2000-AL
P MATH
PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2000

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A and any FOUR questions in Section B.

3. You are provided with one AL(E) answer book and four AL(D) answer books.

   Section A : Write your answers in the AL(E) answer book.

   Section B : Use a separate AL(D) answer book for each question and put the question number on the front cover of each answer book.

4. The four AL(D) answer books should be tied together with the green tag provided. The AL(E) answer book and the four AL(D) answer books must be handed in separately at the end of the examination.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Let $M = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & b & a \\ \mu & c & -b \end{pmatrix}$ where $b^2 + ac = 1$. Show by induction that

$$M^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ n[\lambda(1+b)+\mu a] & 1 & 0 \\ n[\lambda c + \mu(1-b)] & 0 & 1 \end{pmatrix}$$ for all positive integers n .

Hence or otherwise, evaluate $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 2 \\ 1 & -4 & -3 \end{pmatrix}^{2000}$.

(5 marks)

2. (a) Let p and q be positive numbers. Using the fact that $\ln x$ is increasing on $(0, \infty)$, show that $(p-q)(\ln p - \ln q) \geq 0$.

(b) Let a , b and c be positive numbers. Using (a) or otherwise, show that $a\ln a + b\ln b + c\ln c \geq \frac{a+b+c}{3}(\ln a + \ln b + \ln c)$.

(6 marks)

3. Let n be a positive integer.

(a) Expand $\frac{(1+x)^n - 1}{x}$ in ascending powers of x .

(b) Using (a) or otherwise, show that $C_2^n + 2C_3^n + 3C_4^n + \Lambda + (n-1)C_n^n = (n-2)2^{n-1} + 1$.

(5 marks)

4. Consider the circle

$$z\bar{z} = (2+3i)\bar{z} + (2-3i)z + 12 \quad (z \in \mathbf{C}) \quad ....(*).$$

Rewrite (*) in the form of $|z-a| = r$ where $a \in \mathbf{C}$ and $r > 0$.

Hence or otherwise, find the shortest distance between the point $-4-5i$ and the circle.

(5 marks)

5. Let $f(x) = 2x^4 - x^3 + 3x^2 - 2x + 1$ and $g(x) = x^2 - x + 1$.

(a) Show that $f(x)$ and $g(x)$ have no non-constant common factors.

(b) Find a polynomial $p(x)$ of the lowest degree such that $f(x) + p(x)$ is divisible by $g(x)$.

(5 marks)

6. A transformation T in $\mathbf{R}^2$ transforms a vector $\mathbf{x}$ to another vector

$\mathbf{y} = A\mathbf{x} + \mathbf{b}$ where $A = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

(a) Find $\mathbf{y}$ when $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

(b) Describe the geometric meaning of the transformation T .

(c) Find a vector $\mathbf{c}$ such that $\mathbf{y} = A(\mathbf{x} + \mathbf{c})$.

(7 marks)

7. Suppose the equation $x^3 + px^2 + qx + 1 = 0$ has three real roots.

(a) If the roots of the equation can be written as $\frac{a}{r}$, a and ar , show that $p = q$.

(b) If $p = q$, show that -1 is a root of the equation and the three roots of the equation can form a geometric sequence.

(7 marks)

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SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

8. Consider the system of linear equations

$$(S): \begin{cases} x - y - z = a \\ 2x + \lambda y - 2z = b \\ x + (2\lambda + 3)y + \lambda^2 z = c \end{cases} \quad \text{where } \lambda \in \mathbf{R} .$$

(a) Show that (S) has a unique solution if and only if $\lambda \neq -2$.

Solve (S) for $\lambda = -1$.

(7 marks)

(b) Let $\lambda = -2$.

(i) Find the conditions on a , b and c so that (S) has infinitely many solutions.

(ii) Solve (S) when $a = -1$, $b = -2$ and $c = 3$.

(4 marks)

(c) Consider the system of linear equations

$$(T): \begin{cases} x - y - z + 3\mu - 5 = 0 \\ 2x - 2y - 2z + 2\mu - 2 = 0 \\ x - y + 4z - \mu - 1 = 0 \end{cases} \quad \text{where } \mu \in \mathbf{R} .$$

Using the results in (b), or otherwise, solve (T) .

(4 marks)

9. (a) Show that $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$ where n, r are positive integers and $n \geq r+1$.

(2 marks)

(b) Let A, B be two square matrices of the same order. If $AB = BA$, show by induction that for any positive integer n,

$$(A+B)^n = \sum_{r=0}^{n} C_r^n A^{n-r} B^r \quad , \qquad .....(*)$$

where A^0 and B^0 are by definition the identity matrix I.

Would (*) still be valid if $AB \neq BA$? Justify your answer.

(6 marks)

(c) Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where θ is real.

(i) Show that $A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$ for all positive integers n.

(ii) Using (*) and the substitution $B = A^{-1}$, show that

$\sum_{r=0}^{n} C_r^n \cos(n-2r)\theta = 2^n \cos^n \theta$ and

$\sum_{r=0}^{n} C_r^n \sin(n-2r)\theta = 0$.

Hence or otherwise, express $\cos^5 \theta$ in terms of $\cos 5\theta$, $\cos 3\theta$ and $\cos\theta$.

(7 marks)

Go on to the next page

10. A, B, C are the points $(1, 1, 0)$, $(2, -1, 1)$, $(-1, -1, 1)$ respectively and O is the origin. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$.

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.

(3 marks)

(b) Find

(i) the area of ΔOAB, and

(ii) the volume of tetrahedron $OABC$.

(3 marks)

(c) Find the Cartesian equation of the plane π_1 containing A, B and C.

(3 marks)

(d) Let π_2 be the plane $\mathbf{r} \cdot \mathbf{a} = 2$ where $\mathbf{r}$ is any position vector in $\mathbf{R}^3$. P is a point on π_2 such that $\overrightarrow{OP} \times \mathbf{b} = \mathbf{c}$.

(i) Find the coordinates of P.

(ii) Find the length of the orthogonal projection of $\overrightarrow{OP}$ on the plane π_1 in (c).

(6 marks)

11. (a) By considering the derivative of $f(x) = (1+x)^{\alpha} - 1 - \alpha x$, show that $(1+x)^{\alpha} > 1 + \alpha x$ for $\alpha > 1$, $x \geq -1$ and $x \neq 0$.

(4 marks)

(b) Let k and m be positive integers. Show that

(i) $1 - \left(1 - \frac{1}{k}\right)^{\frac{m+1}{m}} < \frac{m+1}{m}\left(\frac{1}{k}\right) < \left(1 + \frac{1}{k}\right)^{\frac{m+1}{m}} - 1$,

(ii) $\frac{m}{m+1}\left[k^{\frac{m+1}{m}} - (k-1)^{\frac{m+1}{m}}\right] < k^{\frac{1}{m}} < \frac{m}{m+1}\left[(k+1)^{\frac{m+1}{m}} - k^{\frac{m+1}{m}}\right]$.

(6 marks)

(c) Using (b) or otherwise, show that

$$\frac{2}{3} < \frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + \Lambda + n^{\frac{1}{2}}}{n^{\frac{3}{2}}} < \frac{2}{3}\left(1 + \frac{1}{n}\right)^{\frac{3}{2}} .$$

Hence or otherwise, find $\lim_{n \to \infty} \frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + \Lambda + n^{\frac{1}{2}}}{n^{\frac{3}{2}}}$.

(5 marks)

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12. (a) Resolve $\frac{x^3 - x^2 - 3x + 2}{x^2(x-1)^2}$ into partial fractions.

(3 marks)

(b) Let $P(x) = m(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ where $m, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{R}$ and $m \neq 0$. Prove that

(i) $\sum_{i=1}^{4} \frac{1}{x-\alpha_i} = \frac{P'(x)}{P(x)}$, and

(ii) $\sum_{i=1}^{4} \frac{1}{(x-\alpha_i)^2} = \frac{[P'(x)]^2 - P(x)\,P''(x)}{[P(x)]^2}$.

(3 marks)

(c) Let $f(x) = ax^4 - bx^2 + a$ where $ab > 0$ and $b^2 > 4a^2$.

(i) Show that the four roots of $f(x) = 0$ are real and none of them is equal to 0 or 1 .

(ii) Denote the roots of $f(x) = 0$ by β_1 , β_2 , β_3 and β_4 . Find $\sum_{i=1}^{4} \frac{\beta_i^3 - \beta_i^2 - 3\beta_i + 2}{\beta_i^2(\beta_i - 1)^2}$ in terms of a and b .

(9 marks)

13. Let $n = 2, 3, 4, \ldots$.

(a) Evaluate $\lim_{x \to 1} \frac{x^{2n} - 1}{x^2 - 1}$.

(2 marks)

(b) Find all the complex roots of $x^{2n} - 1 = 0$.

Hence or otherwise, show that $x^{2n} - 1$ can be factorized as

$$(x^2 - 1)(x^2 - 2x\cos\frac{\pi}{n} + 1)(x^2 - 2x\cos\frac{2\pi}{n} + 1)\cdots(x^2 - 2x\cos\frac{(n-1)\pi}{n} + 1) .$$

(6 marks)

(c) Using (b) or otherwise, show that

$$\lim_{x \to 1} \frac{x^{2n} - 1}{x^2 - 1} = 2^{2n-2} \sin^2 \frac{\pi}{2n} \sin^2 \frac{2\pi}{2n} \cdots \sin^2 \frac{(n-1)\pi}{2n} .$$

(4 marks)

(d) Using (a) and (c), or otherwise, show that

$$\lim_{n \to \infty} \left\{ \frac{1}{\sqrt{n}} \sin\left(\frac{\pi}{2n}\right) \sin\left(\frac{2\pi}{2n}\right) \cdots \sin\left(\frac{(n-1)\pi}{2n}\right) \right\}^{\frac{1}{n}} = \frac{1}{2} .$$

(3 marks)

END OF PAPER

2000-AL
P MATH
PAPER 2

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 2000

PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

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2. Answer ALL questions in Section A and any FOUR questions in Section B.
3. You are provided with one AL(E) answer book and four AL(D) answer books.

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FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \,\mu\, \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \,\mu\, \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$
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$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(E) answer book.

1. Evaluate $\int x \cos x \, dx$.

   Hence evaluate $\int_0^{2\pi} x \,|\cos x|\, dx$.

(4 marks)

2. Show that for $x > 0$, $x \geq 1 + \ln x$.

   Find the necessary and sufficient condition for the equality to hold.

(5 marks)

3. Figure 1 shows the graph of $r = 4 \sin 3\theta$ where $0 \leq \theta \leq \pi$. Find the area of the shaded region.

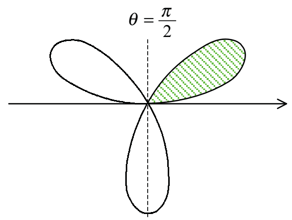

Figure 1

(4 marks)

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4. Let f and g be differentiable functions defined on **R** satisfying the following conditions:

A. $f'(x) = g(x)$ for $x \in \mathbf{R}$;

B. $g'(x) = -f(x)$ for $x \in \mathbf{R}$;

C. $f(0) = 0$ and $g(0) = 1$.

By differentiating $h(x) = [f(x) - \sin x]^2 + [g(x) - \cos x]^2$, or otherwise, show that $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in \mathbf{R}$.

(5 marks)

5. Let k be a positive integer. Evaluate

(a) $\dfrac{d}{dx}\displaystyle\int_0^x \cos t^2 \, dt$,

(b) $\dfrac{d}{dy}\displaystyle\int_0^{y^{2k}} \cos t^2 \, dt$,

(c) $\displaystyle\lim_{y \to 0} \frac{1}{y^{2k}} \int_0^{y^{2k}} \cos t^2 \, dt$.

(6 marks)

6. Use a suitable integral to evaluate $\displaystyle\lim_{n \to \infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right)$

(4 marks)

7. The curve in Figure 2 has parametric equations
$\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$, $0 \le t \le 2\pi$.

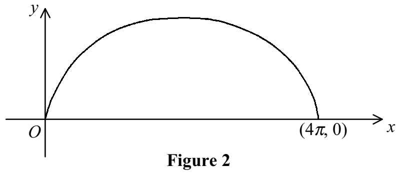

Figure 2

(a) Find the equation of the tangent to the curve at the point where $t = \dfrac{\pi}{2}$.

(b) Find the arc length of the curve.

(6 marks)

8. Let $f(x) = \begin{cases} x^2 + bx + c & \text{if } x \ge 0 , \\ \dfrac{\sin x}{x} + 2x & \text{if } x < 0 . \end{cases}$

(a) If f is continuous at $x = 0$, find c .

(b) If $f'(0)$ exists, find b .

(6 marks)

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SECTION B (60 marks)
Answer any FOUR questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

9. Let $f(x)=\dfrac{x}{(1+x^2)^2}$.

(a) Find $f'(x)$ and $f''(x)$.

(2 marks)

(b) Determine the values of x for each of the following cases:

(i) $f'(x)>0$,

(ii) $f''(x)>0$.

(3 marks)

(c) Find all relative extreme points, points of inflexion and asymptotes of $y=f(x)$.

(4 marks)

(d) Sketch the graph of $f(x)$.

(3 marks)

(e) Let $g(x)=|f(x)|$.

(i) Does $g'(0)$ exist? Why?

(ii) Sketch the graph of $g(x)$.

(3 marks)

10. The equation of the parabola Γ is $y^2=4ax$.

(a) Find the equation of the normal to Γ at the point $(at^2, 2at)$.

Show that if this normal passes through the point (h, k) , then $at^3+(2a-h)t-k=0$.

(4 marks)

(b) Suppose the normals to Γ at three distinct points $(at_1^{\,2}, 2at_1)$, $(at_2^{\,2}, 2at_2)$ and $(at_3^{\,2}, 2at_3)$ are concurrent. Using the result of (a), show that $t_1+t_2+t_3=0$.

(2 marks)

(c) If the circle $x^2+y^2+2gx+2fy+c=0$ intersects Γ at $(as_1^{\,2}, 2as_1)$, $(as_2^{\,2}, 2as_2)$, $(as_3^{\,2}, 2as_3)$ and $(as_4^{\,2}, 2as_4)$, show that $s_1+s_2+s_3+s_4=0$.

(4 marks)

(d) A circle intersects Γ at points A , B , C and D . Suppose A , B and C are distinct and the normals to Γ at these three points are concurrent.

(i) Show that D is the origin.

(ii) If A , B are symmetric about the x-axis, show that the circle touches Γ at the origin.

(5 marks)

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11. (a) In Figure 3, SR is tangent to the curve $y = \ln x$ at $x = r$, where $r \geq 2$. By considering the area of $PQRS$, show that

$$\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \ln x \, dx \leq \ln r \ .$$

Hence show that $\int_{\frac{3}{2}}^{n} \ln x \, dx \leq \ln(n!) - \frac{1}{2}\ln n$ for any integer $n \geq 2$.

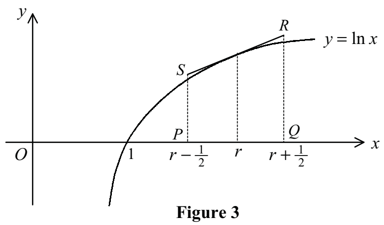

Figure 3

(5 marks)

(b) By considering the graph of $y = \ln x$ and a suitable trapezium, show that for $r \geq 2$, $\int_{r-1}^{r} \ln x \, dx \geq \frac{1}{2}[\ln(r-1) + \ln r]$.

Hence show that $\int_{1}^{n} \ln x \, dx \geq \ln(n!) - \frac{1}{2}\ln n$ for any integer $n \geq 2$.

(4 marks)

(c) Using integration by parts, find $\int \ln x \, dx$.

Using the results of (a) and (b), deduce that $\frac{1}{e} \leq \frac{n^{n+\frac{1}{2}}e^{-n}}{n!} \leq \left(\frac{3}{2e}\right)^{\frac{3}{2}}$ for any integer $n \geq 2$.

(6 marks)

12. (a) Let f be a real-valued function defined on an open interval $\mathbf{I}$, and $f''(x) \geq 0$ for $x \in \mathbf{I}$.

(i) Let $a, b, c \in \mathbf{I}$ with $a < c < b$. Using Mean Value Theorem or otherwise, show that $\frac{f(c) - f(a)}{c - a} \leq \frac{f(b) - f(c)}{b - c}$.

Hence show that $f(c) \leq \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b)$.

(ii) Let $a, b \in \mathbf{I}$ with $a < b$ and $\lambda \in (0, 1)$, show that $a < \lambda a + (1-\lambda)b < b$.

Hence show that $f[\lambda a + (1-\lambda)b] \leq \lambda f(a) + (1-\lambda) f(b)$.

(8 marks)

(b) Let $0 < a < b$. Using (a)(ii) or otherwise, show that

(i) if $p > 1$ and $0 < \lambda < 1$, then $[\lambda a + (1-\lambda)b]^p \leq \lambda a^p + (1-\lambda)b^p$;

(ii) if $0 < \lambda < 1$, then $\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda}$.

(7 marks)

Go on to the next page

13. Let n be a positive integer. Define $f_n(x) = \dfrac{\int_0^x (1-t^4)^n \, dt}{\int_0^1 (1-t^4)^n \, dt}$.

(a) (i) Show that $f_n(x)$ is an odd function.

(ii) Find $f_n'(x)$ and $f_n''(x)$.

(iii) Sketch the graph of $f_n(x)$ for $-1 \le x \le 1$.

(7 marks)

(b) Using the facts

A. $t^3(1-t^4)^n \le (1-t^4)^n$ for $0 \le t \le 1$ and

B. $(1-t^4)^n \le \dfrac{t^3}{x^3}(1-t^4)^n$ for $0 < x \le t \le 1$,

or otherwise, show that $0 \le 1 - f_n(x) \le \dfrac{(1-x^4)^{n+1}}{x^3}$ for $0 < x \le 1$.

(5 marks)

(c) For each $x \in [-1, 1]$, let $g(x) = \lim_{n \to \infty} f_n(x)$. Evaluate $g(x)$ when $0 < x \le 1$ and when $x = 0$ respectively.

Sketch the graph of $g(x)$ for $-1 \le x \le 1$.

(3 marks)

14. Let $a > b > 0$ and define $f(x) = \begin{cases} \left(\dfrac{a^x + b^x}{2} \right)^{\frac{1}{x}} & \text{for } x > 0 , \\ \sqrt{ab} & \text{for } x = 0 . \end{cases}$

(a) (i) Evaluate $\lim_{x \to 0^+} f(x)$.

Hence show that f is continuous at $x = 0$.

(ii) Show that $\lim_{x \to \infty} f(x) = a$.

(6 marks)

(b) Let $h(t) = (1+t)\ln(1+t) + (1-t)\ln(1-t)$ for $0 \le t < 1$ and $g(x) = \ln f(x)$ for $x \ge 0$.

(i) Show that $h(t) > h(0)$ for $0 < t < 1$.

(ii) For $x > 0$, let $t = \dfrac{a^x - b^x}{a^x + b^x}$. Show that $0 < t < 1$ and

$$h(t) = 2\left[\frac{a^x \ln a^x + b^x \ln b^x}{a^x + b^x} + \ln\left(\frac{2}{a^x + b^x} \right) \right] .$$

(iii) Show that for $x > 0$,

$$x^2 g'(x) = \frac{a^x \ln a^x + b^x \ln b^x}{a^x + b^x} + \ln\left(\frac{2}{a^x + b^x} \right) .$$

Hence deduce that $f(x)$ is strictly increasing on $[0, \infty)$.

(9 marks)

END OF PAPER