Solution	Marks
1. $\because\quad M^2=\begin{pmatrix}1&0&0\\ \lambda&b&a\\ \mu&c&-b\end{pmatrix}\begin{pmatrix}1&0&0\\ \lambda&b&a\\ \mu&c&-b\end{pmatrix}$	
$=\begin{pmatrix}1&0&0\\ \lambda+\lambda b+\mu a&b^2+ac&0\\ \mu+\lambda c-\mu b&0&ac+b^2\end{pmatrix}$	1M accept mistakes in 1 entry
$=\begin{pmatrix}1&0&0\\ \lambda(1+b)+\mu a&1&0\\ \lambda c+\mu(1-b)&0&1\end{pmatrix}$	1A
$\therefore$ The statement is true for $n=1$.	
Assume $M^{2k}=\begin{pmatrix}1&0&0\\ k[\lambda(1+b)+\mu a]&1&0\\ k[\lambda c+\mu(1-b)]&0&1\end{pmatrix}$ for some positive integers k, then	1A
$M^{2(k+1)}=\begin{pmatrix}1&0&0\\ k[\lambda(1+b)+\mu a]&1&0\\ k[\lambda c+\mu(1-b)]&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\ \lambda(1+b)+\mu a&1&0\\ \lambda c+\mu(1-b)&0&1\end{pmatrix}$	1A
$=\begin{pmatrix}1&0&0\\ (k+1)[\lambda(1+b)+\mu a]&1&0\\ (k+1)[\lambda c+\mu(1-b)]&0&1\end{pmatrix}$	
By the principle of mathematical induction, the statement holds for all positive integer n.	
Put $M=\begin{pmatrix}1&0&0\\ -2&3&2\\ 1&-4&-3\end{pmatrix}$ which satisfies the condition that $b^2+ac=1$.	
Then $\begin{pmatrix}1&0&0\\ -2&3&2\\ 1&-4&-3\end{pmatrix}^{2000}=\begin{pmatrix}1&0&0\\ 1000[-2(1+3)+2]&1&0\\ 1000[-2(-4)+(1-3)]&0&1\end{pmatrix}$	
$=\begin{pmatrix}1&0&0\\ -6000&1&0\\ 6000&0&1\end{pmatrix}$	1A
	-----(5)

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Solution	Marks
2. (a) If $p-q\ge 0$, then $p\ge q$	
$\Rightarrow \ln p\ge \ln q$ ($\because$ $\ln x$ is increasing for $x>0$)	1M accept missing "="
$\Rightarrow \ln p-\ln q\ge 0$	
$\therefore (p-q)(\ln p-\ln q)\ge 0$	
If $p-q<0$, then $p<q$	
$\Rightarrow \ln p<\ln q$	
$\Rightarrow \ln p-\ln q<0$	
$\therefore (p-q)(\ln p-\ln q)>0$	1
(b) From (a), $p\ln p+q\ln q-(p\ln q+q\ln p)\ge 0$.	
$\therefore$ $a\ln a+b\ln b-(a\ln b+b\ln a)\ge 0$ $b\ln b+c\ln c-(b\ln c+c\ln b)\ge 0$ $c\ln c+a\ln a-(c\ln a+a\ln c)\ge 0$	1M
Summing up the inequalities, we have	1M
$2(a\ln a+b\ln b+c\ln c)-a(\ln b+\ln c)-b(\ln c+\ln a)-c(\ln a+\ln b)\ge 0$	
$3(a\ln a+b\ln b+c\ln c)-a(\ln a+\ln b+\ln c)-b(\ln a+\ln b+\ln c)-c(\ln a+\ln b+\ln c)\ge 0$	1A
$3(a\ln a+b\ln b+c\ln c)-(a+b+c)(\ln a+\ln b+\ln c)\ge 0$	
$a\ln a+b\ln b+c\ln c\ge \dfrac{a+b+c}{3}(\ln a+\ln b+\ln c)$	1
	-----(6)
3. (a) $\because$ $(1+x)^n=1+C_1^n x+C_2^n x^2+\cdots+C_n^n x^n$	1A
$\therefore$ $\dfrac{(1+x)^n-1}{x}=C_1^n+C_2^n x+C_3^n x^2+\cdots+C_n^n x^{n-1}$	1A
(b) From (a), $C_1^n+C_2^n x+C_3^n x^2+\cdots+C_n^n x^{n-1}=\dfrac{(1+x)^n-1}{x}$.	
Differentiating both sides w.r.t. x, we have	1M
$C_2^n+2C_3^n x+3C_4^n x^2+\cdots+(n-1)C_n^n x^{n-2}=\dfrac{nx(1+x)^{n-1}-(1+x)^n+1}{x^2}$	1A accept wrong constant term in (a)
Putting $x=1$,	
$C_2^n+2C_3^n+3C_4^n+\cdots+(n-1)C_n^n=n2^{n-1}-2^n+1$	
$=(n-2)2^{n-1}+1$	1
	-----(5)

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Solution	Marks
4. $z\bar{z}=(2+3i)\bar{z}+(2-3i)z+12$	
$z\bar{z}-(2+3i)\bar{z}-\overline{(2+3i)}z+(2+3i)\overline{(2+3i)}=13+12$	1M adding $(2+3i)\overline{(2+3i)}$ to both sides
$\bar{z}[z-(2+3i)]-\overline{(2+3i)}[z-(2+3i)]=25$	
$[z-(2+3i)][\bar{z}-\overline{(2+3i)}]=25$	
$\lvert z-(2+3i)\rvert=5$	1A
Alternatively,	
Let $z=x+yi$, then (*) becomes	
$(x+yi)(x-yi)=(2+3i)(x-yi)+(2-3i)(x+yi)+12$	
$x^2+y^2=4x+6y+12$	
$(x-2)^2+(y-3)^2=25$ (i.e. $a=2+3i$ and $r=5$)	1M
$\lvert z-(2+3i)\rvert=5$	1A
The circle is centred at $2+3i$ with radius 5.	1A
The shortest distance between the point $-4-5i$ and the circle is $\lvert(2+3i)-(-4-5i)\rvert-5=5$	1M+1A (award 1M also for correct graph) -----(5)
5. (a) Using the Euclidean Algorithm,	1M+1A

$2x^2$	$2x^4 - x^3 + 3x^2 - 2x + 1$	$x^2 - x + 1$	$-x$
	$2x^4 - 2x^3 + 2x^2$	$x^2 + x$	
x	$x^3 + x^2 - 2x + 1$	$-2x + 1$	2
	$x^3 - x^2 + x$	$-2x - 2$	
2	$2x^2 - 3x + 1$	3	
	$2x^2 - 2x + 2$		
	$-x - 1$		

Solution	Marks
f(x) and g(x) have no non-constant common factors.	1
Alternatively,	
∵ The discriminant of g(x) <0	
∴ g(x) has no linear factor.	1M
For f(x) and g(x) to have non-constant common factors, this factor must be g(x).	1M
Using long division, $f(x)=g(x)(2x^2+x+2)-x-1$.	
∴ g(x) is not a factor of f(x).	
Hence f(x) and g(x) have no non-constant common factors.	1
(b) ∵ $f(x)=g(x)(2x^2+x+2)-x-1$,	1M
∴ $f(x)+x+1$ is divisible by g(x).	
Hence $p(x)=x+1$	1A -----(5)

Solution	Marks
6. (a) When $\mathbf{x}=\begin{pmatrix}2\\0\end{pmatrix}$, $\mathbf{y}=\begin{pmatrix}\cos\frac{\pi}{3} & -\sin\frac{\pi}{3}\\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}-3\\1\end{pmatrix}$ $=\begin{pmatrix}-3+2\cos\frac{\pi}{3}\\ 1+2\sin\frac{\pi}{3}\end{pmatrix}$ $=\begin{pmatrix}-2\\1+\sqrt{3}\end{pmatrix}$	1A
(b) A rotation which rotates any vector anticlockwise through $\frac{\pi}{3}$ about the origin, followed by a translation which translates any vector by $\begin{pmatrix}-3\\1\end{pmatrix}$.	① 1A rotation with details 1A translation with details 1A ordering ② 1M rotation and translation 1A order of the composition 1A details of rotation and translation (Choose the one which would give higher marks)
(c) $\because\ \mathbf{y}=A\mathbf{x}+\mathbf{b}=A(\mathbf{x}+A^{-1}\mathbf{b})$	1M for $\mathbf{c}=A^{-1}\mathbf{b}$
$\therefore\ \mathbf{c}=\begin{pmatrix}\cos\frac{\pi}{3} & \sin\frac{\pi}{3}\\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3}\end{pmatrix}\mathbf{b}$	1A for A^{-1}
$=\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-3\\1\end{pmatrix}$	
$=\begin{pmatrix}\frac{-3+\sqrt{3}}{2}\\ \frac{1+3\sqrt{3}}{2}\end{pmatrix}$	1A
	-----(7)

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Solution	Marks
7. (a) If the roots of the equation are $\frac{a}{r}$, a and ar, then	
$\begin{cases} \frac{a}{r}+a+ar=-p \\ \frac{a}{r}\cdot a+a\cdot ar+ar\cdot\frac{a}{r}=q \\ \frac{a}{r}\cdot a\cdot ar=-1 \end{cases}$	1M+1A using relations between roots and coefficients
$\Rightarrow \begin{cases} a\left(\frac{1}{r}+1+r\right)=-p \\ a^2\left(\frac{1}{r}+r+1\right)=q \\ a^3=-1 \end{cases}$	
$\because\ a^3=-1 \Rightarrow a=-1$	
Putting $a=-1$ into $x^3+px^2+qx+1=0$, we have	
$-1+p-q+1=0 \Rightarrow p=q$	1
Alternatively, Putting $a=-1$ into the first two equations, we have $\frac{1}{r}+1+r=p=q$	1
(b) If $p=q$, then the equation becomes $x^3+px^2+px+1=0$.	
Let $x=-1$, then LHS $=-1+p-p+1=0$.	
$\therefore$ -1 is one of the roots of the equation.	1A
Let α and β be the other two real roots, then	
$(\alpha\beta)(-1)=-1$	1M
$\Rightarrow \alpha\beta=1$	1A
Hence the three roots of the equation are $\frac{1}{\alpha}$, -1, α	
which form a geometric sequence.	1
Alternatively, By division, the equation becomes $(x+1)[x^2+(p-1)x+1]=0$.	1A
The roots of the equation are $\frac{(p-1)-\sqrt{p^2-2p-3}}{2}$, -1 and $\frac{(p-1)+\sqrt{p^2-2p-3}}{2}$.	
$\because \left(\frac{(p-1)-\sqrt{p^2-2p-3}}{2}\right)\left(\frac{(p-1)+\sqrt{p^2-2p-3}}{2}\right)$	1M
$=\frac{1}{4}[(p-1)^2-(p^2-2p-3)]$	
$=1$	1A
$=(-1)^2$	
$\therefore$ The roots can form a geometric sequence.	
	-----(7)

Solution	Marks
8. (a) The augmented matrix of (*S*) is	
$\left(\begin{array}{ccc\|c}1 & -1 & -1 & a\\ 2 & \lambda & -2 & b\\ 1 & 2\lambda+3 & \lambda^2 & c\end{array}\right) \sim \left(\begin{array}{ccc\|c}1 & -1 & -1 & a\\ 0 & \lambda+2 & 0 & b-2a\\ 0 & 2\lambda+4 & \lambda^2+1 & c-a\end{array}\right)$	1M
$\sim \left(\begin{array}{ccc\|c}1 & -1 & -1 & a\\ 0 & \lambda+2 & 0 & b-2a\\ 0 & 0 & \lambda^2+1 & c-2b+3a\end{array}\right)$	1A
$\therefore$ (*S*) has a unique solution iff $(\lambda+2)(\lambda^2+1)\neq 0$	
iff $\lambda\neq -2$	1
Alternatively,	
Δ of (*S*) $=\begin{vmatrix}1 & -1 & -1\\ 2 & \lambda & -2\\ 1 & 2\lambda+3 & \lambda^2\end{vmatrix}$	
$=\lambda^3+2-2(2\lambda+3)+\lambda+2(2\lambda+3)+2\lambda^2$	1M
$=(\lambda+2)(\lambda^2+1)$	1A
$\therefore$ (*S*) has a unique solution iff $\Delta\neq 0$	
iff $\lambda\neq -2$	1
For $\lambda=-1$, the augmented matrix of (*S*) becomes	
$\left(\begin{array}{ccc\|c}1 & -1 & -1 & a\\ 0 & 1 & 0 & b-2a\\ 0 & 0 & 2 & c-2b+3a\end{array}\right) \sim \left(\begin{array}{ccc\|c}1 & 0 & 0 & a+(b-2a)+\frac{c-2b+3a}{2}\\ 0 & 1 & 0 & b-2a\\ 0 & 0 & 2 & c-2b+3a\end{array}\right)$	1M
$\therefore\ x=\frac{c+a}{2}$, $y=b-2a$, $z=\frac{c-2b+3a}{2}$	1A+1A+1A
Alternatively,	
Solve (*S*) for $\lambda=-1$ using Crammer's rule:	
$\Delta=\begin{vmatrix}1 & -1 & -1\\ 2 & -1 & -2\\ 1 & 1 & 1\end{vmatrix}=2$	1M (for Δ, Δ_x, Δ_y, Δ_z)
$\Delta_x=\begin{vmatrix}a & -1 & -1\\ b & -1 & -2\\ c & 1 & 1\end{vmatrix}=c+a$	
$\Delta_y=\begin{vmatrix}1 & a & -1\\ 2 & b & -2\\ 1 & c & 1\end{vmatrix}=2b-4a$	
$\Delta_z=\begin{vmatrix}1 & -1 & a\\ 2 & -1 & b\\ 1 & 1 & c\end{vmatrix}=c-2b+3a$	
$\therefore\ x=\frac{c+a}{2}$, $y=b-2a$, $z=\frac{c-2b+3a}{2}$.	1A+1A+1A

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Solution	Marks
(b) When $\lambda=-2$, the augmented matrix of (S) becomes	
$\left(\begin{array}{ccc\|c}1&-1&-1&a\\0&0&0&b-2a\\0&0&5&3a-2b+c\end{array}\right)$ (or $\left(\begin{array}{ccc\|c}1&-1&-1&a\\0&0&0&b-2a\\0&0&5&c-a\end{array}\right)$)	1M
$\therefore$ The system has infinitely many solutions iff $b-2a=0$.	1A
When $a=-1$, $b=-2$ and $c=3$, the augmented matrix becomes	
$\left(\begin{array}{ccc\|c}1&-1&-1&-1\\0&0&0&0\\0&0&5&4\end{array}\right)\sim\left(\begin{array}{ccc\|c}1&-1&0&-1/5\\0&0&1&4/5\\0&0&0&0\end{array}\right)$	1M
S.S. $=\{(t-\frac{1}{5},\ t,\ \frac{4}{5}):t\in\mathbf{R}\}$	1A
(or $\{(t,\ t+\frac{1}{5},\ \frac{4}{5}):t\in\mathbf{R}\}$)	
(c) (T) is equivalent to (S) when $\lambda=-2$ and $a=5-3\mu$, $b=2-2\mu$, $c=1+\mu$.	1M
Using the result of (b)(i), (T) is consistent iff $(2-2\mu)-2(5-3\mu)=0$	
iff $\mu=2$	1A
(i) When $\mu=2$, $a=-1$, $b=-2$ and $c=3$. By (b)(ii),	
S.S. of $(T)=\{(t-\frac{1}{5},\ t,\ \frac{4}{5}):t\in\mathbf{R}\}$	1A
(or $\{(t,\ t+\frac{1}{5},\ \frac{4}{5}):t\in\mathbf{R}\}$)	
(ii) When $\mu\neq2$, (T) is inconsistent.	1A

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Solution	Marks
9. (a) For $n \geq r+1$,	
$C_r^n + C_{r+1}^n = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k-1)!(k+1)!}$	
$= \frac{n!}{(n-k-1)!k!}\left[\frac{1}{n-k} + \frac{1}{k+1}\right]$	1A
$= \frac{n!}{(n-k-1)!k!} \cdot \frac{n+1}{(n-k)(k+1)}$	
$= \frac{(n+1)!}{(n-k)!(k+1)!}$	
$= C_{r+1}^{n+1}$	1
Alternatively,	
$(1+x)(1+x)^n = (1+x)\sum_{k=0}^{n} C_k^n x^k$	1A
$= \sum_{k=0}^{n} C_k^n x^k + \sum_{k=0}^{n} C_k^n x^{k+1}$	
$(1+x)^{n+1} = \sum_{k=0}^{n+1} C_k^{n+1} x^k$	
Comparing the coefficients of x^{k+1} , we have $C_k^n + C_{k+1}^n = C_{k+1}^{n+1}$	1
(b) For $n = 1$, RHS $= C_0^1 A^1 B^0 + C_1^1 A^0 B^1 = A + B =$ LHS $\therefore$ (*) holds for $n = 1$. Assume $(A+B)^k = \sum_{r=0}^{k} C_r^k A^{k-r} B^r$ for some positive integer k .	1
For $n = k+1$,	
LHS $= (A+B)\sum_{r=0}^{k} C_r^k A^{k-r} B^r$	
$= \sum_{r=0}^{k} C_r^k A^{k+1-r} B^r + \sum_{r=0}^{k} C_r^k BA^{k-r} B^r$	
$= \sum_{r=0}^{k} C_r^k A^{k+1-r} B^r + \sum_{r=0}^{k} C_r^k A^{k-r} B^{r+1}$ $(\because AB = BA)$	1M
$= \sum_{r=0}^{k} C_r^k A^{k+1-r} B^r + \sum_{r=1}^{k+1} C_{r-1}^k A^{k-r+1} B^r$	1M
$= C_0^k A^{k+1} + \sum_{r=1}^{k} (C_r^k + C_{r-1}^k) A^{k+1-r} B^r + C_k^k B^{k+1}$	
$= C_0^{k+1} A^{k+1} + \sum_{r=1}^{k} C_r^{k+1} A^{k+1-r} B^r + C_{k+1}^{k+1} B^{k+1}$	
$= \sum_{r=0}^{k+1} C_r^{k+1} A^{k+1-r} B^r$	1
By the principle of mathematical induction, (*) holds for all positive integer n	
Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then	1M
$AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = BA$,	
$(A+B)^2 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix}$ and	
$A^2 + 2AB + B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 + 2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix}$.	
Hence (*) will not be valid when $AB \neq BA$.	1

Solution	Marks
(c) (i) The statement clearly holds for $n=1$. Assume $A^k=\begin{pmatrix}\cos k\theta & -\sin k\theta\\ \sin k\theta & \cos k\theta\end{pmatrix}$ for some positive integer k.	1M
$A^{k+1}=\begin{pmatrix}\cos k\theta & -\sin k\theta\\ \sin k\theta & \cos k\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$	
$=\begin{pmatrix}\cos k\theta\cos\theta-\sin k\theta\sin\theta & -\cos k\theta\sin\theta-\sin k\theta\cos\theta\\ \sin k\theta\cos\theta+\cos k\theta\sin\theta & -\sin k\theta\sin\theta+\cos k\theta\cos\theta\end{pmatrix}$	
$=\begin{pmatrix}\cos(k+1)\theta & -\sin(k+1)\theta\\ \sin(k+1)\theta & \cos(k+1)\theta\end{pmatrix}$	1
By the principle of mathematical induction, $A^n=\begin{pmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{pmatrix}$ for all positive integer n.	
(ii) $B=A^{-1}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$	1A
$(A+B)^n=\begin{pmatrix}2\cos\theta & 0\\ 0 & 2\cos\theta\end{pmatrix}^n$	
$=\begin{pmatrix}2^n\cos^n\theta & 0\\ 0 & 2^n\cos^n\theta\end{pmatrix}$	1A
$\sum_{r=0}^{n}C_r^n A^{n-r}B^r$	
$=\sum_{r=0}^{n}C_r^n\begin{pmatrix}\cos(n-r)\theta & -\sin(n-r)\theta\\ \sin(n-r)\theta & \cos(n-r)\theta\end{pmatrix}\begin{pmatrix}\cos r\theta & \sin r\theta\\ -\sin r\theta & \cos r\theta\end{pmatrix}$	
$=\sum_{r=0}^{n}C_r^n\begin{pmatrix}\cos(n-r)\theta\cos r\theta+\sin(n-r)\theta\sin r\theta & \cos(n-r)\theta\sin r\theta-\sin(n-r)\theta\cos r\theta\\ \sin(n-r)\theta\cos r\theta-\cos(n-r)\theta\sin r\theta & \sin(n-r)\theta\sin r\theta+\cos(n-r)\theta\cos r\theta\end{pmatrix}$	
$=\sum_{r=0}^{n}C_r^n\begin{pmatrix}\cos(n-2r)\theta & -\sin(n-2r)\theta\\ \sin(n-2r)\theta & \cos(n-2r)\theta\end{pmatrix}$	
$\because\ AB=BA=I$	
Using (*), we have $\sum_{r=0}^{n}C_r^n\cos(n-2r)\theta=2^n\cos^n\theta$ and $\sum_{r=0}^{n}C_r^n\sin(n-2r)\theta=0$.	1
Putting $n=5$,	
$32\cos^5\theta=\cos5\theta+5\cos3\theta+10\cos\theta+10\cos(-\theta)+5\cos(-3\theta)+\cos(-5\theta)$	1M
$=2(\cos5\theta+5\cos3\theta+10\cos\theta)$	
$\cos^5\theta=\frac{1}{16}(\cos5\theta+5\cos3\theta+10\cos\theta)$	1A

Solution	Marks
10. (a) Let x, y and z be real numbers such that $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$.	1M
The equation is equivalent to $\begin{pmatrix}1&2&-1\\1&-1&-1\\0&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.	
Since $\begin{vmatrix}1&2&-1\\1&-1&-1\\0&1&1\end{vmatrix}=-3\neq 0$,	1A
$\therefore$ The system has the trivial solution only, i.e. $x=y=z=0$. Hence $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.	1
(b) (i) Area of $\Delta OAB=\frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert$ $=\frac{1}{2}\left\lVert\begin{matrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\2&-1&1\end{matrix}\right\rVert$ $=\frac{1}{2}\lvert\mathbf{i}-\mathbf{j}-3\mathbf{k}\rvert$ $=\frac{\sqrt{11}}{2}$	1A
(ii) Volume of tetrahedron $OABC=\frac{1}{6}\lvert(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\rvert$	1M
$=\frac{1}{6}\left\lvert\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\2&-1&1\end{vmatrix}\cdot(-\mathbf{i}-\mathbf{j}+\mathbf{k})\right\rvert$ $=\frac{1}{6}\lvert(\mathbf{i}-\mathbf{j}-3\mathbf{k})\cdot(-\mathbf{i}-\mathbf{j}+\mathbf{k})\rvert$ $=\frac{1}{2}$	1A
(c) Let (x,y,z) be any point on π_1 and $\mathbf{v}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Then $(\mathbf{v}-\mathbf{a})\cdot((\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{b}))=\mathbf{0}$.	1M
$\therefore$ $\begin{vmatrix}x-1&y-1&z\\1&-2&1\\-3&0&0\end{vmatrix}=0$	1A
$-3(y-1)-6z=0$ $y+2z-1=0$	1A
The Cartesian equation of π_1 is $y+2z-1=0$.	
Alternatively, Let $ax+by+cz=D$ be the Cartesian equation of π_1. $\because$ A, B, C are points on π_1	
$\therefore$ $\begin{cases}a+b=D\\2a-b+c=D\\-a-b+c=D\end{cases}$	1M
Solving the equations in terms of D gives $a=0$, $b=D$ and $c=2D$.	1A
Hence the Cartesian equation of π_1 is $y+2z-1=0$.	1A

Solution	Marks
(d) (i) Let $P=(x,y,z)$, then	
$\overrightarrow{OP}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ 2 & -1 & 1\end{vmatrix} = -\mathbf{i}-\mathbf{j}+\mathbf{k} = \mathbf{c}$.	
$\Leftrightarrow \begin{cases} y + z = -1\\ x - 2z = 1\\ -x - 2y = 1\end{cases}$	1M
$\Leftrightarrow \begin{cases} x + 2y = -1\\ y + z = -1\end{cases}$ .... (*)	1A
$\because$ P lies on $\pi_2 : \mathbf{r}\cdot\mathbf{a}=2$	
$\therefore$ $x+y=2$ .... (**)	1A
Solving (*) and (**), we have $P=(5,-3,2)$.	1A
(ii) $\because$ the equation of π_1 is $y+2z-1=0$	
$\therefore$ $\mathbf{n}=\dfrac{\mathbf{j}+2\mathbf{k}}{\sqrt{5}}$ is a unit normal vector of π_1	
The length of the orthogonal projection of $\overrightarrow{OP}$ on the plane π_1	
$= \lvert\overrightarrow{OP}\times\mathbf{n}\rvert$	1M
$= \left\lvert \dfrac{1}{\sqrt{5}} \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & -3 & 2\\ 0 & 1 & 2\end{vmatrix}\right\rvert$	
$= \left\lvert \dfrac{1}{\sqrt{5}}(-8\mathbf{i}-10\mathbf{j}+5\mathbf{k})\right\rvert$	
$= \sqrt{\dfrac{189}{5}}$ (or $\dfrac{3\sqrt{105}}{5}$, 6.15 (3 s.f.))	1A
Alternatively,	
$\lvert\overrightarrow{OP}\rvert = \sqrt{38}$	
$\overrightarrow{OP}\cdot\mathbf{n} = (5\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot\dfrac{\mathbf{j}+2\mathbf{k}}{\sqrt{5}} = \dfrac{1}{\sqrt{5}}$	
The length of the orthogonal projection of $\overrightarrow{OP}$ on the plane π_1	
$= \sqrt{38-\dfrac{1}{5}}$	1M
$= \sqrt{\dfrac{189}{5}}$	1A

Solution	Marks
11. (a) For $\alpha > 1$,	
$f(x) = (1+x)^{\alpha} - 1 - \alpha x$	
$f'(x) = \alpha[(1+x)^{\alpha-1} - 1]$	1A
$\begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } -1 \le x < 0 \end{cases}$	1M+1A
$\because$ f is continuous at $x = 0$	
$\therefore$ $f(x) > f(0) = 0$ for $x \ge -1$ and $x \ne 0$	
i.e. $(1+x)^{\alpha} > 1 + \alpha x$ for $x \ge -1$ and $x \ne 0$	1
(b) (i) Putting $\alpha = \frac{m+1}{m}$ and $x = \frac{1}{k}$ in the inequality in (a),	1M
$\left(1+\frac{1}{k}\right)^{\frac{m+1}{m}} > 1 + \frac{m+1}{m}\left(\frac{1}{k}\right)$.	
Putting $\alpha = \frac{m+1}{m}$ and $x = -\frac{1}{k}$ in the inequality in (a),	1M
$\left(1-\frac{1}{k}\right)^{\frac{m+1}{m}} > 1 + \frac{m+1}{m}\left(-\frac{1}{k}\right)$.	
Hence $1 - \left(1-\frac{1}{k}\right)^{\frac{m+1}{m}} < \frac{m+1}{m}\left(\frac{1}{k}\right) < \left(1+\frac{1}{k}\right)^{\frac{m+1}{m}} - 1$.	1
(ii) From (b)(i),	
$\frac{m}{m+1}\left[1-\left(1-\frac{1}{k}\right)^{\frac{m+1}{m}}\right]k < 1 < \frac{m}{m+1}\left[\left(1+\frac{1}{k}\right)^{\frac{m+1}{m}} - 1\right]k$	1A
$\frac{m}{m+1}\left[1-\left(1-\frac{1}{k}\right)^{\frac{m+1}{m}}\right]k^{\frac{m+1}{m}} < k^{\frac{1}{m}} < \frac{m}{m+1}\left[\left(1+\frac{1}{k}\right)^{\frac{m+1}{m}} - 1\right]k^{\frac{m+1}{m}}$	1A
$\frac{m}{m+1}\left[k^{\frac{m+1}{m}} - (k-1)^{\frac{m+1}{m}}\right] < k^{\frac{1}{m}} < \frac{m}{m+1}\left[(k+1)^{\frac{m+1}{m}} - k^{\frac{m+1}{m}}\right]$	1

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Solution	Marks
(c) From (b)(ii) and putting $m = 2$,	1M
$\frac{2}{3}\left[k^{\frac{3}{2}} - (k-1)^{\frac{3}{2}}\right] < k^{\frac{1}{2}} < \frac{2}{3}\left[(k+1)^{\frac{3}{2}} - k^{\frac{3}{2}}\right]$	
$\therefore \frac{2}{3}\sum_{k=1}^{n}\left[k^{\frac{3}{2}} - (k-1)^{\frac{3}{2}}\right] < \sum_{k=1}^{n} k^{\frac{1}{2}} < \frac{2}{3}\sum_{k=1}^{n}\left[(k+1)^{\frac{3}{2}} - k^{\frac{3}{2}}\right]$	1M
$\frac{2}{3}\left[\left(1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) + \left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) + \cdots + \left(n^{\frac{3}{2}} - (n-1)^{\frac{3}{2}}\right)\right] < \sum_{k=1}^{n} k^{\frac{1}{2}}$	
$< \frac{2}{3}\left[\left(2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) + \left(3^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) + \cdots + \left((n+1)^{\frac{3}{2}} - n^{\frac{3}{2}}\right)\right]$	
$\frac{2}{3}n^{\frac{3}{2}} < \sum_{k=1}^{n} k^{\frac{1}{2}} < \frac{2}{3}\left[(n+1)^{\frac{3}{2}} - 1\right]$	1A
$\frac{2}{3}n^{\frac{3}{2}} < \sum_{k=1}^{n} k^{\frac{1}{2}} < \frac{2}{3}(n+1)^{\frac{3}{2}}$	
$\frac{2}{3} < \frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + \cdots + n^{\frac{1}{2}}}{n^{\frac{3}{2}}} < \frac{2}{3}\left(1 + \frac{1}{n}\right)^{\frac{3}{2}}$	1
$\because \lim_{n\to\infty} \frac{2}{3}\left(1 + \frac{1}{n}\right)^{\frac{3}{2}} = \frac{2}{3}$	
By sandwich rule, $\lim_{n\to\infty} \frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + \cdots + n^{\frac{1}{2}}}{n^{\frac{3}{2}}} = \frac{2}{3}$	1A

Solution	Marks
12. (a) Let $\dfrac{x^3-x^2-3x+2}{x^2(x-1)^2} \equiv \dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}+\dfrac{D}{(x-1)^2}$, then	1M
$Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2 \equiv x^3-x^2-3x+2$	
$(A+C)x^3+(-2A+B-C+D)x^2+(A-2B)x+B \equiv x^3-x^2-3x+2$	1A
$A=1$, $B=2$, $C=0$, $D=-1$	
i.e. $\dfrac{x^3-6x+3}{x^2(x-1)^2} \equiv \dfrac{1}{x}+\dfrac{2}{x^2}-\dfrac{1}{(x-1)^2}$	1A
(b) (i) $\mathrm{P}(x)=m(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$	
$\mathrm{P}'(x)=m[(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)+(x-\alpha_1)(x-\alpha_3)(x-\alpha_4)$ $+(x-\alpha_1)(x-\alpha_2)(x-\alpha_4)+(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)]$	1A
$\therefore \quad \dfrac{\mathrm{P}'(x)}{\mathrm{P}(x)}=\sum_{i=1}^{4}\dfrac{1}{x-\alpha_i}$	1
(ii) From (b)(i), $\sum_{i=1}^{4}\dfrac{1}{x-\alpha_i}=\dfrac{\mathrm{P}'(x)}{\mathrm{P}(x)}$.	
Differentiating both sides w.r.t. x ,	
$\sum_{i=1}^{4}\dfrac{-1}{(x-\alpha_i)^2}=\dfrac{\mathrm{P}(x)\mathrm{P}''(x)-[\mathrm{P}'(x)]^2}{[\mathrm{P}(x)]^2}$.	
Hence $\sum_{i=1}^{4}\dfrac{1}{(x-\alpha_i)^2}=\dfrac{[\mathrm{P}'(x)]^2-\mathrm{P}(x)\mathrm{P}''(x)}{[\mathrm{P}(x)]^2}$	1
(c) (i) Solving $f(x)=0$, we have	
$x^2=\dfrac{b\pm\sqrt{b^2-4a^2}}{2a}$	
$\because \quad b^2>4a^2$ and $a\neq 0$, $\therefore \quad \lvert b\rvert>\sqrt{b^2-4a^2}>0$	1A
$\because \quad ab>0$, $\therefore \quad (a>0$ and $b>0)$ or $(a<0$ and $b<0)$	
If $b>0$, then $b>\sqrt{b^2-4a^2}>0$ and $a>0$.	
If $b<0$, then $-b>\sqrt{b^2-4a^2}>0$ and $a<0$.	
Both cases imply that $x^2>0$.	1
Hence all the four roots of $f(x)=0$ are real.	
Besides, $f(0)=a\neq 0$ and	1
$f(1)=2a-b\neq 0$ $\quad (\because \quad b^2>4a^2$,	
$\therefore \quad (2a+b)(2a-b)<0$)	1
$\therefore$ Both 0 and 1 are not the roots of $f(x)=0$.	

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Solution		Marks
(ii) $\sum_{i=1}^{4}\dfrac{\beta_i^3-\beta_i^2-3\beta_i+2}{\beta_i^2(\beta_i-1)^2}$		
$=\sum_{i=1}^{4}\dfrac{1}{\beta_i}+2\sum_{i=1}^{4}\dfrac{1}{\beta_i^2}-\sum_{i=1}^{4}\dfrac{1}{(\beta_i-1)^2}$	by (a)	1M
$=-\sum_{i=1}^{4}\dfrac{1}{(0-\beta_i)}+2\sum_{i=1}^{4}\dfrac{1}{(0-\beta_i)^2}-\sum_{i=1}^{4}\dfrac{1}{(1-\beta_i)^2}$		
$=-\dfrac{f'(0)}{f(0)}+2\dfrac{[f'(0)]^2-f(0)f''(0)}{[f(0)]^2}-\dfrac{[f'(1)]^2-f(1)f''(1)}{[f(1)]^2}$	by (b)(i) & (ii)	1M
$\because$ $f(x)=ax^4-bx^2+a$, $f'(x)=4ax^3-2bx$ and $f''(x)=12ax^2-2b$		1A
$\therefore$ $f(0)=a$, $f'(0)=0$, $f''(0)=-2b$ and $f(1)=2a-b$, $f'(1)=2(2a-b)$, $f''(1)=2(6a-b)$		
Hence $\sum_{i=1}^{4}\dfrac{\beta_i^3-\beta_i^2-3\beta_i+2}{\beta_i^2(\beta_i-1)^2}$		
$=0+2\dfrac{-a(-2b)}{a^2}-\dfrac{4(2a-b)^2-2(2a-b)(6a-b)}{(2a-b)^2}$		1A
$=\dfrac{4b}{a}-\dfrac{-4a-2b}{2a-b}$		
$=\dfrac{4a^2+10ab-4b^2}{a(2a-b)}$		1A

Solution	Marks
13. (a) $\lim_{x\to 1}\frac{x^{2n}-1}{x^2-1} = \lim_{x\to 1}\frac{2nx^{2n-1}}{2x}$ by L'Hospital Rule	1M
$= n$	1A
Alternatively,	
$\lim_{x\to 1}\frac{x^{2n}-1}{x^2-1} = \lim_{x\to 1}(x^{2n-2}+x^{2n-4}+x^{2n-6}+\cdots+x^2+1)$	1M
$= n$	1A
(b) $x^{2n}-1=0 \quad \Rightarrow \quad x^{2n}=\cos 0+i\sin 0$	
By De Moivre's Theorem,	
$x = \cos\frac{2k\pi}{2n}+i\sin\frac{2k\pi}{2n}$	1M
$= \cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}$ where $k=0, 1, 2, \ldots, 2n-1$.	1A
Let $x_k=\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}$ for $k=0, 1, 2, \ldots, 2n-1$.	
$x_{2n-k} = \cos\frac{(2n-k)\pi}{n}+i\sin\frac{(2n-k)\pi}{n}$	
$= \cos\frac{k\pi}{n}-i\sin\frac{k\pi}{n}$	
$= \overline{x_k}$	1A
$\therefore \quad x^{2n}-1 = (x-x_0)(x-x_n)\prod_{k=1}^{n-1}(x-x_k)(x-x_{2n-k})$	
$= (x-x_0)(x-x_n)\prod_{k=1}^{n-1}(x-x_k)(x-\overline{x_k})$	1M
$= (x-1)(x+1)\prod_{k=1}^{n-1}[x^2-(x_k+\overline{x_k})x+x_k\overline{x_k}]$	1A
$= (x^2-1)\prod_{k=1}^{n-1}(x^2-2x\cos\frac{k\pi}{n}+1)$	1
(c) When $x\neq 1$,	
$\frac{x^{2n}-1}{x^2-1} = \prod_{k=1}^{n-1}(x^2-2x\cos\frac{k\pi}{n}+1)$ by (b)	1M
$\lim_{x\to 1}\frac{x^{2n}-1}{x^2-1} = \prod_{k=1}^{n-1}\lim_{x\to 1}(x^2-2x\cos\frac{k\pi}{n}+1)$	
$= \prod_{k=1}^{n-1}2(1-\cos\frac{k\pi}{n})$	1A
$= \prod_{k=1}^{n-1}2^2\sin^2\frac{k\pi}{2n}$	1A
$= 2^{2n-2}\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{2n}$	1

Solution	Marks
(d) From (a) and (c), $2^{2n-2}\prod_{k=1}^{n-1}\sin^2\frac{k\pi}{2n}=n$	
$\because$ $\sin\frac{k\pi}{2n}>0$ for $k=1,2,3,\ldots,n-1$	
$\therefore$ $\frac{1}{\sqrt{n}}\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{2\pi}{2n}\right)\cdots\sin\left(\frac{(n-1)\pi}{2n}\right)=\frac{1}{2^{n-1}}$	1M
$\lim_{n\to\infty}\left\{\frac{1}{\sqrt{n}}\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{2\pi}{2n}\right)\cdots\sin\left(\frac{(n-1)\pi}{2n}\right)\right\}^{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{2^{1-\frac{1}{n}}}$	1A
$=\frac{1}{2}$	1

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Solution	Marks
1. $\int x\cos x\,dx = \int x\frac{d\sin x}{dx}dx$	1M Accept $\int x\,d\sin x$. Using integration by parts in the right direction
$= x\sin x - \int \sin x\,dx$	
$= x\sin x + \cos x + c$ for some constant c.	1A no mark if c is missing
$\int_0^{2\pi} x\lvert\cos x\rvert\,dx = \int_0^{\frac{\pi}{2}} x\cos x dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\cos x dx + \int_{\frac{3\pi}{2}}^{2\pi} x\cos x dx$	1M eliminating the absolute sign
$= [x\sin x+\cos x]_0^{\frac{\pi}{2}} - [x\sin x+\cos x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + [x\sin x+\cos x]_{\frac{3\pi}{2}}^{2\pi}$	
$= (\frac{\pi}{2}-1)-(-\frac{3\pi}{2}-\frac{\pi}{2})+(1+\frac{3\pi}{2})$	
$= 4\pi$	1A
	-----(4)
2. Let $f(x) = x - \ln x - 1$ for $x > 0$.	1M
$f'(x) = 1-\frac{1}{x}$	1A finding $f'(x)$
$\begin{cases} <0 & \text{if } 0<x<1 \\ =0 & \text{if } x=1 \\ >0 & \text{if } x>1 \end{cases}$ ....(*)	1
$\therefore$ $f(x) \ge f(1)$ for $x>0$	
$x-\ln x - 1 \ge 0$ for $x>0$	
$x \ge 1+\ln x$ for $x>0$	1 For those who find $x=1$ as a relative minimum but fail to justify that it is an absolute minimum, the last 2 marks will not be awarded.
From (*), $f(x)$ is strictly decreasing on (0, 1) and strictly increasing on $(1, \infty)$.	
$\therefore$ $f(x)$ attains its absolute minimum at $x=1$.	
$f(x)=0$ if and only if $x=1$.	
i.e. $x = \ln x + 1$ if and only if $x=1$.	1
	-----(5)
3. Area of the shaded region $= \int_0^{\frac{\pi}{3}} \frac{1}{2}(4\sin 3\theta)^2\,d\theta$ (or $\frac{1}{3}\int_0^{\pi}\frac{1}{2}(4\sin 3\theta)^2\,d\theta$)	1A integrand; 1A limits
$= \int_0^{\frac{\pi}{3}} 8\sin^2 3\theta\,d\theta$ (or $\frac{8}{3}\int_0^{\pi}\sin^2 3\theta\,d\theta$)	
$= 4\int_0^{\frac{\pi}{3}}(1-\cos 6\theta)\,d\theta$ (or $\frac{4}{3}\int_0^{\pi}(1-\cos 6\theta)\,d\theta$)	
$= 4\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{3}}$ (or $\frac{4}{3}\left[\theta-\frac{\sin 6\theta}{6}\right]_0^{\pi}$)	1A ignore limits
$= \frac{4\pi}{3}$	1A
	-----(4)

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Solution	Marks
4. $h(x) = [f(x) - \sin x]^2 + [g(x) - \cos x]^2$	
$h'(x) = 2[f(x) - \sin x][f'(x) - \cos x] + 2[g(x) - \cos x][g'(x) + \sin x]$	1A
$= 2[f(x) - \sin x][g(x) - \cos x] + 2[g(x) - \cos x][-f(x) + \sin x]$	
$= 0$	1A
Hence $h(x)$ is a constant function for $x \in \mathbf{R}$.	1
$\because$ $h(0) = [f(0) - \sin 0]^2 + [g(0) - \cos 0]^2 = 0$,	1A
$\therefore$ $h(x) = [f(x) - \sin x]^2 + [g(x) - \cos x]^2 = 0$ for $x \in \mathbf{R}$.	
$\because$ both terms of $h(x)$ are non-negative,	
$\therefore$ $f(x) - \sin x = 0$ and $g(x) - \cos x = 0$ for $x \in \mathbf{R}$.	1
i.e. $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in \mathbf{R}$.	
	-----(5)
5. (a) $\frac{d}{dx}\int_0^x \cos t^2 \, dt = \cos x^2$	1A
(b) $\frac{d}{dy}\int_0^{y^{2k}} \cos t^2 \, dt = \cos\left(y^{2k}\right)^2 \cdot \frac{d\,y^{2k}}{dy}$	1M chain rule
$= 2ky^{2k-1} \cos y^{4k}$	1A
(c) $\lim_{y \to 0} \frac{1}{y^{2k}} \int_0^{y^{2k}} \cos t^2 \, dt = \lim_{y \to 0} \frac{2ky^{2k-1} \cos y^{4k}}{2ky^{2k-1}}$ (L'Hospital rule)	1M+1A
$= \lim_{y \to 0} \cos y^{4k}$	
$= 1$	1A
	-----(6)
6. $\lim_{n \to \infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right) = \lim_{n \to \infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{1+\frac{r}{n}}$	1M can be omitted
$= \int_0^1 \frac{1}{1+x} dx$	1A
$= [\ln(1+x)]_0^1$	1A
$= \ln 2$	1A
	-----(4)

Candidates may skip this and jump right to the integral. In this case, 2 marks will be awarded for writing down $\int_0^1 \frac{1}{1+x} dx$.

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Solution	Marks
7. (a) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$	
$= \dfrac{2\sin t}{2(1-\cos t)}$	
$= \dfrac{\sin t}{1-\cos t}$ (or $\cot\dfrac{t}{2}$)	1A
$\left.\dfrac{dy}{dx}\right\|_{t=\frac{\pi}{2}} = 1$	
∴ The equation of the tangent at the point where $t = \dfrac{\pi}{2}$ is	
$\dfrac{y-2}{x-2\left(\frac{\pi}{2}-1\right)} = 1$	
$x - y - \pi + 4 = 0$	1A
(b) Arc length $= \displaystyle\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ (or $2\displaystyle\int_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$)	1M for the correct limits
$= \displaystyle\int_0^{2\pi}\sqrt{4(1-\cos t)^2+4\sin^2 t}\,dt$	
$= 4\displaystyle\int_0^{2\pi}\sqrt{\frac{1-\cos t}{2}}\,dt$ (or $8\displaystyle\int_0^{\pi}\sqrt{\frac{1-\cos t}{2}}\,dt$)	1A
$= 4\displaystyle\int_0^{2\pi}\sin\frac{t}{2}\,dt$ (or $8\displaystyle\int_0^{\pi}\sin\frac{t}{2}\,dt$)	1M using $\dfrac{1-\cos t}{2} = \sin^2\dfrac{t}{2}$
$= 4\left[-2\cos\dfrac{t}{2}\right]_0^{2\pi}$	
$= 16$	1A -----(6)
8. (a) ∵ $f(x)$ is continuous at $x = 0$	
∴ $\displaystyle\lim_{x\to 0^-} f(x) = f(0)$	1M
As $\displaystyle\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin x}{x}+2x\right) = 1$ and	1A
$f(0) = c$	1A
∴ $c = 1$	
(b) ∵ $f'(0)$ exists,	
∴ $\displaystyle\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{f(h)-f(0)}{h}$	1M
As $\displaystyle\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h^2+bh+1-1}{h} = b$ and	1A
$\displaystyle\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{\sin h+2h^2-h}{h^2} = \lim_{h\to 0^-}\frac{\cos h+4h-1}{2h}$	
$= \displaystyle\lim_{h\to 0^-}\frac{-\sin h+4}{2} = 2$	1A
∴ $b = 2$	-----(6)

Solution	Marks

9. (a) $f(x)=\dfrac{x}{(1+x^2)^2}$

$f'(x)=\dfrac{(1+x^2)^2-2x(1+x^2)\cdot 2x}{(1+x^2)^4}$

$=\dfrac{1-3x^2}{(1+x^2)^3}$ — 1A

$f''(x)=\dfrac{(1+x^2)^3(-6x)-(1-3x^2)\cdot 3(1+x^2)^2\cdot 2x}{(1+x^2)^6}$

$=\dfrac{-12x(1-x^2)}{(1+x^2)^4}$ — 1A simplification required

(b) (i) $f'(x)>0 \Leftrightarrow 1-3x^2>0$

$\Leftrightarrow -\dfrac{\sqrt{3}}{3}<x<\dfrac{\sqrt{3}}{3}$ — 1A

$\Leftrightarrow x\in(-\dfrac{\sqrt{3}}{3},\dfrac{\sqrt{3}}{3})$

(ii) $f''(x)>0 \Leftrightarrow x(1-x^2)<0$

$\Leftrightarrow x(1-x)(1+x)<0$ — 1M

$\Leftrightarrow -1<x<0$ or $x>1$ — 1A

$\Leftrightarrow x\in(-1,0)\cup(1,\infty)$

(c)

x	$(-\infty,-1)$	-1	$(-1,-\frac{\sqrt{3}}{3})$	$-\frac{\sqrt{3}}{3}$	$(-\frac{\sqrt{3}}{3},0)$	0
$f(x)$	↓	$-\frac{1}{4}$	↓	$-\frac{3\sqrt{3}}{16}$	↑	0
$f'(x)$	−	−	−	0	+	+
$f''(x)$	−	0	+	+	+	0

x	$(0,\frac{\sqrt{3}}{3})$	$\frac{\sqrt{3}}{3}$	$(\frac{\sqrt{3}}{3},1)$	1	$(1,\infty)$
$f(x)$	↑	$\frac{3\sqrt{3}}{16}$	↓	$\frac{1}{4}$	↓
$f'(x)$	+	0	−	−	−
$f''(x)$	−	−	−	0	+

One relative minimum point: $(-\dfrac{\sqrt{3}}{3},-\dfrac{3\sqrt{3}}{16})$. — 1A $(-0.5774,-0.3248)$

One relative maximum point: $(\dfrac{\sqrt{3}}{3},\dfrac{3\sqrt{3}}{16})$. — 1A $(0.5774, 0.3248)$

Three points of inflexion: $(-1,-\dfrac{1}{4})$, $(0,0)$ and $(1,\dfrac{1}{4})$. — 1A

The curve has no vertical asymptotes since f is continuous on **R**.

$\because \lim\limits_{x\to-\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\dfrac{x}{(1+x^2)^2}=0$

$\therefore y=0$ is a horizontal asymptote. — 1 or let the asymptote be $y=ax+b$ and then obtain $a=0$ and $b=0$ by standard techniques

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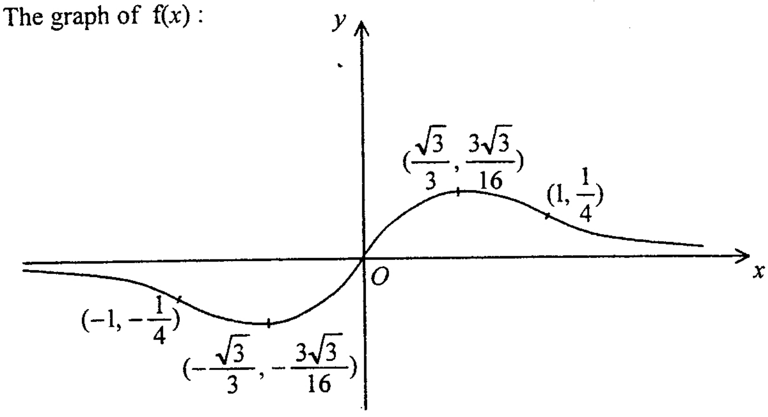
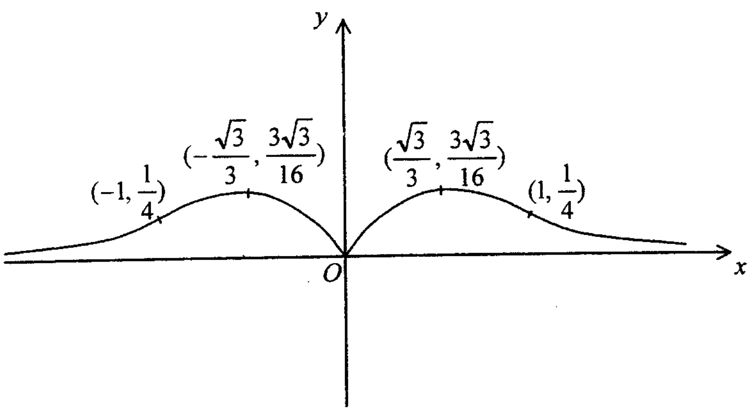
Solution	Marks
(d) The graph of $f(x)$: 	1A Extrema and pts. of inflexion 1A Horizontal asymptote 1A Shape of the curve
(e) (i) $\because$ $\lim_{x \to 0^-} \frac{g(x)-g(0)}{x} = \lim_{x \to 0^-} \frac{\frac{-x}{(1+x^2)^2} - 0}{x} = \lim_{x \to 0^-} \frac{-1}{(1+x^2)^2} = -1$ and $\lim_{x \to 0^+} \frac{g(x)-g(0)}{x} = \lim_{x \to 0^+} \frac{\frac{x}{(1+x^2)^2} - 0}{x} = \lim_{x \to 0^+} \frac{1}{(1+x^2)^2} = 1$	1M
$\therefore$ $g'(0)$ does not exist.	1 justification required
(ii) The graph of $g(x)$: 	1M

Solution	Marks
10. (a) $y^2 = 4ax \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$	1M
Slope of tangent to Γ at $(at^2, 2at) = \frac{2a}{2at} = \frac{1}{t}$	1A
$\therefore$ Equation of the normal to Γ at $(at^2, 2at)$ is	
$\frac{y-2at}{x-at^2} = -t$	
$tx + y - 2at - at^3 = 0$ (or $y = -tx + 2at + at^3$ $y = -tx + at(2+t^2)$ etc.)	1A
If the normal passes through (h, k), then	
$ht + k - 2at - at^3 = 0$	
$at^3 + (2a-h)t - k = 0$	1
(b) Let the normals to Γ at the three points meet at (h, k).	
From (a), t_1, t_2, t_3 satisfy the equation $at^3 + (2a-h)t - k = 0$.	1M
$\because$ the three points are distinct	
$\therefore$ t_1, t_2, t_3 are the roots of $at^3 + (2a-h)t - k = 0$	
Hence sum of roots $= t_1 + t_2 + t_3 = 0$	1
<u>Alternatively</u>,	
Hence $\begin{cases} at_1^3 + (2a-h)t_1 - k = 0 \\ at_2^3 + (2a-h)t_2 - k = 0 \\ at_3^3 + (2a-h)t_3 - k = 0 \end{cases}$.	
For the system of equations to be solvable, we have	
$\begin{vmatrix} t_1^3 & t_1 & 1 \\ t_2^3 & t_2 & 1 \\ t_3^3 & t_3 & 1 \end{vmatrix} = (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 + t_2 + t_3) = 0$.	
$\because$ t_1, t_2, t_3 are distinct	
$\therefore$ $t_1 + t_2 + t_3 = 0$	1

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Solution	Marks
(c) $\because$ $(as_i^2, 2as_i)$, $i = 1, 2, 3, 4$, are points on the circle	
$\therefore$ $(as_i^2)^2 + (2as_i)^2 + 2g(as_i^2) + 2f(2as_i) + c = 0$	1M
$a^2 s_i^4 + 2a(2a+g)s_i^2 + 4afs_i + c = 0$	1A
Hence s_i , $i = 1, 2, 3, 4$, are the roots of the equation	
$a^2 s^4 + 2a(2a+g)s^2 + 4afs + c = 0$ ....(*)	1M
$\therefore$ sum of roots $= s_1 + s_2 + s_3 + s_4 = 0$	1A
Alternatively,	
$\because$ $(as_i^2, 2as_i)$, $i = 1, 2, 3, 4$, are points on the circle	
$\therefore$ s_i , $i = 1, 2, 3, 4$, are the roots of the equation	
$(as^2)^2 + (2as)^2 + 2g(as^2) + 2f(2as) + c = 0$ ....(*')	1M+1A
$\because$ coefficient of s^3 in the equation is zero	1M
$\therefore$ sum of roots $= s_1 + s_2 + s_3 + s_4 = 0$	1A
Alternatively,	
Solving $x^2 + y^2 + 2gx + 2fy + c = 0$ and $y^2 = 4ax$, we have	1M
$\left(\frac{y^2}{4a}\right)^2 + y^2 + 2g\left(\frac{y^2}{4a}\right) + 2fy + c = 0$	
$y^4 + (16a^2 + 8ag)y^2 + 32a^2 fy + 16ca^2 = 0$	1A
$\therefore$ sum of roots $= 2as_1 + 2as_2 + 2as_3 + 2as_4 = 0$	1M
Hence $s_1 + s_2 + s_3 + s_4 = 0$	1A
(d) Let A, B, C and D be the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$ and $(at_4^2, 2at_4)$.	
(i) $\because$ A, B, C and D are the intersection points of a circle and Γ,	
$\therefore$ $t_1 + t_2 + t_3 + t_4 = 0$ from (c).	1
$\because$ the normals to Γ at A, B and C are concurrent,	
$\therefore$ $t_1 + t_2 + t_3 = 0$ from (b).	1
Hence $t_4 = 0 \Rightarrow D = (0, 0)$.	1
(ii) If A, B are symmetric about the x-axis, then	
$t_1 = -t_2 \Rightarrow t_1 + t_2 = 0$	1
$\because$ $t_1 + t_2 + t_3 = 0$	
$\therefore$ $t_3 = 0$	
From (i), $t_3 = t_4 = 0$	
$\therefore$ The circle touches Γ at the origin.	1

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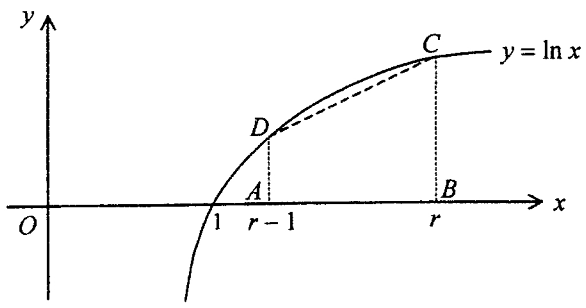
Solution	Marks
11. (a) Area of trapezium $PQRS = \frac{1}{2}(PS+QR)$	
$= \ln r$	1A
$\because$ area under the curve $y=\ln x$ between $x=r-\frac{1}{2}$ and $x=r+\frac{1}{2}$	
$\le$ area of trapezium $PQRS$ for $r \ge 2$	
Hence $\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \ln x \, dx < \ln r$ for $r \ge 2$	1
For any integer $n \ge 2$,	
$\int_{\frac{3}{2}}^{n} \ln x \, dx = \sum_{r=2}^{n} \int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \ln x \, dx - \int_{n}^{n+\frac{1}{2}} \ln x \, dx$	1A
$\le \sum_{r=2}^{n} \ln r - \int_{n}^{n+\frac{1}{2}} \ln n \, dx$ ($\because$ $\ln n \le \ln x$ for $n \le x \le n+\frac{1}{2}$ or $y=\ln x$ is increasing)	1M
$= \ln(n!) - \frac{1}{2}\ln n$	
$\int_{\frac{3}{2}}^{n} \ln x dx \le \ln(n!) - \frac{1}{2}\ln n$	1
(b) Refer to the figure below, area under the curve $y=\ln x$ between $x=r-1$ and $x=r$	
$\ge$ area of trapezium $ABCD$ for $r \ge 2$	
	1M
Hence $\int_{r-1}^{r} \ln x \, dx \ge \frac{1}{2}[\ln(r-1)+\ln r]$ for $r \ge 2$	1
For any integer $n \ge 2$,	
$\int_{1}^{n} \ln x \, dx = \sum_{r=2}^{n} \int_{r-1}^{r} \ln x \, dx$	
$\ge \frac{1}{2}\sum_{r=2}^{n} [\ln(r-1)+\ln r]$	1M
$= \sum_{r=1}^{n} \ln r - \frac{1}{2}(\ln 1 + \ln n)$	1
$= \ln(n!) - \frac{1}{2}\ln n$	

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Solution	Marks
(c) $\int \ln x\,\mathrm{d}x = x\ln x - \int x(\frac{1}{x})\mathrm{d}x$	
$= x\ln x - x + c$	1A ignore c
$\therefore \int_{\frac{3}{2}}^{n} \ln x\,\mathrm{d}x = n\ln n - n - \frac{3}{2}\ln\frac{3}{2} + \frac{3}{2}$ and	
$\int_{1}^{n} \ln x\,\mathrm{d}x = n\ln n - n + 1$	1A 1 mark awarded for getting one of the definite integrals right
Using the results of (a) and (b), we have	
$n\ln n - n - \frac{3}{2}\ln\frac{3}{2} + \frac{3}{2} \le \ln(n!) - \frac{1}{2}\ln n \le n\ln n - n + 1$	
$-\frac{3}{2}\,\text{in}\,\frac{3}{2} + \frac{3}{2} \le \ln(n!) - \frac{1}{2}\ln n - n\ln n + n \le 1$	1M
$\frac{3}{2}(1 - \ln\frac{3}{2}) \le \ln(n!) - \left(n + \frac{1}{2}\right)\ln n + n \le 1$	
$\frac{3}{2}\ln\frac{2e}{3} \le \ln(n!) - \left(n + \frac{1}{2}\right)\ln n + n\ln e \le \ln e$	1M
$\because$ e^x is strictly increasing	1M
$\therefore \left(\frac{2e}{3}\right)^{\frac{3}{2}} \le \frac{n!}{n^{n+\frac{1}{2}}e^{-n}} \le e$	
$\frac{1}{e} \le \frac{n^{n+\frac{1}{2}}e^{-n}}{n!} \le \left(\frac{3}{2e}\right)^{\frac{3}{2}}$	1

Solution	Marks
12. (a) (i) By Mean Value Theorem, there exists $\xi_1 \in (a, c)$ and $\xi_2 \in (c, b)$ such that	
$\dfrac{f(c)-f(a)}{c-a} = f'(\xi_1)$ and $\dfrac{f(b)-f(c)}{b-c} = f'(\xi_2)$.	1M+1A
$\because$ $f''(x) \geq 0$ for $x \in \mathbf{I}$	
$\therefore$ f' is increasing on $\mathbf{I}$	1M
$\because$ $a < \xi_1 < c < \xi_2 < b$	
$\therefore$ $f'(\xi_1) \leq f'(\xi_2)$	
$\dfrac{f(c)-f(a)}{c-a} \leq \dfrac{f(b)-f(c)}{b-c}$	1
$(b-c)[f(c)-f(a)] \leq (c-a)[f(b)-f(c)]$	
$[(b-c)+(c-a)]f(c) \leq (b-c)f(a)+(c-a)f(b)$	
$f(c) \leq \dfrac{b-c}{b-a}f(a) + \dfrac{c-a}{b-a}f(b)$	1
(ii) $\because$ $\lambda a+(1-\lambda)b = a+(1-\lambda)(b-a) > a$ and	1A
$\lambda a+(1-\lambda)b = b-\lambda(b-a) < b$	1A
Alternatively,	
$\because$ $\lambda a+(1-\lambda)b-a = (1-\lambda)b-(1-\lambda)a = (1-\lambda)(b-a) > 0$ and	1A
$b-[\lambda a+(1-\lambda)b] = \lambda(b-a) > 0$	1A
Alternatively,	
Let $g(t) = ta+(1-t)b$ where $t \in [0, 1]$.	
$\because$ $g'(t) = a-b < 0$	1A
$\therefore$ $g(1) < g(\lambda) < g(0)$ for $\lambda \in (0, 1)$	1A
$\therefore$ $a < \lambda a+(1-\lambda)b < b$	
Let $c = \lambda a+(1-\lambda)b$, then $c \in (a, b)$.	
Using the result of (a)(i),	
$f[\lambda a+(1-\lambda)b] = f(c)$	
$\leq \dfrac{b-c}{b-a}f(a) + \dfrac{c-a}{b-a}f(b)$	
$= \dfrac{\lambda(b-a)}{b-a}f(a) + \dfrac{(1-\lambda)(b-a)}{b-a}f(b)$	
$= \lambda f(a)+(1-\lambda)f(b)$	1
(b) (i) Let $g(x) = x^p$ for $x > 0$.	1A −1 mark for not mentioning $x > 0$
Then $g''(x) = p(p-1)x^{p-2} \geq 0$.	1M for checking $g''(x) \geq 0$
From (a)(ii) and for $0 < a < b$, $0 < \lambda < 1$,	
$[\lambda a+(1-\lambda)b]^p \leq \lambda a^p + (1-\lambda)b^p$	1
(ii) Let $h(x) = -\ln x$ for $x > 0$.	1A
Then $h''(x) = \dfrac{1}{x^2} \geq 0$.	1A
From (a)(ii) and for $0 < a < b$, $0 < \lambda < 1$,	
$-\ln[\lambda a+(1-\lambda)b] \leq \lambda[-\ln a]+(1-\lambda)[-\ln b]$	
$\ln[\lambda a+(1-\lambda)b] \geq \lambda \ln a+(1-\lambda)\ln b$	
$\ln[\lambda a+(1-\lambda)b] \geq \ln a^{\lambda}b^{1-\lambda}$	
$\lambda a+(1-\lambda)b \geq a^{\lambda}b^{1-\lambda}$ ($\because$ e^x is increasing)	2

Solution	Marks

13. (a) (i) $\because$ $$f_n(-x) = \frac{\int_0^{-x}(1-t^4)^n\,dt}{\int_0^1 (1-t^4)^n\,dt}$$

$$= \frac{-\int_0^{x}(1-(-u)^4)^n\,du}{\int_0^1 (1-t^4)^n\,dt} \quad \text{(putting } t=-u\text{)}$$

$$= -f_n(x)$$

$\therefore$ $f_n(x)$ is an odd function. — 1

Alternatively,
$\because$ $(1-t^4)^n$ is even
$\therefore$ $\int_0^x (1-t^4)^n dt = \int_{-x}^0 (1-t^4)^n dt$
$= -\int_0^{-x}(1-t^4)^n dt$

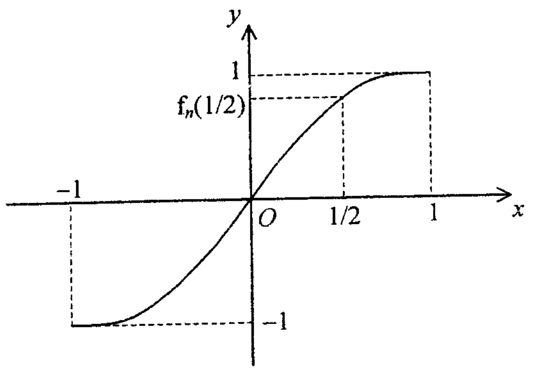
(ii) $$f_n'(x) = \frac{(1-x^4)^n}{\int_0^1 (1-t^4)^n\,dt}$$ — 1A

$$f_n''(x) = \frac{-4nx^3(1-x^4)^{n-1}}{\int_0^1 (1-t^4)^n\,dt}$$ — 1A

(iii) For $0<x<1$, $f_n'(x)>0$ and $f_n''(x)<0$. — 1M

$f_n(0)=0$, $f_n(1)=1$, $f_n'(1)=0$ $f_n''(0)=0$ and $f_n''(1)=0$.

The graph of $f_n(x)$ for $-1\le x\le 1$.

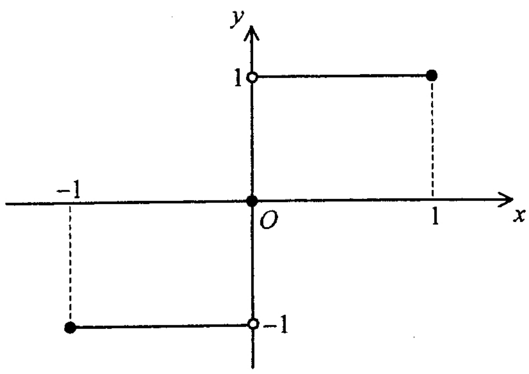


1A increasing, passes through (0, 0), (1, 1) for $0\le x\le 1$
1A concave downward for $0\le x\le 1$
1A all correct

(b) $$1-f_n(x) = \frac{\int_x^1 (1-t^4)^n\,dt}{\int_0^1 (1-t^4)^n\,dt}$$

When $0<x\le 1$, $1-f_n(x)\ge 0$ since $(1-t^4)^n\ge 0$ for $0\le t\le 1$. — 1

Alternatively,
$\because$ $f_n(x)$ is increasing on $[-1, 1]$ and
$f_n(1)=1$
$\therefore$ $f_n(x)\le 1$ on $[-1, 1]$

Solution	Marks
Again for $0 < x \le 1$,	
$(1-t^4)^n \le \dfrac{t^3}{x^3}(1-t^4)^n$ for $0 < x \le t \le 1$	
$\Rightarrow \int_x^1 (1-t^4)^n \mathrm{d}t \le \int_x^1 \dfrac{t^3}{x^3}(1-t^4)^n \mathrm{d}t$	1M
$= \dfrac{1}{x^3}\left[-\dfrac{(1-t^4)^{n+1}}{4(n+1)}\right]_x^1$	1M
$= \dfrac{(1-x^4)^{n+1}}{4(n+1)x^3}$	
$t^3(1-t^4)^n \le (1-t^4)^n$ for $0 \le t \le 1$	
$\Rightarrow \int_0^1 (1-t^4)^n \mathrm{d}t \ge \int_0^1 t^3(1-t^4)^n \mathrm{d}t$	1M
$= \left[-\dfrac{(1-t^4)^{n+1}}{4(n+1)}\right]_0^1$	
$= \dfrac{1}{4(n+1)}$	
Hence $1-\mathrm{f}_n(x) \le \dfrac{(1-x^4)^{n+1}}{4(n+1)x^3}\cdot 4(n+1) = \dfrac{(1-x^4)^{n+1}}{x^3}$	1
(c) For each $x \in (0,1]$	
$\because$ $0 \le 1-\mathrm{f}_n(x) \le \dfrac{(1-x^4)^{n+1}}{x^3}$ and $\lim_{n\to\infty}(1-x^4)^{n+1} = 0$	
$\therefore$ $\lim_{n\to\infty}[1-\mathrm{f}_n(x)] = 0$ by sandwich principle	1
$\Rightarrow$ $\mathrm{g}(x) = 1$ for $0 < x \le 1$.	
$\because$ $\mathrm{f}_n(0) = 0$ for all n	
$\therefore$ $\mathrm{g}(0) = 0$	1
Using that fact that $\mathrm{f}_n(x)$ is an odd function, the graph of $\mathrm{g}(x)$ for $-1 \le x \le 1$ is sketched below.	
	1A

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Solution	Marks
14. (a) (i) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$	
Let $y=\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$ for $x>0$, $\ln y=\frac{1}{x}\ln\left(\frac{a^x+b^x}{2}\right)$	1M for taking logarithm
$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln\left(\frac{a^x+b^x}{2}\right)}{x}$	
$= \lim_{x\to 0^+}\frac{a^x\ln a+b^x\ln b}{a^x+b^x}$ by L'Hospital rule	1A
$=\frac{\ln a+\ln b}{2}$	
$=\ln\sqrt{ab}$	
$\therefore \lim_{x\to 0^+} f(x)=\sqrt{ab}$	1A
$\because \lim_{x\to 0^+} f(x)=f(0)$ and f is defined on $[0,\infty]$	
$\therefore$ f is continuous at $x=0$.	1
(ii) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$	
$=\lim_{x\to\infty} 2^{-\frac{1}{x}}a\left[1+\left(\frac{b}{a}\right)^x\right]^{\frac{1}{x}}$	1A
$=a$ $\left(\because \lim_{x\to\infty}2^{-\frac{1}{x}}=1\right.$ and $\left.\lim_{x\to\infty}\left[1+\left(\frac{b}{a}\right)^x\right]^{\frac{1}{x}}=\lim_{x\to\infty}e^{\frac{1}{x}\ln\left[1+\left(\frac{b}{a}\right)^x\right]}=1\right)$	1A
Alternatively,	
Let $y=\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$ for $x>0$, $\ln y=\frac{1}{x}\ln\left(\frac{a^x+b^x}{2}\right)$	
$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(\frac{a^x+b^x}{2}\right)}{x}$	
$=\lim_{x\to\infty}\frac{a^x\ln a+b^x\ln b}{a^x+b^x}$ by L'Hospital rule	
$=\lim_{x\to\infty}\frac{a^x\ln a}{a^x+b^x}+\lim_{x\to\infty}\frac{b^x\ln b}{a^x+b^x}$	
$=\lim_{x\to\infty}\frac{\ln a}{1+\left(\frac{b}{a}\right)^x}+\lim_{x\to\infty}\frac{\ln b}{\left(\frac{a}{b}\right)^x+1}$	1A
$=\ln a$	1A

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Solution	Marks

(b) (i) For $0 < t < 1$,

$$h(t) = (1+t)\ln(1+t) + (1-t)\ln(1-t)$$

$$h'(t) = 1 + \ln(1+t) - 1 - \ln(1-t)$$ **1A**

$$= \ln\left(\frac{1+t}{1-t}\right) > 0$$

Hence $h(t)$ is strictly increasing on $(0, 1)$.

$\because$ h is continuous at $t = 0$,

$\therefore$ $h(t) > h(0)$ for $0 < t < 1$. **1**

(ii) Let $t = \dfrac{a^x - b^x}{a^x + b^x}$.

$\because$ $0 < a^x - b^x < a^x + b^x$

$\therefore$ $0 < t < 1$ **1**

$$h(t) = \frac{2a^x}{a^x+b^x}\ln\left(\frac{2a^x}{a^x+b^x}\right) + \frac{2b^x}{a^x+b^x}\ln\left(\frac{2b^x}{a^x+b^x}\right)$$

$$= \frac{2a^x}{a^x+b^x}\left[\ln a^x + \ln\left(\frac{2}{a^x+b^x}\right)\right] + \frac{2b^x}{a^x+b^x}\left[\ln b^x + \ln\left(\frac{2}{a^x+b^x}\right)\right]$$

$$= 2\left[\frac{a^x \ln a^x + b^x \ln b^x}{a^x+b^x} + \ln\left(\frac{2}{a^x+b^x}\right)\right]$$ **1**

(iii) For $x > 0$,

$$g(x) = \ln f(x) = \frac{1}{x}\ln\left(\frac{a^x+b^x}{2}\right)$$

$$g'(x) = \frac{f'(x)}{f(x)} = \frac{1}{x}\cdot\frac{a^x \ln a + b^x \ln b}{a^x+b^x} - \frac{1}{x^2}\ln\left(\frac{a^x+b^x}{2}\right)$$ **1M**

$$x^2 g'(x) = \frac{x(a^x \ln a + b^x \ln b)}{a^x+b^x} - \ln\left(\frac{a^x+b^x}{2}\right)$$

$$= \frac{a^x \ln a^x + b^x \ln b^x}{a^x+b^x} + \ln\left(\frac{2}{a^x+b^x}\right)$$ **1**

Now, $2x^2 g'(x) = h(t)$ where $t = \dfrac{a^x - b^x}{a^x + b^x}$.

$\because$ $0 < t < 1$

$\therefore$ $h(t) > h(0) = 0$

$\because$ $2x^2 > 0$ for $x > 0$

$\therefore$ $g'(x) > 0$ for $x > 0$ **1**

$\because$ $f(x) = e^{g(x)}$

$f'(x) = e^{g(x)} g'(x)$ **1A**

$\therefore$ $f'(x) > 0$ for $x > 0$

$\because$ f is continuous at $x = 0$ by (a)(i)

$\therefore$ f is strictly increasing on $[0, \infty)$. **1**

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