Solution	Marks
1. (a) $\begin{vmatrix} 1 & 1 & -\lambda \\ 1 & \lambda & -1 \\ \lambda & 1 & -1 \end{vmatrix} = \lambda^3 - 3\lambda + 2$	1A
$\because$ (*) has non-trivial solutions	
$\therefore$ $\lambda^3 - 3\lambda + 2 = 0$	1M
$(\lambda+2)(\lambda-1)^2 = 0$	
$\lambda = -2$ or 1	1A
Alternatively,	
$\begin{pmatrix} 1 & 1 & -\lambda \\ 1 & \lambda & -1 \\ \lambda & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -\lambda \\ 0 & \lambda-1 & \lambda-1 \\ 0 & 1-\lambda & -1+\lambda^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -\lambda \\ 0 & \lambda-1 & \lambda-1 \\ 0 & 0 & \lambda^2+\lambda-2 \end{pmatrix}$	1A
$\therefore$ (*) has non-trivial solutions if $\lambda - 1 = 0$ or $\lambda^2 + \lambda - 2 = 0$	1M
$\lambda = -2$ or 1	1A
(b) When $\lambda = -2$, (*) becomes $\begin{cases} x+y+2z=0 \\ y+z=0 \end{cases}$	1M for substitution
S.S. = $\{(t, t, -t) : t \in \mathbb{R}\}$	1A
When $\lambda = 1$, (*) becomes $x+y-z=0$.	
S.S. = $\{(s, t, s+t) : s, t \in \mathbb{R}\}$	1A
	(6)
2 (a) $\because$ $\dfrac{n!}{(n-k)!k!} + \dfrac{n!}{(n-k-1)!(k+1)!} = \dfrac{n!}{(n-k-1)!k!}\left[\dfrac{1}{n-k} + \dfrac{1}{k+1}\right]$	1A
$= \dfrac{n!}{(n-k-1)!k!} \cdot \dfrac{n+1}{(n-k)(k+1)}$	
$= \dfrac{(n+1)!}{(n-k)!(k+1)!}$	
$\therefore$ $C_k^n + C_{k+1}^n = C_{k+1}^{n+1}$	1
Alternatively,	
$(1+x)(1+x)^n = (1+x)\sum_{k=0}^{n} C_k^n x^k$	1A
$= \sum_{k=0}^{n} C_k^n x^k + \sum_{k=0}^{n} C_k^n x^{k+1}$	
$(1+x)^{n+1} = \sum_{k=0}^{n+1} C_k^{n+1} x^k$	
Comparing the coefficients of x^{k+1}, we have $C_k^n + C_{k+1}^n = C_{k+1}^{n+1}$	1
(b) When $m = 0$, LHS = $C_n^n = 1$	
RHS = $C_{n+1}^{n+1} = 1$	
$\therefore$ The statement holds for $m = 0$.	1
Assume $C_n^n + C_n^{n+1} + C_n^{n+2} + \cdots + C_n^{n+k} = C_{n+1}^{n+k+1}$ for some non-negative integer k.	
Then $C_n^n + C_n^{n+1} + C_n^{n+2} + \cdots + C_n^{n+k+1} = C_{n+1}^{n+k+1} + C_n^{n+k+1}$	1
$= C_{n+1}^{n+k+2}$ (by (a))	1
By the principle of mathematical induction, the statement is true for any integer $m \geq 0$.	
	(5)

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Solution	Marks
3. If $\alpha(\mathbf{a}+\mathbf{b})+\beta(\mathbf{b}+\mathbf{c})+\gamma(\mathbf{c}+\mathbf{a})=0$, then	1A
$(\alpha+\gamma)\mathbf{a}+(\alpha+\beta)\mathbf{b}+(\beta+\gamma)\mathbf{c}=0$	
$\because$ $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent	
$\therefore \begin{cases}\alpha+\gamma=0\\ \alpha+\beta=0\\ \beta+\gamma=0\end{cases}$.	1M+1A
$\Rightarrow \alpha=\beta=\gamma=0$	1
Hence $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$ and $\mathbf{c}+\mathbf{a}$ are linearly independent.	(4)
<u>Alternatively,</u>	
Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$,	
then $\Delta=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}\neq 0$.	1A
Consider $\Delta'=\begin{vmatrix}a_1+b_1&a_2+b_2&a_3+b_3\\b_1+c_1&b_2+c_2&b_3+c_3\\c_1+a_1&c_2+a_2&c_3+a_3\end{vmatrix}$.	1A
Using row operation, $\Delta'\neq 0$.	1M+1A follow through
4. (a) $\because$ $x^2-4x-21=(x+3)(x-7)$	
$x^2-6x-7=(x+1)(x-7)$	
$\therefore$ $x-7$ is a common factor.	1A
$\because$ $P(x)=(x+3)(x-7)\,Q_1(x)+11x-10$ for some polynomial $Q_1(x)$	1M
$P(x)=(x+1)(x-7)\,Q_2(x)+9x+c$ for some polynomial $Q_2(x)$	
Putting $x=7$, we have $77-10=63+c$	
$c=4$	1A
(b) $x^2+4x+3=(x+3)(x+1)$	
Let (1): $P(x)=(x+3)(x+1)\,Q_3(x)+Ax+B$	1A
for some polynomial $Q_3(x)$ and $A, B\in \mathbb{R}$	
$\because$ (2): $P(x)=(x+3)(x-7)\,Q_1(x)+11x-10$	
(3): $P(x)=(x+1)(x-7)\,Q_2(x)+9x+4$	
Putting $x=-3$ in (1) and (2), $-3A+B=-43$	} 1M
Putting $x=-1$ in (1) and (3), $-A+B=-5$	
$\Rightarrow A=19,\quad B=14$	1A
The remainder is $19x+14$.	(6)

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Solution	Marks
5. $\because$ ΔABC is equilateral and $z_1 = 0$	
$\therefore$ $z_3 = z_2(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3})$	1A+1A
$= (1+i)(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i)$	1M
$= \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i$ or $\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i$	1A+1A
<u>Alternatively,</u>	
Let ABC be a equilateral triangle in the xy-plane where $A = (0, 0)$ and $B = (1, 1)$.	
C lies on the perpendicular bisector of AB: $x + y = 1$.	1A
Putting $C = (a, 1 - a)$ and using the fact that $AB = AC$, we have	
$a^2 + (1-a)^2 = 1^2 + 1^2$	1M
$2a^2 - 2a - 1 = 0$	
$a = \frac{1 \pm \sqrt{3}}{2}$	1A
$C = \left(\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$ or $\left(\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$	1A
Hence $z_3 = \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i$ or $\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i$	1A
<u>Alternatively,</u>	
Let $z_3 = a + bi$.	
$\because$ $\lvert z_3 - z_1\rvert^2 = \lvert z_3 - z_2\rvert^2 = \lvert z_2 - z_1\rvert^2$	1M
$\therefore$ $a^2 + b^2 = (a-1)^2 + (b-1)^2 = 2$	
$\Rightarrow \begin{cases} a^2 + b^2 = 2 \\ a + b = 1 \end{cases}$	1A+1A
$\Rightarrow 2a^2 - 2a - 1 = 0$	
$a = \frac{1 \pm \sqrt{3}}{2}$	1A
Hence $z_3 = \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i$ or $\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i$	1A
	(5)

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Solution	Marks
6. (a) $\because \tan\alpha = \frac{1}{2}$	
$\therefore \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-\frac{1}{4}}{1+\frac{1}{4}} = \frac{3}{5}$	1M or looking for α
$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$	
Alternatively,	
$\because \tan\alpha = \frac{1}{2} \Rightarrow \begin{cases}\cos\alpha = \frac{2}{\sqrt{5}} \\ \sin\alpha = \frac{1}{\sqrt{5}}\end{cases}$ or $\begin{cases}\cos\alpha = -\frac{2}{\sqrt{5}} \\ \sin\alpha = -\frac{1}{\sqrt{5}}\end{cases}$	
$\therefore \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{3}{5}$	1M
$\sin 2\alpha = 2\cos\alpha\sin\alpha = \frac{4}{5}$	
The matrix representation of T $= \begin{pmatrix}\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5}\end{pmatrix}$ (or $\frac{1}{5}\begin{pmatrix}3 & 4 \\ 4 & -3\end{pmatrix}$)	1A+1A 1A for any two
(b) $\frac{1}{5}\begin{pmatrix}3 & 4 \\ 4 & -3\end{pmatrix}\begin{pmatrix}4 \\ 7\end{pmatrix} = \begin{pmatrix}8 \\ -1\end{pmatrix}$	1M+1A
$\therefore x_1 = 8,\ y_1 = -1$	
(c) $\frac{1}{5}\begin{pmatrix}3 & 4 \\ 4 & -3\end{pmatrix}\left[\begin{pmatrix}4 \\ 10\end{pmatrix} - \begin{pmatrix}0 \\ 3\end{pmatrix}\right] + \begin{pmatrix}0 \\ 3\end{pmatrix} = \begin{pmatrix}8 \\ -1\end{pmatrix} + \begin{pmatrix}0 \\ 3\end{pmatrix}$	1M
$= \begin{pmatrix}8 \\ 2\end{pmatrix}$	1A
$\therefore x_2 = 8,\ y_2 = 2$	(7)

Solution	Marks
7. (a) When $n=1$, $\dfrac{1}{n^2+4n} = \dfrac{1}{1^2+4} = \dfrac{1}{5} = a_1$.	1
Assume $a_k = \dfrac{1}{k^2+4k}$ for some positive integer k.	
$\because \dfrac{1}{a_{k+1}} - \dfrac{1}{a_k} = 2k+5$	
$\therefore \dfrac{1}{a_{k+1}} = k^2+4k+2k+5$	
$= (k+1)^2+4(k+1)$	
$a_{k+1} = \dfrac{1}{(k+1)^2+4(k+1)}$	1
By the principle of mathematical induction, the result follows.	
Alternatively,	
$\because \dfrac{1}{a_n} - \dfrac{1}{a_{n-1}} = 2(n-1)+5$	
$\dfrac{1}{a_{n-1}} - \dfrac{1}{a_{n-2}} = 2(n-2)+5$	
$\vdots$	
$\dfrac{1}{a_2} - \dfrac{1}{a_1} = 2+5$	
$\therefore \dfrac{1}{a_n} - \dfrac{1}{a_1} = 2[(n-1)+(n-2)+\cdots+1]+5(n-1)$	1
$\dfrac{1}{a_n} - 5 = n(n-1)+5(n-1)$	
$a_n = \dfrac{1}{n^2+4n}$	1
(b) Let $\dfrac{x+2}{(x^2+4x)^2} = \dfrac{x+2}{x^2(x+4)^2} = \dfrac{A}{x^2}+\dfrac{B}{x}+\dfrac{C}{(x+4)^2}+\dfrac{D}{x+4}$	1M
$x+2 = Cx^2+D(x+4)x^2+A(x+4)^2+B(x+4)^2x$	1A
$x+2 = (B+D)x^3+(A+8B+C+4D)x^2+(8A+16B)x+16A$	
$A=\dfrac{1}{8}$, $C=-\dfrac{1}{8}$, $B=D=0$	1A
$\therefore \dfrac{x+2}{(x^2+4x)^2} = \dfrac{1}{8}\left(\dfrac{1}{x^2}-\dfrac{1}{(x+4)^2}\right)$	
Hence $\dfrac{k+2}{(k^2+4k)^2} = \dfrac{1}{8}\left(\dfrac{1}{k^2}-\dfrac{1}{(k+4)^2}\right)$	
$\sum_{k=1}^{n}(k+2)a_k^2 = \dfrac{1}{8}\sum_{k=1}^{n}\left(\dfrac{1}{k^2}-\dfrac{1}{(k+4)^2}\right)$	
$= \dfrac{1}{8}\left(1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}-\dfrac{1}{(n+1)^2}-\dfrac{1}{(n+2)^2}-\dfrac{1}{(n+3)^2}-\dfrac{1}{(n+4)^2}\right)$	1M
$\lim\limits_{n\to\infty}\sum_{k=1}^{n}(k+2)a_k^2 = \dfrac{1}{8}\left(1+\dfrac{1}{4}+\dfrac{1}{9}+\dfrac{1}{16}\right) = \dfrac{205}{1152}$	1A
	(7)

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Solution	Marks	
8. (a) The augmented matrix of (E) is		
$\left(\begin{array}{ccc\|c} 1 & \lambda & 1 & \lambda \\ 3 & -1 & \lambda+2 & 7 \\ 1 & -1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -1 & 1 & 3 \\ 0 & 2 & \lambda-1 & -2 \\ 0 & \lambda+1 & 0 & \lambda-3 \end{array}\right)$	1M	
$\sim \left(\begin{array}{ccc\|c} 1 & -1 & 1 & 3 \\ 0 & 2 & \lambda-1 & -2 \\ 0 & 0 & (\lambda-1)(\lambda+1) & -4(\lambda-1) \end{array}\right)$	1A	
(E) has a unique solution $\Leftrightarrow (\lambda-1)(\lambda+1) \neq 0$ $\Leftrightarrow \lambda \neq \pm 1$	1	
Alternatively,		
Δ of $(E) = \begin{vmatrix} 1 & \lambda & 1 \\ 3 & -1 & \lambda+2 \\ 1 & -1 & 1 \end{vmatrix}$		
$= -1+\lambda(\lambda+2)-3+1+(\lambda+2)-3\lambda$	1M	
$= \lambda^2-1$	1A	
$\therefore$ (E) has unique solution iff $\Delta \neq 0$ iff $\lambda \neq \pm 1$	1	
(b) (i) If $\lambda \neq \pm 1$, the augmented matrix of (E) may further be reduced to		
$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 3+\frac{\lambda-3}{\lambda+1}-\frac{-4}{\lambda+1} \\ 0 & 1 & 0 & \frac{\lambda-3}{\lambda+1} \\ 0 & 0 & 1 & \frac{-4}{\lambda+1} \end{array}\right)$	2M	
$\therefore$ $x=4$, $y=\frac{\lambda-3}{\lambda+1}$, $z=-\frac{4}{\lambda+1}$.	1A+1A	1A for any one
Alternatively, Solve (E) for $\lambda \neq \pm 1$ using Crammer's rule:		
$\Delta_x = \begin{vmatrix} \lambda & \lambda & 1 \\ 7 & -1 & \lambda+2 \\ 3 & -1 & 1 \end{vmatrix} = -\lambda+3\lambda(\lambda+2)-7+3+\lambda(\lambda+2)-7\lambda = 4(\lambda^2-1)$ $\Delta_y = \begin{vmatrix} 1 & \lambda & 1 \\ 3 & 7 & \lambda+2 \\ 1 & 3 & 1 \end{vmatrix} = 7+\lambda(\lambda+2)+9-7-3(\lambda+2)-3\lambda = \lambda^2-4\lambda+3$ $\Delta_z = \begin{vmatrix} 1 & \lambda & \lambda \\ 3 & -1 & 7 \\ 1 & -1 & 3 \end{vmatrix} = -3+7\lambda-3\lambda+\lambda+7-9\lambda = -4(\lambda-1)$	2M	
$\therefore$ $x=\frac{\Delta_x}{\Delta}=4$, $y=\frac{\Delta_y}{\Delta}=\frac{\lambda-3}{\lambda+1}$, $z=\frac{\Delta_z}{\Delta}=-\frac{4}{\lambda+1}$.	1A+1A	1A for any one

Solution	Marks
(b) (ii) When $\lambda=-1$, the augmented matrix of (E) becomes $\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 3 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 0 & -8 \end{array}\right)$.	1A } 1 mark for conclusion only
$\therefore$ (E) is inconsistent.	1
Alternatively, When $\lambda=-1$, (E) becomes $\begin{cases} x-y+z=-1 & \ldots(1) \\ 3x-y+z=7 & \ldots(2) \\ x-y+z=3 & \ldots(3) \end{cases}$	1A
$\because$ (1) and (3) are contradictory $\therefore$ (E) has no solution	1
(iii) When $\lambda=1$, the augmented matrix of (E) becomes $\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$	1M
S.S. $=\{(t,-1,2-t): t\in\mathbb{R}\}$ or $\{(2-t,-1,t): t\in\mathbb{R}\}$	1A
(c) The first three equations form (E) when $\lambda=1$. Sub. $(t,-1,2-t)$ into the fourth equation, then $at-b+c(2-t)=d$	1M
$(a-c)t=b-2c+d$	1A
$\therefore$ The system of equations is consistent when $a\neq c$ or $b-2c+d=0$.	1A+1A
Alternatively, The first three equations form (E) when $\lambda=1$. Thus the system of equations is equivalent to $\left(\begin{array}{ccc\|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ a & b & c & d \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & b & c-a & d-2a \end{array}\right)$	1M
$\sim \left(\begin{array}{ccc\|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & c-a & d-2a+b \end{array}\right)$	1A
$\therefore$ The system is consistent when $a\neq c$ or $d=2a-b$	1A+1A

Solution	Marks
9. (a) The statement clearly holds for $n = 1$.	
Assume $(A+B)^k = A^k + B^k$ for some positive integer k, then	1A
$(A+B)^{k+1} = (A+B)^k(A+B)$	1A
$= (A^k + B^k)(A+B)$	
$= A^{k+1} + A^kB + B^kA + B^{k+1}$	1A −1 for AB^k
$= A^{k+1} + B^{k+1}$	1
$\therefore$ By the principle of mathematical induction, $(A+B)^n = A^n + B^n$ for $n=1,2,3,\dots$.	
(2 marks for using binomial expansion without mentioning $AB = BA$. 2 marks for proving the particular case on 2×2 matrices only.)	
(b) $\because$ $AB = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ 0 & 0 \end{pmatrix}$	1A
$BA = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ap & bp \\ ar & br \end{pmatrix}$	1A
$\therefore$ $AB = BA = 0$	
iff $ap+br = aq+bs = ap = ar = bp = br = 0$.	1M } 1 mark for sufficiency or necessity only
iff $p = r = 0$ ($\because$ $a \neq 0$ or $b \neq 0$) and $aq + bs = 0$.	1
(c) Let $D = \begin{pmatrix} x & k \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & y-k \\ 0 & z \end{pmatrix}$ for some constant k.	1M for $D = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$
D and E are non-zero matrices since x, z are non-zero.	
For $x(y-k)+kz = 0$, we need $k = \dfrac{xy}{x-z}$.	
Since $x \neq z$, we can take $D = \begin{pmatrix} x & \dfrac{xy}{x-z} \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & \dfrac{-yz}{x-z} \\ 0 & z \end{pmatrix}$,	1A+1A
then $D+E = C$ and $DE = ED = 0$.	

Solution	Marks
(d) Using (c), $\begin{pmatrix}2&5\\0&1\end{pmatrix}=\begin{pmatrix}2&10\\0&0\end{pmatrix}+\begin{pmatrix}0&-5\\0&1\end{pmatrix}$	1A
where $\begin{pmatrix}2&10\\0&1\end{pmatrix}\begin{pmatrix}0&-5\\0&1\end{pmatrix}=\begin{pmatrix}0&-5\\0&1\end{pmatrix}\begin{pmatrix}2&10\\0&1\end{pmatrix}=0$.	
$\therefore \begin{pmatrix}2&5\\0&1\end{pmatrix}^{99}=\begin{pmatrix}2&10\\0&0\end{pmatrix}^{99}+\begin{pmatrix}0&-5\\0&1\end{pmatrix}^{99}$ (by (a))	
$\because \begin{pmatrix}2&10\\0&0\end{pmatrix}^{2}=2^2\begin{pmatrix}1&5\\0&0\end{pmatrix}^{2}=2^2\begin{pmatrix}1&5\\0&0\end{pmatrix}$	
$\vdots$	
$\begin{pmatrix}2&10\\0&0\end{pmatrix}^{99}=2^{99}\begin{pmatrix}1&5\\0&0\end{pmatrix}$	1A
and $\begin{pmatrix}0&-5\\0&1\end{pmatrix}^{2}=\begin{pmatrix}0&-5\\0&1\end{pmatrix}\begin{pmatrix}0&-5\\0&1\end{pmatrix}=\begin{pmatrix}0&-5\\0&1\end{pmatrix}$	
$\vdots$	
$\begin{pmatrix}0&-5\\0&1\end{pmatrix}^{99}=\begin{pmatrix}0&-5\\0&1\end{pmatrix}$	1A
$\therefore \begin{pmatrix}2&5\\0&1\end{pmatrix}^{99}=\begin{pmatrix}2^{99}&2^{99}\cdot 5\\0&0\end{pmatrix}+\begin{pmatrix}0&-5\\0&1\end{pmatrix}$	
$=\begin{pmatrix}2^{99}&5(2^{99}-1)\\0&1\end{pmatrix}$	1A
<u>Alternatively</u>,	
$\begin{pmatrix}2&5\\0&1\end{pmatrix}^{2}=\begin{pmatrix}2^2&5\cdot 3\\0&1\end{pmatrix}$	
$\begin{pmatrix}2&5\\0&1\end{pmatrix}^{3}=\begin{pmatrix}2^3&5\cdot 7\\0&1\end{pmatrix}$	
Assume $\begin{pmatrix}2&5\\0&1\end{pmatrix}^{k}=\begin{pmatrix}2^k&5(2^k-1)\\0&1\end{pmatrix}$ for some positive integer k.	2A
Then $\begin{pmatrix}2&5\\0&1\end{pmatrix}^{k+1}=\begin{pmatrix}2&5\\0&1\end{pmatrix}\begin{pmatrix}2^k&5(2^k-1)\\0&1\end{pmatrix}=\begin{pmatrix}2^{k+1}&5(2^{k+1}-1)\\0&1\end{pmatrix}$.	
By the principle of mathematical induction,	
$\begin{pmatrix}2&5\\0&1\end{pmatrix}^{n}=\begin{pmatrix}2^n&5(2^n-1)\\0&1\end{pmatrix}$ for all positive integers n.	1
Hence $\begin{pmatrix}2&5\\0&1\end{pmatrix}^{99}=\begin{pmatrix}2^{99}&5(2^{99}-1)\\0&1\end{pmatrix}$	1A

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Solution	Marks	
10. (a) $\because$ $\mathbf{a}\cdot\mathbf{b}=(1)(-1)+(-1)(1)+(2)(1)=0$		
$\therefore$ a , b are orthogonal to each other.	1	
(b) $\because$ $\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{vmatrix}=-3\mathbf{i}-3\mathbf{j}$	1A	
$\therefore$ The volume of the parallelepiped formed by a , b and c		
$= \lvert(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\rvert$		
$= \lvert(-3\mathbf{i}-3\mathbf{j})\cdot(\mathbf{j}+2\mathbf{k})\rvert$	1M	−1 for missing absolute sign
$= 3$	1A	
Alternatively,		
The volume required $= \lvert(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\rvert$		
$= \left\lvert \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{vmatrix}\cdot(\mathbf{j}+2\mathbf{k}) \right\rvert$	1M	
$= \left\lvert \begin{vmatrix} 0 & 1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{vmatrix} \right\rvert$	1A	
$= 3$	1A	
(c) $\mathbf{n}=\dfrac{\mathbf{a}\times\mathbf{b}}{\lvert\mathbf{a}\times\mathbf{b}\rvert}$ is a unit vector normal to π. (or $\mathbf{n}=-\dfrac{\mathbf{a}\times\mathbf{b}}{\lvert\mathbf{a}\times\mathbf{b}\rvert}$)	1M	Alternatively, 1 mark for $\mathbf{a}\times\mathbf{b}$, 1 mark for $\dfrac{\mathbf{a}\times\mathbf{b}}{\lvert\mathbf{a}\times\mathbf{b}\rvert}$
$\because$ $\lvert\mathbf{a}\times\mathbf{b}\rvert=3\sqrt{2}$		
$\therefore$ $\mathbf{n}=-\dfrac{\sqrt{2}}{2}\mathbf{i}-\dfrac{\sqrt{2}}{2}\mathbf{j}$ (or $\mathbf{n}=\dfrac{\sqrt{2}}{2}\mathbf{i}+\dfrac{\sqrt{2}}{2}\mathbf{j}$)	1A	
(d) The angle between c and n $=\cos^{-1}\left(\dfrac{\mathbf{c}\cdot\mathbf{n}}{\lvert\mathbf{c}\rvert\lvert\mathbf{n}\rvert}\right)$		
$=\cos^{-1}\dfrac{(\mathbf{j}+2\mathbf{k})\left(-\dfrac{\sqrt{2}}{2}\mathbf{i}-\dfrac{\sqrt{2}}{2}\mathbf{j}\right)}{\sqrt{5}}$	1M	
$=\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{\sqrt{5}}\right)$		
$=\cos^{-1}\left(-\dfrac{\sqrt{10}}{10}\right)$		
The angle between c and P $=\cos^{-1}\left(-\dfrac{\sqrt{10}}{10}\right)-\dfrac{\pi}{2}$ (or $\dfrac{\pi}{2}-\cos^{-1}\dfrac{\sqrt{10}}{10}$)	1M+1A	
$\approx 18.4^\circ$ or 0.322		

Solution	Marks
(e) Choose $\mathbf{u}$ to be the projection of $\mathbf{c}$ onto $\mathbf{n}$, then $\mathbf{u}$ is perpendicular to P and	
$\mathbf{u} = (\mathbf{c}\cdot\mathbf{n})\mathbf{n}$	1M
$= -\frac{\sqrt{2}}{2}(-\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j})$	
$= \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$	
Let $\mathbf{v} = \mathbf{c} - \mathbf{u}$.	
Then $\mathbf{u} + \mathbf{v} = \mathbf{c}$ and $\mathbf{v}$ lies on P for $\mathbf{v}\cdot\mathbf{n} = (\mathbf{c}-\mathbf{u})\cdot\mathbf{n} = (\mathbf{c}\cdot\mathbf{n}) - (\mathbf{c}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{n}) = 0$.	1M
$\mathbf{v} = (\mathbf{j} + 2\mathbf{k}) - (\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j})$	
$= -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$	1A
(f) $\mathbf{c} = \left(\frac{\mathbf{c}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert^2}\right)\mathbf{a} + \left(\frac{\mathbf{c}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)\mathbf{b} + (\mathbf{c}\cdot\mathbf{n})\cdot\mathbf{n}$	1M+1M
$= \frac{1}{6}[(\mathbf{j}+2\mathbf{k})(\mathbf{i}-\mathbf{j}+2\mathbf{k})]\mathbf{a} + \frac{1}{3}[(\mathbf{j}+2\mathbf{k})(-\mathbf{i}+\mathbf{j}+\mathbf{k})]\mathbf{b} - \frac{\sqrt{2}}{2}\mathbf{n}$	
$= \frac{1}{2}\mathbf{a} + \mathbf{b} - \frac{\sqrt{2}}{2}\mathbf{n}$ if $\mathbf{n} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$	1A
(or $\frac{1}{2}\mathbf{a} + \mathbf{b} + \frac{\sqrt{2}}{2}\mathbf{n}$ if $\mathbf{n} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$)	
<u>Alternatively,</u>	
$\mathbf{c} = \mathbf{u} + \mathbf{v}$	
$= (\mathbf{c}\cdot\mathbf{n})\mathbf{n} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert^2}\right)\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)\mathbf{b}$	1M+1M
$= -\frac{\sqrt{2}}{2}\mathbf{n} + \frac{1}{6}\left[\left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}\right)(\mathbf{i}-\mathbf{j}+2\mathbf{k})\right]\mathbf{a} + \frac{1}{3}\left[\left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}\right)(-\mathbf{i}+\mathbf{j}+\mathbf{k})\right]\mathbf{b}$	
$= \frac{1}{2}\mathbf{a} + \mathbf{b} - \frac{\sqrt{2}}{2}\mathbf{n}$ if $\mathbf{n} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$	1A

Solution	Marks
11. (a) (i) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$	
$= a^2+b^2+c^2+\omega(ab+bc+ca)+\omega^2(ab+bc+ca)$ $\quad(\because\ \omega^3=1)$	1M+1　1M for $\omega^3=1$
$= a^2+b^2+c^2+(\omega+\omega^2)(ab+bc+ca)$	
$= a^2+b^2+c^2-ab-bc-ca$ $\quad(\because\ \omega^2+\omega+1=0)$	1M for $\omega^2+\omega+1=0$
(ii) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$	
$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$	1
$= a^3+ab^2+ac^2-a^2b-a^2c-abc+a^2b+b^3+bc^2-ab^2-abc-b^2c$ $+a^2c+b^2c+c^3-abc-ac^2-bc^2$	
$= a^3+b^3+c^3-3abc$	
<u>Alternatively,</u>	
$(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$	
$= a^3+a^2b\omega^2+a^2c\omega+a^2b\omega+ab^2\omega^3+abc\omega^2+a^2c\omega^2+abc\omega^4+ac^2\omega^3$ $+a^2b+ab^2\omega^2+abc\omega+ab^2\omega+b^3\omega^3+b^2c\omega^2+abc\omega^2+b^2c\omega^4+bc^2\omega^3$ $+a^2c+abc\omega^2+ac^2\omega+abc\omega+b^2c\omega^3+bc^2\omega^2+ac^2\omega^2+bc^2\omega^4+c^3\omega^3$	
$= a^3+a^2b\omega^2+a^2c\omega+a^2b\omega+ab^2+abc\omega^2+a^2c\omega^2+abc\omega+ac^2$ $+a^2b+ab^2\omega^2+abc\omega+ab^2\omega+b^3+b^2c\omega^2+abc\omega^2+b^2c\omega+bc^2$ $+a^2c+abc\omega^2+ac^2\omega+abc\omega+b^2c+bc^2\omega^2+ac^2\omega^2+bc^2\omega+c^3$	
$= a^3+b^3+c^3+3abc(\omega^2+\omega)$ $+(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)(\omega^2+\omega+1)$	
$= a^3+b^3+c^3-3abc$	1
(b) (i) $p^3+\left(\dfrac{3}{p}\right)^3=12$	
$(p^3)^2-12p^3+27=0$	
$p^3=3$ or 9	1A
$\begin{cases}p=\sqrt[3]{3}\\ q=\sqrt[3]{9}\end{cases}$ or $\begin{cases}p=\sqrt[3]{9}\\ q=\sqrt[3]{3}\end{cases}$	1A
(ii) $\because\quad z^3-9z+12$	
$= z^3+p^3+q^3-3zpq$	1A
$= (z+p+q)(z+p\omega+q\omega^2)(z+p\omega^2+q\omega)$	1A
$\therefore$ Roots of (*) are	
$-p-q=-\sqrt[3]{3}-\sqrt[3]{9}$,	
$-p\omega-q\omega^2=-\sqrt[3]{3}\omega-\sqrt[3]{9}\omega^2$,	
$-p\omega^2-q\omega=-\sqrt[3]{3}\omega^2-\sqrt[3]{9}\omega$.	1A

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Solution	Marks
(c) Sub. $y=z-h$ into $y^3+3y^2-12y+10\sqrt{5}-14=0$,	
$(z-h)^3+3(z-h)^2-12(z-h)+10\sqrt{5}-14=0$	
The term in z^2 will vanish if $-3h+3=0$.	1M
Choosing $h=1$, (**) becomes	
$z^3-3z^2+3z-1+3(z^2-2z+1)-12(z-1)+10\sqrt{5}-14=0$	
$z^3-15z+10\sqrt{5}=0$	1A
Let $\begin{cases} p^3+q^3=10\sqrt{5} \\ pq=5 \end{cases}$	1M
then $p^3+\left(\frac{5}{p}\right)^3=10\sqrt{5}$	
$(p^3)^2-10\sqrt{5}p^3+125=0$	
$p^3=5\sqrt{5}$	
$p=q=\sqrt{5}$	1A
Using (b),	
$z=-p-q\ ,\ -p\omega-q\omega^2$ or $-p\omega-q\omega^2$	
$=-\sqrt{5}-\sqrt{5}\ ,\ -\sqrt{5}\omega-\sqrt{5}\omega^2$ or $-\sqrt{5}\omega-\sqrt{5}\omega^2$	
$=-2\sqrt{5}$ or $\sqrt{5}$ (repeated)	1A
<u>Alternatively,</u>	
$z^3-15z+10\sqrt{5}=0$	
$(z-\sqrt{5})(z^2+\sqrt{5}z-10)=0$	2A
$z=-2\sqrt{5}$ or $\sqrt{5}$ (repeated)	1A
$\because\ y=z-1$	
$\therefore\ y=-2\sqrt{5}-1$ or $\sqrt{5}-1$ (repeated)	1A

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Solution		Marks
12. (a) $\because \sum_{i=1}^{n}(a_i t - b_i)^2 \geq 0$	for all $t \in \mathbf{R}$	1M
$\therefore \left(\sum_{i=1}^{n} a_i^2\right)t^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right)t + \sum_{i=1}^{n} b_i^2 \geq 0$	for all $t \in \mathbf{R}$	1A
The required inequality is trivial when $\sum_{i=1}^{n} a_i^2 = 0$. Suppose $\sum_{i=1}^{n} a_i^2 > 0$.		1
For the quadratic expression to be non-negative for values of t, it is necessary that $4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leq 0$		1M
$\Leftrightarrow \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$		
(b) Let $a_i = x_i^{\frac{k+2}{2}}$ and $b_i = x_i^{\frac{k}{2}}$ for $i = 1, 2, \ldots, n$.	$(\because x_i > 0)$	1M+1A
By (a), $\left(\sum_{i=1}^{n} x_i^{k+1}\right)^2 = \left[\sum_{i=1}^{n}\left(x_i^{\frac{k+2}{2}}\right)\left(x_i^{\frac{k}{2}}\right)\right]^2$		
$\leq \left(\sum_{i=1}^{n} x_i^{k+2}\right)\left(\sum_{i=1}^{n} x_i^{k}\right)$		1
(c) For $p = 0$, LHS $= \sum_{i=1}^{n} x_i^{p} = n$		
RHS $= n\sum_{i=1}^{n} x_i^{p+1} = n\sum_{i=1}^{n} x_i = n$	$(\because \sum_{i=1}^{n} x_i = 1)$	
$\therefore$ the inequality holds for $p = 0$.		1
Assume $\sum_{i=1}^{n} x_i^{k} \leq n\sum_{i=1}^{n} x_i^{k+1}$ for some positive integer k.		1
For $p = k+1$,		
$\left(\sum_{i=1}^{n} x_i^{k+1}\right)^2 \leq \left(\sum_{i=1}^{n} x_i^{k+2}\right)\left(\sum_{i=1}^{n} x_i^{k}\right)$	(by (b))	1
$\leq \left(\sum_{i=1}^{n} x_i^{k+2}\right)\left(n\sum_{i=1}^{n} x_i^{k+1}\right)$	(by induction hypothesis)	1
$\therefore \sum_{i=1}^{n} x_i^{k+1} \leq n\sum_{i=1}^{n} x_i^{k+2}$	$(\because x_1, x_2, \ldots, x_n > 0)$	1
By the principle of mathematical induction, the inequality holds for any non-negative integer p.		

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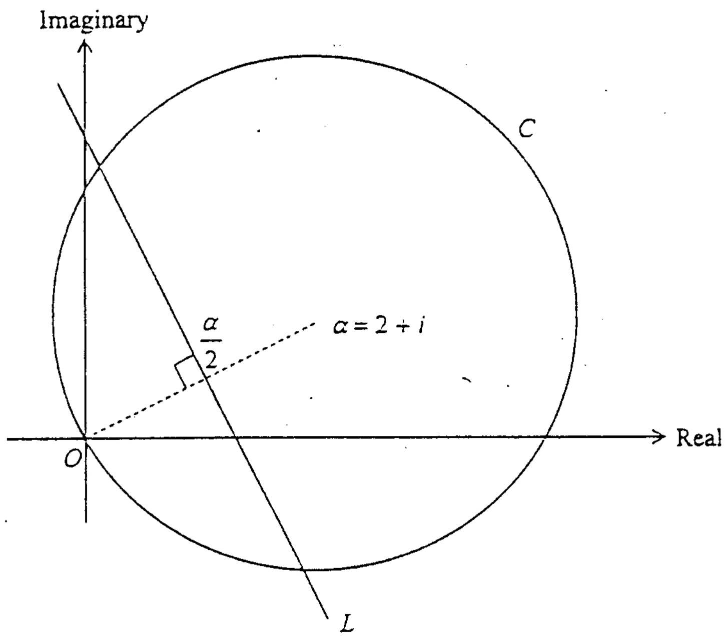
Solution	Marks
(d) Let $x_i = \dfrac{y_i}{\sum_{k=1}^{n} y_k}$, $\quad (\because\ y_1, y_2, \dots, y_n > 0)$	
then $\sum_{i=1}^{n} x_i = 1$.	1
From (c), $\sum_{i=1}^{n} \dfrac{y_i^{\,p}}{\left(\sum_{k=1}^{n} y_k\right)^p} \le n \sum_{i=1}^{n} \dfrac{y_i^{\,p+1}}{\left(\sum_{k=1}^{n} y_k\right)^{p+1}}$	
$\dfrac{\sum_{i=1}^{n} y_i^{\,p}}{\left(\sum_{k=1}^{n} y_k\right)^p} \le \dfrac{n\sum_{i=1}^{n} y_i^{\,p+1}}{\left(\sum_{k=1}^{n} y_k\right)^{p+1}}$	1A
$\left(\sum_{k=1}^{n} y_k\right)\left(\sum_{i=1}^{n} y_i^{\,p}\right) \le n \sum_{i=1}^{n} y_i^{\,p+1}$ $\quad (\because\ y_1, y_2, \dots, y_n > 0)$	
$\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} y_i^{\,p}\right) \le n \sum_{i=1}^{n} y_i^{\,p+1}$	1

Solution	Marks	
13. (a) (i) $\overline{\alpha}u + \alpha\overline{u} = \alpha\overline{\alpha} \Leftrightarrow u\overline{u} - \overline{\alpha}u - \alpha\overline{u} + \alpha\overline{\alpha} = u\overline{u}$	1M	for either
$\Leftrightarrow (u-\alpha)(\overline{u}-\overline{\alpha}) = u\overline{u}$		
$\Leftrightarrow \lvert u-\alpha\rvert = \lvert u\rvert$	1	
(ii) $\overline{\alpha}v + \alpha\overline{v} = v\overline{v} \Leftrightarrow v\overline{v} - \overline{\alpha}v - \alpha\overline{v} + \alpha\overline{\alpha} = \alpha\overline{\alpha}$	1M	for either
$\Leftrightarrow (v-\alpha)(\overline{v}-\overline{\alpha}) = \alpha\overline{\alpha}$		
$\Leftrightarrow \lvert v-\alpha\rvert = \lvert\alpha\rvert$	1	
(b) 	1A+1A 1A+1A	for L (shape / position) for C (shape / position)

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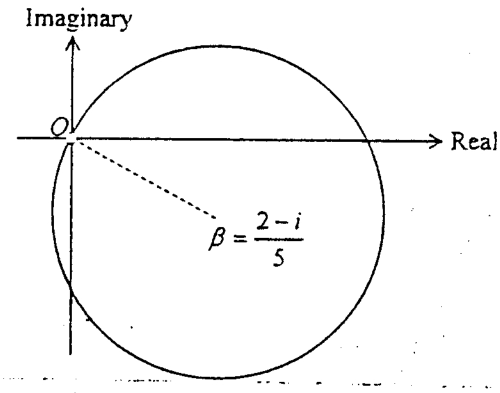
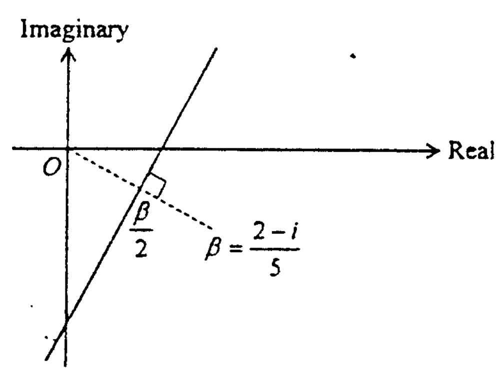
Solution	Marks
(c) (i) $z = \frac{1}{u} \Rightarrow u = \frac{1}{z} \quad (z \neq 0)$	
$\because$ u satisfies $\bar{\alpha}u + \alpha\bar{u} = \alpha\bar{\alpha}$	
$\therefore$ $\bar{\alpha}\cdot\frac{1}{z} + \alpha\cdot\frac{1}{\bar{z}} = \alpha\bar{\alpha}$	
$\overline{\alpha z} + \alpha z = \alpha\bar{\alpha}z\bar{z}$	1M
$\left(\frac{1}{\bar{\alpha}}\right)z + \left(\frac{1}{\alpha}\right)\bar{z} = z\bar{z}$	1A
Hence z satisfies $\bar{\beta}z + \beta\bar{z} = z\bar{z}$ where $\beta = \frac{1}{\alpha}$.	
When $\alpha = 2 + i$, $\beta = \frac{2-i}{5}$.	
	1A+1A
(ii) $z = \frac{1}{v} \Rightarrow v = \frac{1}{z} \quad (v \neq 0)$	
$\because$ v satisfies $\bar{\alpha}v + \alpha\bar{v} = v\bar{v}$	
$\therefore$ $\bar{\alpha}\cdot\frac{1}{z} + \alpha\cdot\frac{1}{\bar{z}} = \frac{1}{z\bar{z}}$	
$\overline{\alpha z} + \alpha z = 1$	
$\left(\frac{1}{\bar{\alpha}}\right)z + \left(\frac{1}{\alpha}\right)\bar{z} = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\bar{\alpha}}\right)$	1A
Hence z satisfies $\bar{\beta}z + \beta\bar{z} = \beta\bar{\beta}$ where $\beta = \frac{1}{\alpha}$.	
	1A+1A

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Solution	Marks
1. (a) $\lim_{x\to 0}\frac{x\sin x}{1-\cos x} = \lim_{x\to 0}\frac{x\cos x+\sin x}{\sin x}$ (by L'Hospital Rule)	1M+1A
$= \lim_{x\to 0}\left(\frac{x}{\tan x}+1\right)$	
$= 2$	1A
(b) Let $y=(1-\cos x)^{\frac{1}{\ln x}}$	
$\ln y = \frac{\ln(1-\cos x)}{\ln x}$	1M
$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\frac{\sin x}{1-\cos x}}{\frac{1}{x}}$ (by L'Hospital Rule)	1A
$= 2$ (by (a))	
$\therefore \lim_{x\to 0^+} y = e^2$	1A
<u>Alternatively,</u>	
$\lim_{x\to 0^+}(1-\cos x)^{\frac{1}{\ln x}} = \lim_{x\to 0^+} e^{\frac{\ln(1-\cos x)}{\ln x}}$	1M
$= \lim_{x\to 0^+} e^{\frac{\frac{\sin x}{1-\cos x}}{\frac{1}{x}}}$ (by L'Hospital Rule)	1A
$= \lim_{x\to 0^+} e^{\frac{x\sin x}{1-\cos x}}$	
$= e^2$ (by (a))	1A
	(6)
2. (a) Let $t=\pi-x$, then	1A
$\int_0^\pi f(\pi-x)\,dx = -\int_\pi^0 f(t)\,dt = \int_0^\pi f(x)\,dx$	1
(b) $\because \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$	1M
$= \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$	
$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx$	1A
$= -\frac{\pi}{2}\int_0^\pi \frac{1}{1+\cos^2 x}\cdot\frac{d\cos x}{dx}\cdot dx$	
$= -\frac{\pi}{2}\left[\tan^{-1}(\cos x)\right]_0^\pi$	1A
$= -\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)$	
$= \frac{\pi^2}{4}$	1A
	(6)

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Solution	Marks
3. Area $= 2\int_0^{\frac{\pi}{3}} \frac{1}{2} r^2 \, d\theta$ (or $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2} r^2 \, d\theta$)	1A for limits
$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \cos 2\theta\right)^2 d\theta$	1A for integrand
$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{4} + \cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$	1A for handling $\int \cos^2 2\theta \, d\theta$
$= \left[\frac{3\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{3}}$	
$= \frac{\pi}{4} + \frac{3\sqrt{3}}{16}$	1A
	(4)
4. (a) $f(x) = \frac{2x}{x^2-1}$	
$= \frac{1}{x-1} + \frac{1}{x+1}$	1A
(b) $f'(x) = \frac{-1}{(x-1)^2} + \frac{-1}{(x+1)^2}$	
$f''(x) = \frac{(-1)(-2)}{(x-1)^3} + \frac{(-1)(-2)}{(x+1)^3}$ $\vdots$	3A (1A for working out $f'(x)$)
$f^{(n)}(x) = \frac{(-1)(-2)\cdots(-n)}{(x-1)^{n+1}} + \frac{(-1)(-2)\cdots(-n)}{(x+1)^{n+1}}$ for $n \ge 1$	
$\therefore \quad f^{(n)}(0) = \begin{cases} -2n! & \text{when } n \text{ is odd,} \\ 0 . & \text{when } n \text{ is even.} \end{cases}$ (or $n![-1+(-1)^n]$)	1A
Alternatively,	
Let $y = \frac{2x}{x^2-1}$, then $(x^2-1)y = 2x$.	
Using Leibniz's Theorem, $\sum_{r=0}^{n} C_r^n (x^2-1)^{(r)} y^{(n-r)} = (2x)^{(n)}$.	1M for using the Leibniz formula
When $n=1$, $(x^2-1)y' + 2xy = 2$ $y'(0) = -2$	1A
When $n \ge 2$, $(x^2-1)y^{(n)} + n(2x)y^{(n-1)} + \frac{n(n-1)}{2}(2)y^{(n-2)} = 0$	
$y^{(n)}(0) = n(n-1)y^{(n-2)}(0)$	1A
Hence $y^{(n)}(0) = \begin{cases} -2n! & \text{when } n \text{ is odd,} \\ 0 & \text{when } n \text{ is even.} \end{cases}$	1A
	(5)

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Solution	Marks
5. f(x) is differentiable at $x=0 \Rightarrow$ f(x) is continuous at $x=0$	1 for mentioning continuity at 0, or the equivalent
$\therefore$ $f(0)=\lim_{x\to 0^+} f(x)$	
$b = \lim_{x\to 0^+} x^2\cos\frac{1}{x}$	1M
$=0$ $\quad(\because \left\lvert\cos\frac{1}{x}\right\rvert \le 1)$	1A with reason
$\because$ f(x) is differentiable at $x=0$	
$\therefore$ $\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$	1M
As $\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h^2\cos\frac{1}{h}}{h} = 0$ (reason not required for $\lim_{h\to 0^+} h\cos\frac{1}{h}=0$)	1A for either
$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{ah}{h} = a$	
$\therefore$ $a=0$	1A (6)
6. (a) $f(x+w) = f(x+2a-2b)$	
$= f(a+(x+a-2b))$	1M
$= f(a-(x+a-2b))$	
$= f(2b-x)$	1A
$= f(b+(b-x))$	
$= f(b-(b-x))$	
$= f(x)$ for all $x \in \mathbb{R}$.	1
$\therefore$ w is a period of f.	
(b) Let $c=\frac{T}{2}$, then for all $x\in\mathbb{R}$,	1A
$g(c+x) = g(\frac{T}{2}+x)$	
$= g(-\frac{T}{2}-x)$ $\quad(\because g(x)=g(-x))$	1M for either
$= g(T-\frac{T}{2}-x)$ $\quad(\because g(x)=g(T+x))$	
$= g(\frac{T}{2}-x)$	
$= g(c-x)$	1 (6)

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Solution	Marks	
7. (a) $\begin{cases} x+y+z=1 \\ x-y-z=5 \end{cases} \Leftrightarrow \begin{cases} x=3 \\ y+z=-2 \end{cases}$	1A	for solving the linear system or finding the direction ratio
$(3, t, -t-2)$ is a point on L_1 for all $t \in \mathrm{R}$.		
$\therefore$ Equation of L_1 is $\begin{cases} x=3 \\ y=t \\ z=-t-2 \end{cases}, t \in \mathrm{R}$.	1A	or equivalent
(b) Let the point of intersection be $(3, t, -t-2)$.		
$\because$ $L_1 \perp L_2$		
$\therefore$ $(0,1,-1)\cdot[(3,t,-t-2)-(1,1,-1)]=0$	1M+1M	1M for direction of L_2 1M for dot product
$(0,1,-1)\cdot(2,t-1,-t-1)=0$		
$t-1+t+1=0$		
$t=0$		
$\therefore$ The point of intersection is $(3, 0, -2)$	1A	
Direction ratios of $L_2 = 2:-1:-1$		
$\therefore$ Equation of L_2 is $\begin{cases} x=2s+1 \\ y=-s+1 \\ z=-s-1 \end{cases}, s \in \mathrm{R}$ $\left(\text{or } \begin{cases} x=2s+3 \\ y=-s \\ z=-s-2 \end{cases}, s \in \mathrm{R} \right)$	1M+1A	or equivalent, 1M for writing down th equation of a straight li with given direction rat
Alternatively,		
Let the equation of L_2 be $\begin{cases} x=as+1 \\ y=bs+1 \\ z=cs-1 \end{cases}, s \in \mathrm{R}$, where a, b and c are constants.	1M	
$\because$ $L_1 \perp L_2$		
$\therefore$ $(0,-1,1)\cdot(a,b,c)=0$	1M	
$b=c$		
L_1 intersects L_2		
$\therefore$ $\begin{cases} as+1=3 \\ bs+1=t \\ cs-1=-t-2 \end{cases}$ is solvable	1M	
$\Rightarrow$ $\begin{cases} as=2 \\ bs-t=-1 \\ cs+t=-1 \end{cases}$ is solvable		
$\Rightarrow$ $t=0$ $(\because b=c)$		
Hence the point of intersection of L_1 and L_2 is $(3, 0, -2)$.	1A	
Now, $a:b:c=3-1:0-1:-2+1=2:-1:-1$		
$\therefore$ Equation of L_2 is $\begin{cases} x=2s+1 \\ y=-s+1 \\ z=-s-1 \end{cases}, s \in \mathrm{R}$.	1A	
	(7)	

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Solution	Marks	
8. (a) $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} xe^{\frac{1}{x}} = 0$	1A	
$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} xe^{\frac{1}{x}}$		
$= \lim_{x\to0^+} \dfrac{e^{\frac{1}{x}}}{\frac{1}{x}}$		
$= \lim_{x\to0^+} \dfrac{e^{\frac{1}{x}}\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'}$ (by L'Hospital rule)	1M	
$= \lim_{x\to0^+} e^{\frac{1}{x}} \to \infty$	1	
(1 mark if candidate wrote $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} e^{\frac{1}{x}} = \infty$)		
Alternatively,		
$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} xe^{\frac{1}{x}}$		
$= \lim_{y\to\infty} \dfrac{1}{y}e^y$		
$= \lim_{y\to\infty} e^y$ (by L'Hospital rule)	1M	
$\to \infty$	1	
(b) For $x \neq 0$,		
$f'(x) = xe^{\frac{1}{x}}\left(-\dfrac{1}{x^2}\right) + e^{\frac{1}{x}} = \left(1-\dfrac{1}{x}\right)e^{\frac{1}{x}}$	1A	
$f''(x) = \left(1-\dfrac{1}{x}\right)e^{\frac{1}{x}}\left(-\dfrac{1}{x^2}\right) + e^{\frac{1}{x}}\left(\dfrac{1}{x^2}\right) = \dfrac{1}{x^3}e^{\frac{1}{x}}$	1A	
(c) (i) $f'(x) > 0 \Leftrightarrow \dfrac{x-1}{x} > 0 \Leftrightarrow x < 0$ or $x > 1$. i.e. $f'(x) > 0$ on $(-\infty, 0)\cup(1, \infty)$.	2A	1A if only $(1, \infty)$ was given
(ii) $f''(x) > 0 \Leftrightarrow \dfrac{1}{x^3} > 0 \Leftrightarrow x > 0$. i.e. $f''(x) > 0$ on $(0, \infty)$.	1A	

(d)

x	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$f(x)$	↑	Undefined	↓	e	↑
$f'(x)$	+	Undefined	−	0	+
$f''(x)$	−	Undefined	+	+	+

$(1, e)$ is a relative minimum point. — 2A (1A for correct answer, 1A for justification)

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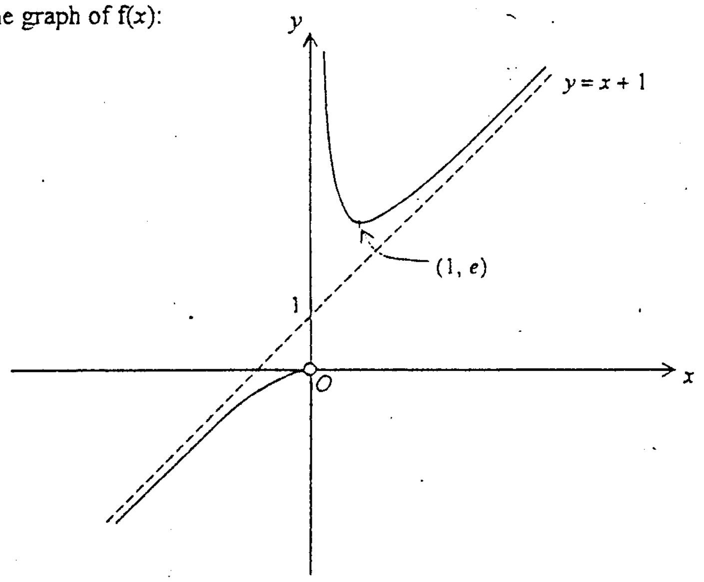
Solution	Marks
(e) From (a)(i), $\because$ $f(x) \to \infty$ as $x \to 0^+$	
$\therefore$ $x = 0$ is a vertical asymptote.	1A
$\because$ $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{1}{x}} = 1$	1A
$\lim_{x\to\infty}\left(f(x) - x\right) = \lim_{x\to\infty}\left(xe^{\frac{1}{x}} - x\right)$	
$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}}$	
$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'}$	
$= 1$	1A
$\therefore$ $y = x + 1$ is an oblique asymptote.	
(f) The graph of $f(x)$: 	1 for the shape of the curve 1 for all being correct

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Solution		Marks	
9. (a) For $t^2 \neq 1$,			
$\left(1+t^2+\cdots+t^{2n-2}\right)+\dfrac{t^{2n}}{1-t^2}=\dfrac{1-(t^2)^n}{1-t^2}+\dfrac{t^{2n}}{1-t^2}$		1M	for any reasonable method
$=\dfrac{1}{1-t^2}$		1	
(b) (i) $\displaystyle\int_0^x \frac{t}{1-t^2}\,dt=\frac{1}{2}\int_0^{x^2}\frac{1}{1-u}\,du$	(putting $u=t^2$)	1A	must get the limits of integration right
$=-\dfrac{1}{2}\left[\ln\lvert 1-u\rvert\right]_0^{x^2}$		1M	ignore limits
$=\ln\dfrac{1}{\sqrt{1-x^2}}$	for $-1<x<1$	1	
(ii) From (a),			
$\dfrac{t}{1-t^2}=\left(t+t^3+\cdots+t^{2n-1}\right)+\dfrac{t^{2n+1}}{1-t^2}$	for $t^2\neq 1$		
$\displaystyle\int_0^x\frac{t}{1-t^2}\,dt=\int_0^x\left(t+t^3+\cdots+t^{2n-1}\right)dt+\int_0^x\frac{t^{2n+1}}{1-t^2}\,dt$	for $-1<x<1$	1M+1A	
$\displaystyle\ln\frac{1}{\sqrt{1-x^2}}=\int_0^x\left(t+t^3+\cdots+t^{2n-1}\right)dt+\int_0^x\frac{t^{2n+1}}{1-t^2}\,dt$	(by (b)(i))	2 (for these three lines)	
$\displaystyle\ln\frac{1}{\sqrt{1-x^2}}=\left[\frac{t^2}{2}+\frac{t^4}{4}+\cdots+\frac{t^{2n}}{2n}\right]_0^x+\int_0^x\frac{t^{2n+1}}{1-t^2}\,dt$			
$\displaystyle\ln\frac{1}{\sqrt{1-x^2}}-\left(\frac{x^2}{2}+\frac{x^4}{4}+\cdots+\frac{x^{2n}}{2n}\right)=\int_0^x\frac{t^{2n+1}}{1-t^2}\,dt$			

Solution	Marks
(c) Putting $x^2 = \frac{8}{9}$ in (b)(ii), we have	1M
$$\ln 3 - \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k = \int_0^{\sqrt{\frac{8}{9}}} \frac{t^{2n+1}}{1-t^2}\,dt$$	1A
For $0 \le t \le \sqrt{\frac{8}{9}}$,	
$\because \frac{t^{2n+1}}{1-t^2} \ge 0$ $\therefore \int_0^{\sqrt{\frac{8}{9}}} \frac{t^{2n+1}}{1-t^2}\,dt \ge 0$	1
$\because \frac{t^{2n+1}}{1-t^2} \le \frac{t^{2n+1}}{1-\frac{8}{9}} = 9t^{2n+1}$	1M
$\therefore \int_0^{\sqrt{\frac{8}{9}}} \frac{t^{2n+1}}{1-t^2}\,dt \le 9\int_0^{\sqrt{\frac{8}{9}}} t^{2n+1}\,dt$ $= \left[\frac{9}{2n+2}t^{2n+2}\right]_0^{\sqrt{\frac{8}{9}}}$	
$= \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}$	1
Hence $0 \le \ln 3 - \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k \le \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1}$	
$\because \lim_{n\to\infty} \frac{9}{2n+2}\left(\frac{8}{9}\right)^{n+1} = 0$	
$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2k}\left(\frac{8}{9}\right)^k = \ln 3$	1A

Solution	Marks
10. (a) (i) Suppose on the contrary that f^{-1} is not strictly increasing.	
Then there exist y_1 and y_2 such that $y_1 < y_2$ and	
$f^{-1}(y_1) \ge f^{-1}(y_2)$	1M
$\Rightarrow f(f^{-1}(y_1)) \ge f(f^{-1}(y_2))$ ($\because$ f is strictly increasing)	
$\Rightarrow y_1 \ge y_2$	
which contradicts that $y_1 < y_2$.	1
(1 mark if candidate used $(f^{-1})'(y) = \dfrac{1}{f'(x)}$)	
(ii) $\because$ $t_1, t_2, \ldots, t_n \in [a, b]$ and f is strictly increasing	
$\therefore$ $f(a) \le f(t_i) \le f(b)$ for $i = 1, 2, \ldots, n$	1M
$\Rightarrow n f(a) \le \sum_{i=1}^{n} f(t_i) \le n f(b)$	1M
$\Rightarrow f(a) \le \dfrac{1}{n}\sum_{i=1}^{n} f(t_i) \le f(b)$	
$\Rightarrow a \le f^{-1}\left(\dfrac{1}{n}\sum_{i=1}^{n} f(t_i)\right) \le b$ ($\because f^{-1}$ is strictly increasing)	1
$\because$ f is strictly increasing	
$\therefore$ For $t_1, t_2, \ldots, t_n \in [a, b]$,	
$f(a) = \dfrac{1}{n}\sum_{i=1}^{n} f(t_i)$ iff $t_1 = t_2 = \cdots = t_n = a$ and	1A
$\dfrac{1}{n}\sum_{i=1}^{n} f(t_i) = f(b)$ iff $t_1 = t_2 = \cdots = t_n = b$.	1A
Again since f^{-1} is strictly increasing,	
$a < f^{-1}\left(\dfrac{1}{n}\sum_{i=1}^{n} f(t_i)\right) < b$	
iff $t_1, t_2, \ldots, t_n$ not all equal to a and not all equal to b.	1A
(or equivalently, there is a $t_i \ne a$ and there is a $t_j \ne b$, $1 \le i, j \le n$.)	

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Solution	Marks
(b) (i) For any $x_1, x_2 \in R$,	
$g(x_1) = g(x_2) \Rightarrow x_1^{\frac{1}{3}} = x_2^{\frac{1}{3}} \Rightarrow x_1 = x_2$	
$\therefore$ g is injective.	1
For any $y \in R$, let $x = y^3$, then $g(x) = y$.	
$\therefore$ g is surjective.	1
Alternatively, Let $f(x) = x^3$ for $x \in R$.	
Then $g \circ f(x) = g(x^3) = (x^3)^{\frac{1}{3}} = x$ and $f \circ g(x) = f(x^{\frac{1}{3}}) = \left(x^{\frac{1}{3}}\right)^3 = x$	1
$\therefore$ f is the inverse of g. Hence g is bijective.	1
$\because$ $g'(x) = \dfrac{1}{3x^{\frac{2}{3}}} > 0$ for $x \in R \setminus \{0\}$ and g is continuous on R	
$\therefore$ g is strictly increasing.	1
Alternatively, Since $g = f^{-1}$ and f is strictly increasing, so is g.	1
Using (a) and putting $t_i = i$ where $i = 1, 2, \ldots, n$,	
$\because$ $t_1, t_2, \ldots, t_n$ not all equal to 1 and not all equal to n	
$\therefore$ $1 < g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(i)\right) < n$	
$\Rightarrow 1 < \left(\dfrac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n}\right)^3 < n$	1
(ii) Area enclosed by g and g^{-1} $= 2\int_0^1 (g - g^{-1})dx$	1M for correct limits of integration
$= 2\int_0^1 \left(x^{\frac{1}{3}} - x^3\right)dx$	1A
$= 2\left[\frac{3}{4}x^{\frac{4}{3}} - \frac{1}{4}x^4\right]_0^1$	
$= 1$	1A

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Solution	Marks
11. (a) $I_n(y) = \int_0^y t^n e^{-t}\,dt$	
$= -\left\{\left[t^n e^{-t}\right]_0^y - \int_0^y n t^{n-1} e^{-t}\,dt\right\}$	1M for integration by
$= -\dfrac{y^n}{e^y} + n\int_0^y t^{n-1}e^{-t}\,dt$	1A
$= -\dfrac{y^n}{e^y} + n\,I_{n-1}(y)$ for $n \ge 1$ and $y \ge 0$	1
$I_0(y) = \int_0^y e^{-t}\,dt = \left[-e^{-t}\right]_0^y = 1 - \dfrac{1}{e^y} \le 1$	1A
Suppose $I_k(y) \le k!$ for some non-negative integer k.	
$I_{k+1}(y) = -\dfrac{y^n}{e^y} + (k+1)I_k(y)$	
$\le (k+1)I_k(y)$	
$\le (k+1)!$	1
By the principle of mathematical induction, the result follows.	
(b) (i) $\because$ $g'(x) = \dfrac{n}{n+x} + \dfrac{n}{n-x} - 2 = \dfrac{2x^2}{n^2 - x^2} \ge 0$ for $0 \le x < n$.	1A
$\therefore$ g is increasing for $0 \le x < n$	1
i.e. $g(x) \ge g(0)$ for $0 \le x < n$	
Hence for $0 \le x < n$,	
$n\ln(n+x) - n\ln(n-x) - 2x \ge g(0) = 0$	1
$n\ln(n+x) \ge n\ln(n-x) + 2x$	
$(n+x)^n \ge (n-x)^n e^{2x}$	
$(n+x)^n e^{-(n+x)} \ge (n-x)^n e^{-(n-x)}$	1
(ii) $\int_n^{2n} u^n e^{-u}\,du = \int_0^n (n+x)^n e^{-(n+x)}\,dx$ (putting $u = n+x$)	
$\ge \int_0^n (n-x)^n e^{-(n-x)}\,dx$ (by (b)(i))	1M
$= \int_n^0 y^n e^{-y}(-dy)$ (putting $y = n-x$)	
$= \int_0^n u^n e^{-u}\,du$	1
Hence $\int_0^{2n} t^n e^{-t}\,dt = \int_0^n t^n e^{-t}\,dt + \int_n^{2n} t^n e^{-t}\,dt$	1M
$\ge \int_0^n t^n e^{-t}\,dt + \int_0^n t^n e^{-t}\,dt$	
$= 2\int_n^0 (n-x)^n e^{-n+x}(-dx)$ (putting $t = n-x$)	
$= 2e^{-n}\int_0^n (n-t)^n e^t\,dt$	1

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Solution	Marks
(c) For $n > 0$,	
$\frac{1}{n!}\int_0^n (n-t)^n e^t \mathrm{d}t \le \frac{1}{2e^{-n}(n!)}\int_0^{2n} t^n e^{-t} \mathrm{d}t$ (by (b)(ii))	1
$= \frac{1}{2e^{-n}(n!)} I_n(2n)$	
$\le \frac{1}{2e^{-n}(n!)}(n!)$ (by (a))	
$= \frac{e^n}{2}$	1

Solution	Marks
12. (a) (i) $\frac{(x-h)^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\left(1-\frac{(x-h)^2}{a^2}\right)$	
$\therefore$ Area enclosed by $E = 2\int_{h-a}^{h+a} b\sqrt{1-\frac{(x-h)^2}{a^2}}\,dx$	1A limits of integration must be correct
$= 2\int_{-a}^{a} b\sqrt{1-\frac{y^2}{a^2}}\,dy$ (by putting $y=x-h$)	
$= \frac{4b}{a}\int_0^a \sqrt{a^2-y^2}\,dy$	
$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta$ (by putting $y=a\sin\theta$)	1M
$= 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,dx$	
$= 2ab\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$	
$= ab\pi$	1A
<u>Alternatively,</u> Transforming E to standard position by $x \mapsto x-h$, E becomes $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.	
Area enclosed by $E = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$	1A
$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$	
$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$	
$= \frac{4b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta$ (by putting $x=a\sin\theta$)	1M
$= 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,dx$	
$= 2ab\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$	
$= ab\pi$	1A
(ii) Sub. $y=mx$ into the equation of E, then $\frac{(x-h)^2}{a^2}+\frac{m^2x^2}{b^2}=1$	
$(a^2m^2+b^2)x^2-2b^2hx+b^2h^2-a^2b^2=0$	1A
$\because$ $y=mx$ is tangent to E	
$\therefore$ $4b^4h^2-4(a^2m^2+b^2)b^2(h^2-a^2)=0$	1M
$b^2h^2-(a^2m^2+b^2)(h^2-a^2)=0$	
$b^2-m^2(h^2-a^2)=0$	
$m^2=\frac{b^2}{h^2-a^2}$	1

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Solution	Marks
(b) (i) $\because$ E_n and E_{n+1} touch each other externally	
$\therefore$ $h_n - a_n = h_{n+1} + a_{n+1}$	
$h_n - h_{n+1} = a_n + a_{n+1}$	1A
(ii) By (a)(ii), $m^2 = \dfrac{p^2 a_n^2}{h_n^2 - a_n^2}$	1M
$\therefore$ $\dfrac{p^2 a_n^2}{h_n^2 - a_n^2} = \dfrac{p^2 a_1^2}{h_1^2 - a_1^2}$	1M
$a_n^2 h_1^2 = a_1^2 h_n^2$	
$h_n = \dfrac{h_1}{a_1} a_n$	1A
Hence $\dfrac{h_1}{a_1} a_n - \dfrac{h_1}{a_1} a_{n+1} = a_n + a_{n+1}$ (by (b)(i))	
$\left(\dfrac{h_1}{a_1} - 1\right) a_n = \left(\dfrac{h_1}{a_1} + 1\right) a_{n+1}$	
$\dfrac{a_{n+1}}{a_n} = \dfrac{h_1 - a_1}{h_1 + a_1}$	1
(iii) $\sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} a_n (p a_n) \pi$ (by (a)(i))	1M for using the result of a(i)
$= p\pi \sum_{n=1}^{\infty} a_n^2$	
$\because$ $\dfrac{a_{n+1}^2}{a_n^2} = \left(\dfrac{h_1 - a_1}{h_1 + a_1}\right)^2 \in (0, 1)$	
$\therefore$ The summation is an infinite sum of a geometric sequence with common ratio $\left(\dfrac{h_1 - a_1}{h_1 + a_1}\right)^2$.	1M for realizing the series is a geometric series
Hence $\sum_{n=1}^{\infty} S_n = p\pi \cdot \dfrac{a_1^2}{1 - \left(\dfrac{h_1 - a_1}{h_1 + a_1}\right)^2}$	1A
$= p\pi \cdot \dfrac{a_1^2 (h_1 + a_1)^2}{(h_1 + a_1)^2 - (h_1 - a_1)^2}$	
$= \dfrac{p a_1 (h_1 + a_1)^2 \pi}{4 h_1}$	1A

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Solution	Marks	
13. (a) (i) By Mean Value Theorem, there exists $\xi_1 \in (a, x)$ such that		
$f(x) - f(a) = f'(\xi_1)(x-a)$ for $x \in (a, a+1)$	1	for using mean value theorem properly
$\because$ $f(a) = 0$ and $x - a > 0$		
$\therefore$ $\lvert f(x)\rvert = \lvert f'(\xi_1)\rvert(x-a) \le \lvert f(\xi_1)\rvert(x-a)$	1	
(ii) Using (a)(i), the statement holds for $n = 1$.		
Assume that there exists $\xi_k \in (a, x)$ such that $\lvert f(x)\rvert \le \lvert f(\xi_k)\rvert(x-a)^k$.	1	
By Mean Value Theorem, there exists $\xi_{k+1} \in (a, \xi_k)$ such that		
$f(\xi_k) - f(a) = f'(\xi_{k+1})(\xi_k - a)$		
$\lvert f(\xi_k)\rvert = \lvert f'(\xi_{k+1})\rvert(\xi_k - a) \le \lvert f(\xi_{k+1})\rvert(x-a)$	1	
$\therefore$ $\lvert f(x)\rvert \le \lvert f(\xi_k)\rvert(x-a)^k \le \lvert f(\xi_{k+1})\rvert(x-a)^{k+1}$	1	
By the principle of mathematical induction, the result follows.		
(iii) $\because$ there exists $M > 0$ such that		
$\lvert f(x)\rvert \le M$ for all $x \in [a, a+1]$		
$\therefore$ $0 \le \lvert f(x)\rvert \le M(x-a)^n$ for all $n = 1, 2, 3, \ldots$ and $x \in (a, a+1)$	1	
$x \in (a, a+1) \Rightarrow \lim_{n\to\infty} M(x-a)^n = 0$	1A	
Hence $f(x) = 0$ for all $x \in (a, a+1)$	1	
$\because$ f is continuous on $[a, a+1]$		
$\therefore$ $f(x) = 0$ for all $x \in [a, a+1]$	1	
(b) (i) $f(0) = 0 \Rightarrow f(x) = 0$ for all $x \in [0, 1]$ (by (a)(iii))	1	
Assume $f(x) = 0$ for all $x \in [k, k+1]$ where k is a non-negative integer.		
Hence $f(k+1) = 0 \Rightarrow f(x) = 0$ for all $x \in [k+1, k+2]$.		
By the principle of mathematical induction,		induction not mandatory
$f(x) = 0$ for all $x \in [n, n+1]$ where $n = 0, 1, 2, \ldots$.	1	
i.e. $f(x) = 0$ for all $x \in [0, 1] \cup [1, 2] \cup [2, 3] \cup \cdots$		
$f(x) = 0$ for all $x \in [0, \infty)$	1	
(ii) Now define $g(x) = f(-x)$ for $x \in R$, then	1M	
g is a differentiable function on R with		
$g(0) = f(0) = 0$ and		
$\lvert g'(x)\rvert = \lvert -f'(-x)\rvert = \lvert f'(-x)\rvert \le \lvert f(-x)\rvert = \lvert g(x)\rvert$	1	
Using (b)(i),		
$g(x) = 0$ for all $x \in [0, \infty)$		
$\Rightarrow$ $f(x) = 0$ for all $x \in (-\infty, 0]$		
Hence $f(x) = 0$ for all $x \in R$	1	

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