Let a_1, a_2, …, a_n be n $(n \ge 2)$ positive real numbers. Using (b) or otherwise, show that $\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n}$ and the equality holds if and only if $a_1 = a_2 = \cdots = a_n$.
(3 marks)

(d) For $u \ge 0$ and $n = 2, 3, 4, \ldots$, using the identity $(1+u)^n - 1 = u[(1+u)^{n-1} + (1+u)^{n-2} + \cdots + 1]$ or otherwise, show that $(1+u)^n \ge 1 + nu(1+u)^{\frac{n-1}{2}}$ and the equality holds if and only if $u = 0$.
(3 marks)

END OF PAPER

2001 Paper 1
Section A

1. (a) $\frac{1}{x-2} - \frac{2}{x} + \frac{1}{x+2}$

2. (b) $n(n+1)(n+2)$

4. (a) $bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}$

5. (a) $1 \le k \le \frac{2n+1}{4}$

(b) $T_8 = C_7^{30}\left(\frac{1}{3}\right)^7$

7. (a) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

(b) $\lambda = \sqrt{2}$

A rotation which rotates any vector anticlockwise through $\frac{\pi}{4}$ about the origin, followed by an enlargement with factor $\sqrt{2}$.

2001-AL
P MATH
PAPER 2

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2001

PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A and any FOUR questions in Section B.
3. You are provided with one AL(E) answer book and four AL(D) answer books.
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$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. (a) Find $\lim_{x \to 0} \frac{(e^x - e^{-x})^2}{1 - \cos 2x}$.

   (b) Prove that $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$.

(6 marks)

2. Evaluate

   (a) $\int \frac{x^3}{1+x^2} \, dx$,

   (b) $\int x^2 \tan^{-1} x \, dx$.

(5 marks)

3. Let $a_1 = 1$ and $a_{n+1} = \sqrt{\frac{4 + a_n{}^2}{2}}$ for $n \in \mathbf{N}$. Show that $1 \le a_n < 2$ for all $n \in \mathbf{N}$.

   Hence show that $\{a_n\}$ is convergent and find its limit.

(6 marks)

4. Let P be the parabola $y^2 = 4ax$ where a is a non-zero constant, and $A(at_1{}^2, 2at_1)$, $B(at_2{}^2, 2at_2)$ be two distinct points on P .

   (a) Find the equation of chord AB .

   (b) If A and B move in such a way that chord AB always passes through $(a, 0)$, find the equation of the locus of the mid-point of AB .

(5 marks)

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5. Let f be a real-valued function continuous on $[0, 1]$ and differentiable in $(0, 1)$. Suppose f satisfies

   A. $f(0) = 0$,

   B. $f(1) = \frac{1}{2}$,

   C. $0 < f'(t) < 1$ for $t \in (0, 1)$.

   Define $F(x) = 2\int_0^x f(t)dt - [f(x)]^2$ for $x \in [0, 1]$.

   (a) Show that $F'(x) > 0$ for $x \in (0, 1)$.

   (b) Show that $\int_0^1 f(t)dt > \frac{1}{8}$.

(6 marks)

6. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \begin{cases} x^3 - 12x & \text{when } x \ge 2 , \\ \frac{x}{e^2} + \frac{x}{e^x} & \text{when } x < 2 . \end{cases}$

   (a) Show that $\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x) = 0$.

   (b) Is f differentiable at $x = 2$? Explain your answer.

(5 marks)

7. Figure 1 shows the curve

   $\Gamma : \begin{cases} x = \cos^3 t , \\ y = \sin^3 t , \end{cases} \quad 0 \le t \le 2\pi$.

   (a) Find the length of Γ .

   (b) Find the area enclosed by Γ .

(7 marks)

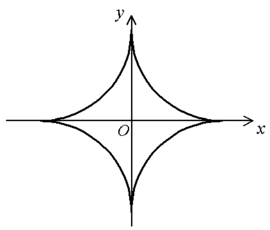

Figure 1

SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

8. Let $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$.

(a) (i) Find $f'(x)$ for $x \neq 0, 6$.

(ii) Show that $f'(0)$ and $f'(6)$ do not exist.

(iii) Show that $f''(x) = \dfrac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$ for $x \neq 0, 6$.

(4 marks)

(b) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$, (ii) $f'(x) < 0$,

(iii) $f''(x) > 0$, (iv) $f''(x) < 0$.

(3 marks)

(c) Find all relative extreme points and points of inflexion of $f(x)$.
(3 marks)

(d) Find all asymptotes of the graph of $f(x)$.
(2 marks)

(e) Sketch the graph of $f(x)$.
(3 marks)

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9. In Figure 2, C is the circle $x^2 + (y+1)^2 = 4^2$ and F is the point $(0, 1)$. For any point P on C , let L_p be the perpendicular bisector of the line segment FP . It appears that as P moves on C , all the L_p 's are tangents to an ellipse inside C .

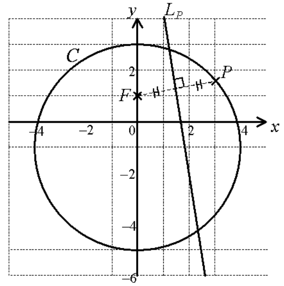

Figure 2

(a) Suppose for every P on C , the line L_p is tangent to the same ellipse. Write down the equations of the two horizontal tangents and the two vertical tangents to the ellipse.

Hence guess the equation of the ellipse.
(Note that C is symmetric about the y-axis and F lies on this line of symmetry.)

(5 marks)

(b) Let E be the ellipse found in (a).

(i) For any point $P(p, q)$ on C , let M be the point (m, n) where $m = \dfrac{3p}{7-q}$ and $n = \dfrac{4(q-1)}{7-q}$. Show that

(I) M lies on E ,

(II) the tangent at M to E is the perpendicular bisector of FP .

(ii) For any point M on E , show that there is a point P on C such that the perpendicular bisector of FP is the tangent to E at M .
(10 marks)

10. (a) Prove by induction that $\lim_{x \to 0^+} x(\ln x)^n = 0$ for any non-negative integer n.

(3 marks)

(b) Let n be a positive integer.

(i) Show that $\int (\ln x)^n \mathrm{d}x = x(\ln x)^n - n\int (\ln x)^{n-1} \mathrm{d}x$.

(ii) Show that the improper integral $\int_0^1 \ln x \, \mathrm{d}x$ is convergent (i.e. $\lim_{h \to 0^+} \int_h^1 \ln x \, \mathrm{d}x$ exists) and find its value.

Hence deduce that the improper integral $\int_0^1 (\ln x)^n \mathrm{d}x$ is convergent and find its value.

(8 marks)

(c) Let n be a positive integer and α be a positive real number. For $0 < h < 1$, show that $\int_h^1 x^{\alpha-1} (\ln x)^n \mathrm{d}x = \frac{1}{\alpha^{n+1}} \int_{h^\alpha}^1 (\ln x)^n \mathrm{d}x$.

Hence show that the improper integral $\int_0^1 x^{\alpha-1} (\ln x)^n \mathrm{d}x$ is convergent and find its value.

(4 marks)

Go on to the next page

11. (a) Let f and g be real-valued functions continuous on $[a, b]$ and differentiable in (a, b) .

(i) By considering the function $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$ on $[a, b]$, or otherwise, show that there is $c \in (a, b)$ such that $f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$.

(ii) Suppose $g'(x) > 0$ for all $x \in (a, b)$. Show that $g(x) - g(a) > 0$ for any $x \in (a, b)$.

If, in addition, $\frac{f'(x)}{g'(x)}$ is increasing on (a, b) , show that $P(x) = \frac{f(x) - f(a)}{g(x) - g(a)}$ is also increasing on (a, b) .

(9 marks)

(b) Let $Q(x) = \begin{cases} \dfrac{e^x \cos x - 1}{\sin x + \cos x - 1} & \text{if } x \in \left(0, \dfrac{\pi}{4}\right], \\ 1 & \text{if } x = 0 . \end{cases}$

Show that Q is continuous at $x = 0$ and increasing on $\left[0, \frac{\pi}{4}\right]$.

Hence or otherwise, deduce that for $x \in \left[0, \frac{\pi}{4}\right]$, $\int_0^x Q(t) \, \mathrm{d}t \geq x$.

(6 marks)

12. (a) Evaluate

(i) $\int \frac{1}{x^2-x+1}\,dx$,

(ii) $\int \frac{x^2+1}{(x^2-x+1)(x^2+x+1)}\,dx$.

(6 marks)

(b) For $n = 1, 2, 3, \ldots$ and $0 \le x < 1$, define $g_n(x) = \sum_{r=1}^{n} \frac{x^{6r-5}}{6r-5}$ and $h_n(x) = \sum_{r=1}^{n} \frac{x^{6r-1}}{6r-1}$. For any fixed $x \in [0, 1)$,

(i) show that the sequences $\{g_n(x)\}$ and $\{h_n(x)\}$ are increasing;

(ii) deduce that $\lim_{n\to\infty} g_n(x)$ and $\lim_{n\to\infty} h_n(x)$ exist.

(3 marks)

(c) For $n = 1, 2, 3, \ldots$ and $0 \le x < 1$, define $f_n(x) = \sum_{r=1}^{n} \left(\frac{x^{6r-5}}{6r-5} - \frac{x^{6r-1}}{6r-1} \right)$ and let $f(x) = \lim_{n\to\infty} f_n(x)$.

(i) For any fixed $x \in (0, 1)$, evaluate $f_n'(x)$ and show that $\lim_{n\to\infty} f_n'(x)$ exists.

(ii) Assuming that $f'(x) = \lim_{n\to\infty} f_n'(x)$ for any fixed $x \in (0, 1)$ and f is continuous on $[0, 1)$, show that $f'(x) = \frac{1+x^2}{1+x^2+x^4}$ for $x \in (0, 1)$.

Hence find $f(x)$.

(6 marks)

Go on to the next page

13. Let $f : (0, \infty) \to (0, \infty)$ be a continuous function satisfying $f[f(x)] = x$ and $f(1+x) = \frac{f(x)}{1+f(x)}$ for all x .

(a) Show that for any $n \in \mathbf{N}$ and $x \in \mathbf{R}$, $f(n+x) = \frac{f(x)}{1+n\,f(x)}$.

(3 marks)

(b) Define $x_n = 1 + \frac{f(1)}{1+n\,f(1)}$ for any $n \in \mathbf{N}$. Show that $f(x_n) = \frac{n+1}{n+2}$.

Hence, by considering $\lim_{n\to\infty} x_n$, show that $f(1) = 1$.

Deduce that $f(n) = \frac{1}{n}$ and $f\left(\frac{1}{n}\right) = n$.

(5 marks)

(c) For any $q \in \mathbf{N}$ and $q \ge 2$, let $S(q)$ be the statement "$f\left(\frac{p}{q}\right) = \frac{q}{p}$ for all $p \in \mathbf{N}$ with $0 < p < q$."

(i) Show that $S(2)$ is true.

(ii) Assume that $S(h)$ is true for $2 \le h \le q$ and $h \in \mathbf{N}$. Use (a) to show that $f\left(\frac{q+1}{p}\right) = \frac{p}{q+1}$ for $0 < p < q+1$.

Hence deduce that $S(q+1)$ is true.

(iii) Use (a) to show that for any positive rational number x , $f(x) = \frac{1}{x}$.

(7 marks)

END OF PAPER