99-AL
P MATHS
PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1999

PURE MATHEMATICS A-LEVEL PAPER 1

8.30 am – 11.30 am (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A and any FOUR questions in Section B.
3. You are provided with one AL(C)1 answer book and four AL(D) answer books.

   Section A : Write your answers in the AL(C)1 answer book.

   Section B : Use a separate AL(D) answer book for each question and put the question number on the front cover of each answer book.
4. The four AL(D) answer books should be tied together with the green tag provided. The AL(C)1 answer book and the four AL(D) answer books must be handed in separately at the end of the examination.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C)1 answer book.

1. Suppose the system of linear equations

$$(*): \begin{cases} x + y - \lambda z = 0 \\ x + \lambda y - z = 0 \\ \lambda x + y - z = 0 \end{cases}$$

has nontrivial solutions.

(a) Find all the values of λ.

(b) Solve (*) for each of the values of λ obtained in (a).

(6 marks)

2. For any positive integer n, let C_k^n be the coefficient of x^k in the expansion of $(1+x)^n$.

(a) Show that $C_k^n + C_{k+1}^n = C_{k+1}^{n+1}$.

(b) By induction on m or otherwise, show that
$C_n^n + C_n^{n+1} + C_n^{n+2} + \cdots + C_n^{n+m} = C_{n+1}^{n+m+1}$
for any $m \geq 0$.

(5 marks)

3. If vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent, show that $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$ and $\mathbf{c}+\mathbf{a}$ are also linearly independent.

(4 marks)

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4. Let $\mathrm{P}(x)$ be a polynomial. When $\mathrm{P}(x)$ is divided by $x^2-4x-21$, the remainder is $11x-10$. When $\mathrm{P}(x)$ is divided by x^2-6x-7, the remainder is $9x+c$, where c is a constant.

(a) Find a common factor of $x^2-4x-21$ and x^2-6x-7.
Hence find c.

(b) Find the remainder when $\mathrm{P}(x)$ is divided by x^2+4x+3.

(6 marks)

5. A, B and C are three points in the Argand diagram representing the complex numbers z_1, z_2 and z_3 respectively. If $z_1 = 0$, $z_2 = 1+i$ and ΔABC is equilateral, find z_3.

(5 marks)

6. It is given that the matrix representing the reflection in the line $y = (\tan\alpha)x$ is
$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$.

Let T be the reflection in the line $y = \frac{1}{2}x$.

(a) Find the matrix representation of T.

(b) The point $(4, 7)$ is transformed by T to another point (x_1, y_1). Find x_1 and y_1.

(c) The point $(4, 10)$ is reflected in the line $y = \frac{1}{2}x + 3$ to another point (x_2, y_2). Find x_2 and y_2.

(7 marks)

7. A sequence $\{a_n\}$ is defined as follows:

$$a_1 = \frac{1}{5} \quad \text{and} \quad \frac{1}{a_{n+1}} - \frac{1}{a_n} = 2n+5 \quad \text{for } n = 1, 2, 3, \dots .$$

(a) Show that $a_n = \dfrac{1}{n^2+4n}$ for $n = 1, 2, 3, \dots$.

(b) Resolve $\dfrac{x+2}{(x^2+4x)^2}$ into partial fractions.

Hence or otherwise, evaluate $\lim\limits_{n\to\infty} \sum_{k=1}^{n} (k+2){a_k}^2$.

(7 marks)

SECTION B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

8. Consider the system of linear equations

$$(E): \begin{cases} x + \lambda y + z = \lambda \\ 3x - y + (\lambda+2)z = 7 \\ x - y + z = 3 \end{cases} \quad \text{where } \lambda \in \mathbf{R} .$$

(a) Show that (E) has a unique solution if and only if $\lambda \neq \pm 1$.

(3 marks)

(b) Solve (E) for

(i) $\lambda \neq \pm 1$,

(ii) $\lambda = -1$,

(iii) $\lambda = 1$.

(8 marks)

(c) Find the conditions on a , b, c and d so that the system of linear equations

$$\begin{cases} x + y + z = 1 \\ 3x - y + 3z = 7 \\ x - y + z = 3 \\ ax + by + cz = d \end{cases}$$

is consistent.

(4 marks)

9. (a) Let A and B be two square matrices of the same order. If $AB = BA = 0$, show that $(A+B)^n = A^n + B^n$ for any positive integer n .

(4 marks)

(b) Let $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where a , b are not both zero. If $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, show that $AB = BA = 0$ if and only if $p = r = 0$ and $aq + bs = 0$.

(4 marks)

(c) Let $C = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ where x , z are non-zero and distinct. Find non-zero matrices D and E such that $C = D + E$ and $DE = ED = 0$.

(3 marks)

(d) Evaluate $\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^{99}$.

(4 marks)

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10. Let $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{j} + 2\mathbf{k}$ be three vectors in $\mathbf{R}^3$ and P be the plane containing $\mathbf{a}$ and $\mathbf{b}$.

(a) Prove that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

(1 mark)

(b) Find the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(3 marks)

(c) Find a unit vector $\mathbf{n}$ perpendicular to the plane P .

(2 marks)

(d) Find the angle between $\mathbf{c}$ and P .

(3 marks)

(e) Find two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} + \mathbf{v} = \mathbf{c}$, where $\mathbf{u}$ is perpendicular to P , and $\mathbf{v}$ lies on P .

(3 marks)

(f) Express $\mathbf{c}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{n}$ where $\mathbf{n}$ is the vector obtained in (c).

(3 marks)

11. Let ω be a complex cube root of 1 .

(a) Prove the identities

(i) $a^2+b^2+c^2-ab-bc-ca=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$,

(ii) $a^3+b^3+c^3-3abc=(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$.

(4 marks)

(b) Consider the equation

$z^3-9z+12=0$ .... (*) .

(i) Find real numbers p and q such that
$p^3+q^3=12$ and $pq=3$.

(ii) Using (a) or otherwise, find the roots of (*) in terms of ω.

(5 marks)

(c) Consider the equation

$y^3+3y^2-12y+10\sqrt{5}-14=0$ .... (**) .

Using the substitution $y=z-h$ with a suitable constant h , rewrite (**) as $z^3+sz+t=0$ where s , t are constants.

Hence solve (**).

(6 marks)

Go on to the next page

12. (a) Let a_i , b_i be real numbers, where $i=1, 2, \ldots, n$. By considering the function $f(t)=\sum_{i=1}^{n}(a_i t-b_i)^2$, or otherwise, prove Schwarz's inequality $\left(\sum_{i=1}^{n}a_i b_i\right)^2 \le \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)$.

(4 marks)

(b) Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Show that

$\left(\sum_{i=1}^{n}x_i^{k+1}\right)^2 \le \left(\sum_{i=1}^{n}x_i^{k+2}\right)\left(\sum_{i=1}^{n}x_i^{k}\right)$

for any non-negative integer k .

(3 marks)

(c) Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^{n}x_i=1$.

Prove by induction on p that $\sum_{i=1}^{n}x_i^{p} \le n\sum_{i=1}^{n}x_i^{p+1}$ for any non-negative integer p .

(5 marks)

(d) Let $y_1, y_2, \ldots, y_n$ be positive real numbers. Show that

$\left(\sum_{i=1}^{n}y_i\right)\left(\sum_{i=1}^{n}y_i^{p}\right) \le n\sum_{i=1}^{n}y_i^{p+1}$

for any non-negative integer p .

(3 marks)

13. Let α be a complex number and u, v be variable complex numbers satisfying

$$\bar{\alpha}u + \alpha\bar{u} = \alpha\bar{\alpha} \quad \text{and} \quad \bar{\alpha}v + \alpha\bar{v} = v\bar{v}$$

respectively. Let L be the locus of u and C be the locus of v.

(a) Show that

(i) the equation of L can be written as $|u-\alpha| = |u|$,

(ii) the equation of C can be written as $|v-\alpha| = |\alpha|$.

(4 marks)

(b) For $\alpha = 2+i$, sketch L and C on an Argand diagram.

(4 marks)

(c) (i) Let $z = \frac{1}{u}$. Show that z satisfies $\bar{\beta}z + \beta\bar{z} = z\bar{z}$ for some constant β. Hence sketch the locus of z on an Argand diagram for $\alpha = 2+i$.

(ii) Let $z = \frac{1}{v}$. Sketch the locus of z on an Argand diagram for $\alpha = 2+i$.

(7 marks)

END OF PAPER

1999 Paper 1
Section A

1. (a) $\lambda = -2$ or 1

(b) When $\lambda = -2$, S.S. $= \{(t, t, -t) : t \in \mathbf{R}\}$.
When $\lambda = 1$, S.S. $= \{(s, t, s+t) : s, t \in \mathbf{R}\}$.

4. (a) $x-7$ is a common factor.
$c = 4$

(b) $19x+14$

5. $\frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i$ or $\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i$

6. (a) $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

(b) $x_1 = 8$, $y_1 = -1$

(c) $x_2 = 8$, $y_2 = 2$

7. (b) $\frac{x+2}{(x^2+4x)^2} = \frac{1}{8}\left(\frac{1}{x^2} - \frac{1}{(x+4)^2}\right)$

$\lim_{n\to\infty} \sum_{k=1}^{n} (k+2)a_k^{\,2} = \frac{205}{1152}$

99-AL
P MATHS
PAPER 2

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1999

PURE MATHEMATICS A-LEVEL PAPER 2

1.30 pm – 4.30 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A and any FOUR questions in Section B.
3. You are provided with one AL(C)1 answer book and four AL(D) answer books.

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FORMULAS FOR REFERENCE

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$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C)1 answer book.

1. (a) Evaluate $\lim_{x \to 0} \frac{x \sin x}{1 - \cos x}$.

   (b) Using (a) or otherwise, evaluate $\lim_{x \to 0^+} (1 - \cos x)^{\frac{1}{\ln x}}$.

(6 marks)

2. (a) Let f be a continuous function. Show that $\int_0^\pi \mathrm{f}(x)\,\mathrm{d}x = \int_0^\pi \mathrm{f}(\pi - x)\,\mathrm{d}x$.

   (b) Evaluate $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\,\mathrm{d}x$.

(6 marks)

3. Figure 1 shows the graph of $r = \frac{1}{2} + \cos 2\theta$. Find the area of the shaded region.

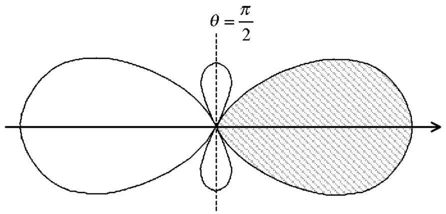

Figure 1

(4 marks)

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4. Let $\mathrm{f}(x) = \frac{2x}{x^2 - 1}$.

   (a) Resolve $\mathrm{f}(x)$ into partial fractions.

   (b) Find $\mathrm{f}^{(n)}(0)$, where $n = 1, 2, 3, \ldots$.

(5 marks)

5. Let $\mathrm{f}(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x > 0, \\ ax + b & \text{if } x \le 0 \end{cases}$
   be differentiable at $x = 0$. Find a and b .

(6 marks)

6. (a) Suppose $\mathrm{f} : \mathbf{R} \to \mathbf{R}$ is a function satisfying $\mathrm{f}(a + x) = \mathrm{f}(a - x)$ and $\mathrm{f}(b + x) = \mathrm{f}(b - x)$ for all x, where a, b are constants and $a > b$. Let $w = 2(a - b)$. Show that w is a period of f , i.e., $\mathrm{f}(x + w) = \mathrm{f}(x)$ for all $x \in \mathbf{R}$.

   (b) Suppose $\mathrm{g} : \mathbf{R} \to \mathbf{R}$ is a periodic function with period $T > 0$ satisfying $\mathrm{g}(x) = \mathrm{g}(-x)$ for all x . Show that there exists c with $0 < c < T$ such that $\mathrm{g}(c + x) = \mathrm{g}(c - x)$ for all x .

(6 marks)

7. Let L_1 be the line of intersection of the planes $x + y + z = 1$ and $x - y - z = 5$, and L_2 be the line passing through $(1, 1, -1)$ and intersecting L_1 at right angle.

   (a) Find a parametric equation of L_1 .

   (b) Find the coordinates of the point of intersection of L_1 and L_2 , and a parametric equation of L_2 .

(7 marks)

SECTION B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Use a separate AL(D) answer book for each question.

8. Let $f(x) = xe^{\frac{1}{x}}$ for $x \neq 0$.

(a) Find $\lim_{x \to 0^-} f(x)$ and show that $f(x) \to \infty$ as $x \to 0^+$.

(3 marks)

(b) Find $f'(x)$ and $f''(x)$ for $x \neq 0$.

(2 marks)

(c) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$,

(ii) $f''(x) > 0$.

(3 marks)

(d) Find all relative extrema of $f(x)$.

(2 marks)

(e) Find all asymptotes of the graph of $f(x)$.

(3 marks)

(f) Sketch the graph of $f(x)$.

(2 marks)

Go on to the next page

9. Let n be a positive integer.

(a) Show that $\frac{1}{1-t^2} = (1 + t^2 + \cdots + t^{2n-2}) + \frac{t^{2n}}{1-t^2}$ for $t^2 \neq 1$.

(2 marks)

(b) For $-1 < x < 1$, show that

(i) $\int_0^x \frac{t}{1-t^2}\,dt = \ln \frac{1}{\sqrt{1-x^2}}$,

(ii) $\int_0^x \frac{t^{2n+1}}{1-t^2}\,dt = \ln \frac{1}{\sqrt{1-x^2}} - \left(\frac{x^2}{2} + \frac{x^4}{4} + \cdots + \frac{x^{2n}}{2n} \right)$.

(7 marks)

(c) Show that $0 \leq \ln 3 - \sum_{k=1}^{n} \frac{1}{2k} \left(\frac{8}{9} \right)^k \leq \frac{9}{2n+2} \left(\frac{8}{9} \right)^{n+1}$.

Hence evaluate $\sum_{k=1}^{\infty} \frac{1}{2k} \left(\frac{8}{9} \right)^k$.

(6 marks)

10. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be a strictly increasing bijective function.

(i) Show that the inverse function f^{-1} is also strictly increasing.

(ii) Let $a < b$ and t_1 , t_2 , ... , $t_n \in [a, b]$, $n \geq 2$.

Show that $a \leq f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(t_i)\right) \leq b$.

Find a necessary and sufficient condition on t_1 , t_2 , ... , t_n such that $a < f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(t_i)\right) < b$.

(8 marks)

(b) Let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = x^{\frac{1}{3}}$.

(i) Show that g is bijective and strictly increasing.

Hence show that $1 < \left(\frac{1 + 2^{\frac{1}{3}} + \cdots + n^{\frac{1}{3}}}{n}\right)^3 < n$ for $n \geq 2$.

(ii) Find the area enclosed by the graphs of $y = g(x)$ and $y = g^{-1}(x)$.

(7 marks)

Go on to the next page

11. (a) For $n = 0, 1, 2, \ldots$ and $y \geq 0$, define $I_n(y) = \int_0^y t^n e^{-t}\,dt$.

Prove that $I_n(y) = -y^n e^{-y} + nI_{n-1}(y)$ for $n \geq 1$ and $y \geq 0$.

Hence deduce that $I_n(y) \leq n!$ for $n \geq 0$ and $y \geq 0$.

(5 marks)

(b) Let n be a positive integer.

(i) By considering $g(x) = n\ln(n+x) - n\ln(n-x) - 2x$ for $0 \leq x < n$, show that $(n+x)^n e^{-(n+x)} \geq (n-x)^n e^{-(n-x)}$ for $0 \leq x < n$.

(ii) Use (b)(i) to show that $\int_n^{2n} u^n e^{-u}\,du \geq \int_0^n u^n e^{-u}\,du$.

Hence deduce that $\int_0^{2n} t^n e^{-t}\,dt \geq 2e^{-n}\int_0^n (n-t)^n e^t\,dt$.

(8 marks)

(c) Using the above results or otherwise, show that $\frac{1}{n!}\int_0^n (n-t)^n e^t\,dt \leq \frac{e^n}{2}$ for all positive integers n .

(2 marks)

12. (a) The equation of the ellipse E is $\dfrac{(x-h)^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a, b, h \in \mathbf{R}$ and $a, b > 0$.

(i) By integration, find the area enclosed by E .

(ii) If the straight line $y = mx$ is tangent to E , show that

$$m^2 = \frac{b^2}{h^2 - a^2} .$$

(6 marks)

(b) For $n = 1, 2, 3, \ldots$, let E_n be the ellipse given by

$$\frac{(x-h_n)^2}{a_n{}^2}+\frac{y^2}{p^2 a_n{}^2}=1 ,$$

where $p > 0$, $h_n > h_{n+1}$ and $h_n > a_n > 0$. Suppose for all n , E_n and E_{n+1} touch each other externally and the straight line $y = mx$ is a common tangent to all E_n as shown in Figure 2.

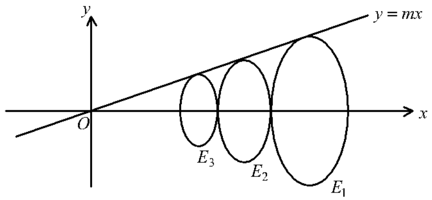

Figure 2

(i) Express $h_n - h_{n+1}$ in terms of a_n and a_{n+1} .

(ii) Using (a)(ii) and the result of (b)(i), or otherwise, show that

$$\frac{a_{n+1}}{a_n} = \frac{h_1 - a_1}{h_1 + a_1} .$$

(iii) Let S_n be the area enclosed by the ellipse E_n . Evaluate $\sum_{n=1}^{\infty} S_n$ in terms of a_1 , h_1 and p .

(9 marks)

Go on to the next page

13. Let $f(x)$ be a differentiable function on $\mathbf{R}$ such that $|f'(x)| \le |f(x)|$ for all $x \in \mathbf{R}$.

(a) Suppose $a \ge 0$ and $f(a) = 0$. Let $x \in (a, a+1)$.

(i) Using Mean Value Theorem or otherwise, show that there exists $\xi_1 \in (a, x)$ such that $|f(x)| \le |f(\xi_1)|(x-a)$.

(ii) Using (a)(i) or otherwise, show that for each $n = 1, 2, 3, \ldots$, there exists $\xi_n \in (a, x)$ such that $|f(x)| \le |f(\xi_n)|(x-a)^n$.

(iii) Using (a)(ii) or otherwise, show that $f(x) = 0$ for all $x \in [a, a+1]$.

You may use the fact that there is $M > 0$ such that $|f(x)| \le M$ for all $x \in [a, a+1]$.

(9 marks)

(b) Suppose $f(0) = 0$.

(i) Using (a) or otherwise, show that $f(x) = 0$ for all $x \in [0, \infty)$.

(ii) Show that $f(x) = 0$ for all $x \in \mathbf{R}$.

(6 marks)

END OF PAPER