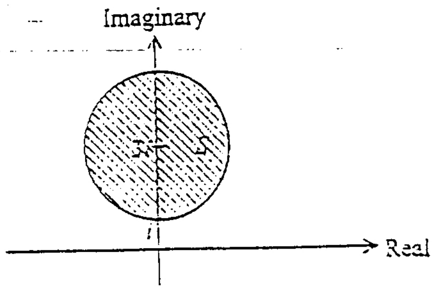
Solution	Marks
1. (a) If (*) has infinitely many solutions, then	
$\begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & k+1 \\ k & -1 & 4 \end{vmatrix} = 0$	1M
$k(k+1)-2+2(k+1)-4=0$	
$k^2+3k-4=0$	1A
$(k+4)(k-1)=0$	
$k=-4$ or 1	1A
(b) If $k=-4$,	
$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & -3 \\ -4 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{pmatrix}$	1M
S.S. = $\{(3t,\ -8t,\ t) : t \in R\}$	1A
If $k=1$,	
$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$	
S.S. = $\{(-2t,\ 2t,\ t) : t \in R\}$	1A
	(6)
2. (a) $\because$ $\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$	
$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$	1M
$\therefore$ The matrix representation is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.	1A
(b) $\because$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$	
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$	1M
$\therefore$ The matrix representation is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.	1A
	(4)

Solution	Marks
3. (a) $\mathbf{m}\cdot\mathbf{n} = 2\times 1\times\cos\frac{2\pi}{3} = -1$	1A
(b) $\lvert\mathbf{p}\rvert^2 = (3\mathbf{m}+4\mathbf{n})\cdot(3\mathbf{m}+4\mathbf{n})$	1M
$= 9\mathbf{m}\cdot\mathbf{m}+24\mathbf{m}\cdot\mathbf{n}+16\mathbf{n}\cdot\mathbf{n}$	
$= 9(2^2)+24(-1)+16(1^2)$	
$= 28$	
$\therefore \lvert\mathbf{p}\rvert = 2\sqrt{7}$	1A
$\lvert\mathbf{q}\rvert^2 = (2\mathbf{m}-\mathbf{n})\cdot(2\mathbf{m}-\mathbf{n})$	
$= 4\mathbf{m}\cdot\mathbf{m}-4\mathbf{m}\cdot\mathbf{n}+\mathbf{n}\cdot\mathbf{n}$	
$= 4(2^2)-4(-1)+(1^2)$	
$= 21$	
$\therefore \lvert\mathbf{q}\rvert = \sqrt{21}$	1A
(c) Area of the parallelogram formed by the vectors p and q	
$= \lvert\mathbf{p}\times\mathbf{q}\rvert$	1M
$= \lvert(3\mathbf{m}+4\mathbf{n})\times(2\mathbf{m}-\mathbf{n})\rvert$	
$= \lvert -3\mathbf{m}\times\mathbf{n}+8\mathbf{n}\times\mathbf{m}\rvert$	1A
$= 11\lvert\mathbf{m}\times\mathbf{n}\rvert$	
$= 11(2\times 1\times\sin\frac{2\pi}{3})$	
$= 11\sqrt{3}$	1A
<u>Alternatively,</u>	
Let θ be the angle between p and q.	
$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert}$	
$= \frac{(3\mathbf{m}+4\mathbf{n})\cdot(2\mathbf{m}-\mathbf{n})}{(2\sqrt{7})(\sqrt{21})}$	
$= \frac{15}{14\sqrt{3}}$	
$\sin\theta = \sqrt{1-\cos^2\theta}$	
$= \frac{11}{14}$	1A
$\therefore$ Area of the parallelogram $= \lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\sin\theta$	
$= (2\sqrt{7})(\sqrt{21})\left(\frac{11}{14}\right)$	1M
$= 11\sqrt{3}$	1A
	(7)

Solution	Marks
4. (a) Let $z = x + yi$ where $x, y \in \mathbb{R}$	
$\because \quad 2\lvert z-2i\rvert = \lvert z+i\rvert$	
$\therefore \quad 2\lvert x+(y-2)i\rvert = \lvert x+(y+1)i\rvert$	
$4x^2+4(y-2)^2 = x^2+(y+1)^2$	1M
$x^2+y^2-6y+5=0$	1A or equivalent
$x^2+(y-3)^2=2^2$	
Alternatively,	
$\because \quad 2\lvert z-2i\rvert = \lvert z+i\rvert$	
$\therefore \quad 4(z-2i)(\bar{z}+2i) = (z+i)(\bar{z}-i)$	1M
$4(z\bar{z}+2iz-2i\bar{z}+4) = z\bar{z}-iz+i\bar{z}+1$	
$z\bar{z}+3iz-3i\bar{z}+5=0$	
$(z-3i)(\bar{z}+3i)=4$	
$\lvert z-3i\rvert = 2$	1A
$\therefore$ The locus of z is a circle of radius 2 centred at $3i$.	1A accept (0, 3)
(b) From (a),	
$2\lvert z-2i\rvert \le \lvert z+i\rvert \Leftrightarrow 4x^2+4(y-2)^2 \le x^2+(y+1)^2$	
$\Leftrightarrow x^2+(y-3)^2 \le 2^2$	
Alternatively,	
$2\lvert z-2i\rvert \le \lvert z+i\rvert \Leftrightarrow 4(z-2i)(\bar{z}+2i) \le (z+i)(\bar{z}-i)$	
$\Leftrightarrow \lvert z-3i\rvert \le 2$	
$\therefore$ The region inside the circle including the boundary represents S on the Argand diagram.	
	1M+1A (1M for shading interior part 1A for circle)
Let $z_0 = i$, then $\lvert z\rvert \ge \lvert z_0\rvert$ for all $z \in S$.	1A (6)

Solution	Marks
5. For $n=1$, $\alpha+\beta=14$ which is divisible by 2.	1A
For $n=2$, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$	
$=14^2-2\cdot36$	
$=124$ which is divisible by 2^2.	1A
Assume $\alpha^n+\beta^n$ is divisible by 2^n for $n=k-1,\ k$ (where $k\geq 2$).	1M
$\alpha^{k+1}+\beta^{k+1}=(\alpha+\beta)(\alpha^k+\beta^k)-\alpha\beta(\alpha^{k-1}+\beta^{k-1})$	1A
$=14\cdot2^k p-36\cdot2^{k-1}q$ (for some p, $q\in Z$)	
$=2^{k+1}(7p-9q)$	1
$\therefore$ $\alpha^{k+1}+\beta^{k+1}$ is divisible by 2^{k+1}.	
By the principle of mathematical induction, $\alpha^n+\beta^n$ is divisible by 2^n for $n=1,2,3,\dots$.	
	(5)
6. (a) $f(x)=x^p-px+p-1$	
$f'(x)=px^{p-1}-p$	1A
$=p(x^{p-1}-1)$	
$\begin{cases} >0 & \text{if } 0<x<1 \\ =0 & \text{if } x=1 \\ <0 & \text{if } x>1 \end{cases}$	
$\therefore$ $f(x)\leq f(1)$ for $x>0$	
The absolute maximum value of $f(x)$ for $x>0$	
$=f(1)$	1A
$=0$	1A
(b) $\because$ $a, b>0$ $\therefore$ $\frac{a}{b}>0$	1
By (a), $f(\frac{a}{b})\leq 0$	1
$(\frac{a}{b})^p-p(\frac{a}{b})+p-1\leq 0$	
$a^pb^{1-p}-pa+pb-b\leq 0$	
$a^pb^{1-p}\leq pa+(1-p)b$	1
	(6)

Solution	Marks
7. (a) $\underline{f(x)}$ $\underline{g(x)}$	
$\begin{array}{r\|rrrrr\|rrrr\|l} 2x+1 & 2x^4 & +x^3 & +10x^2 & +2x & +15 & x^3 & & +2x & -3 & \frac{1}{6}x-\frac{1}{6} \\ & 2x^4 & & +4x^2 & -6x & & x^3 & +x^2 & +3x & & \\ \hline & & x^3 & +6x^2 & +8x & +15 & & -x^2 & -x & -3 & \\ & & x^3 & & +2x & -3 & & -x^2 & -x & -3 & \\ \hline & & & 6x^2 & +6x & +18 & & & & 0 & \end{array}$	1M+1A
$\therefore \quad d(x) = x^2 + x + 3$	1A or equivalent
Alternatively,	
$g(x) = (x-1)(x^2+x+3)$	1A
$f(x) = (2x^2-x+5)(x^2+x+3)$	1A
$\therefore \quad d(x) = x^2+x+3$	1A
(b) $\because \quad f(x) = (2x+1)g(x) + 6d(x)$	1M
$\therefore \quad \frac{1}{6}f(x) + \frac{-(2x+1)}{6}g(x) = d(x)$	1A+1A for u(x) and v(x)
Alternatively,	
Let $u(x) = ax+b$, $v(x) = cx+d$	
If $u(x)f(x) + v(x)g(x) = d(x)$	
then $(ax+b)(2x^2-x+5)+(cx+d)(x-1)=1$	1M
$2ax^3+(-a+2b+c)x^2+(5a-b-c+d)x+(5b-d-1)=0$	
giving $a=0$, $b=\frac{1}{6}$, $c=-\frac{1}{3}$, $d=-\frac{1}{6}$	1A+1A
Take $u(x)=\frac{1}{6}$, $v(x)=-\frac{2x+1}{6}$, the result follows.	
	(6)

Solution — Marks

8. (a) The augmented matrix of (E) is

$$\left(\begin{array}{ccc|c} a & 1 & b & 1 \\ 1 & a & b & 1 \\ 1 & 1 & ab & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & ab & b \\ 0 & a-1 & (1-a)b & 1-b \\ 0 & 1-a & (1-a^2)b & 1-ab \end{array}\right)$$

1M

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & ab & b \\ 0 & a-1 & (1-a)b & 1-b \\ 0 & 0 & (1-a)(2+a)b & 2-b-ab \end{array}\right)$$

1A

(E) has a unique solution $\Leftrightarrow (1-a)(2+a)b \neq 0$

$\Leftrightarrow a \neq 1\ ,\ a \neq -2$ and $b \neq 0$ — 1

If $a \neq 1\ ,\ a \neq -2$ and $b \neq 0$, then

$$z = \frac{2-b-ab}{(1-a)(2+a)b}$$

1M+1A

$$y = \frac{-1}{(1-a)}\left[1-b-(1-a)b\cdot\frac{2-b-ab}{(1-a)(2+a)b}\right] = \frac{-(a-b)}{(1-a)(2+a)}$$

1A

$$x = b + \frac{a-b}{(1-a)(2+a)} - \frac{ab(2-b-ab)}{(1-a)(2+a)b} = \frac{-(a-b)}{(1-a)(2+a)}$$

1A

Alternatively,

$$\Delta \text{ of (E)} = \begin{vmatrix} a & 1 & b \\ 1 & a & b \\ 1 & 1 & ab \end{vmatrix} = b\begin{vmatrix} a-1 & 1 & 1 \\ 1-a & a & 1 \\ 0 & 1 & a \end{vmatrix} = (1-a)b\begin{vmatrix} -1 & 1 & 1 \\ 1 & a & 1 \\ 0 & 1 & a \end{vmatrix}$$

1M

$$= (1-a)b[-a^2+1+1-a]$$

$$= (1-a)^2(2+a)b$$

1A+1

∴ (E) has unique solution iff $\Delta \neq 0$

iff $a \neq 1,\ a \neq -2$ and $b \neq 0$

Solve (E) for $a \neq 1,\ a \neq -2$ and $b \neq 0$ using Crammer's rule:

$$\Delta x = \begin{vmatrix} 1 & 1 & b \\ 1 & a & b \\ b & 1 & ab \end{vmatrix} = -(1-a)(a-b)b$$

$$\Delta y = \begin{vmatrix} a & 1 & b \\ 1 & 1 & b \\ 1 & b & ab \end{vmatrix} = -(1-a)(a-b)b$$

$$\Delta z = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & b \end{vmatrix} = (1-a)(2-b-ab)$$

} 1M

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{-(a-b)}{(1-a)(2+a)}$$

1A

$$y = \frac{\Delta y}{\Delta} = \frac{-(a-b)}{(1-a)(2+a)}$$

1A

$$z = \frac{\Delta z}{\Delta} = \frac{2-b-ab}{(1-a)(2+a)b}$$

1A

Solution — Marks

(b) (i) When $a=-2$, the augmented matrix of (E) becomes $\left(\begin{array}{ccc|c} 1 & 1 & -2b & b \\ 0 & -3 & 3b & 1-b \\ 0 & 0 & 0 & 2+b \end{array}\right)$. 1M

(E) is consistent when $b=-2$. 1A

When $a=-2$ and $b=-2$, the augmented matrix of (E) becomes

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

S.S. $=\{(-1-2t,-1-2t,t):t\in\mathrm{R}\}$ 1A

or $\{(t,t,-\frac{1+t}{2}):t\in\mathrm{R}\}$

(ii) When $a=1$, the augmented matrix of (E) becomes $\left(\begin{array}{ccc|c} 1 & 1 & b & b \\ 0 & 0 & 0 & 1-b \\ 0 & 0 & 0 & 0 \end{array}\right)$. 1M

(E) is consistent when $b=1$. 1A

When $a=1$ and $b=1$, (E) reduces to $x+y+z=1$.

S.S. $=\{(s,t,1-s-t):s,t\in\mathrm{R}\}$ 1A

or $\{(1-s-t,s,t):s,t\in\mathrm{R}\}$ or $\{(s,1-s-t,t):s,t\in\mathrm{R}\}$

(c) When $b=0$, the augmented matrix of (E) becomes $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & a-1 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$. 1M

Hence (E) is inconsistent for $b=0$. 1

Solution	Marks
9. (a) Since $\det A = 0$, $\therefore\ ad = bc$	1M
Let $k = \dfrac{c}{a}$ $(\because\ a \neq 0)$	1
then $c = ka$ and $d = \dfrac{bc}{a} = kb$	
$\therefore\ A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$	1
Accept	
$\because \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$ $\therefore \begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ are linearly dependent	1
Hence $k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$ for some $k \in \mathbb{R}$.	1
$\Rightarrow c = ka$ and $d = kb$.	1
(b) $\because \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\begin{pmatrix} a & b \\ ka & kb \end{pmatrix} = \begin{pmatrix} a & b \\ (r+k)a & (r+k)b \end{pmatrix}$	1M
$\therefore$ Take $P = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$, then $PA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$.	1A
$PAP^{-1} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$	
$= \begin{pmatrix} a+kb & b \\ 0 & 0 \end{pmatrix}$	1A
$\because \begin{pmatrix} a+kb & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+kb & s(a+kb)+b \\ 0 & 0 \end{pmatrix}$	1M
and $a + kb = a + d \neq 0$	
$\therefore$ Take $Q = \begin{pmatrix} 1 & -\dfrac{b}{a+kb} \\ 0 & 1 \end{pmatrix}$, then	
$PAP^{-1}Q = \begin{pmatrix} a+kb & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\dfrac{b}{a+kb} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+kb & 0 \\ 0 & 0 \end{pmatrix}$	1A

Solution	Marks
(c) Let $A = \begin{pmatrix}3 & 7\\ 6 & 14\end{pmatrix} = \begin{pmatrix}3 & 7\\ 2\times 3 & 2\times 7\end{pmatrix}$.	
Let $P = \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$, then $PAP^{-1} = \begin{pmatrix}17 & 7\\ 0 & 0\end{pmatrix}$ by (b)	1A or for $P^{-1} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$
Let $Q = \begin{pmatrix}1 & -\frac{7}{17}\\ 0 & 1\end{pmatrix}$, then $PAP^{-1}Q = \begin{pmatrix}17 & 0\\ 0 & 0\end{pmatrix}$ by (b)	1A
and $Q^{-1}PAP^{-1}Q = \begin{pmatrix}1 & \frac{7}{17}\\ 0 & 1\end{pmatrix}\begin{pmatrix}17 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}17 & 0\\ 0 & 0\end{pmatrix}$	1A
$\therefore$ Let $S = Q^{-1}P$	
$= \begin{pmatrix}1 & \frac{7}{17}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$	
$= \begin{pmatrix}\frac{3}{17} & \frac{7}{17}\\ -2 & 1\end{pmatrix}$	1A
then $S^{-1} = P^{-1}Q$	
and $SAS^{-1} = Q^{-1}PAP^{-1}Q = \begin{pmatrix}17 & 0\\ 0 & 0\end{pmatrix}$.	
Hence $A^n = \left[S^{-1}\begin{pmatrix}17 & 0\\ 0 & 0\end{pmatrix}S\right]^n$	
$= S^{-1}\begin{pmatrix}17 & 0\\ 0 & 0\end{pmatrix}^n S$	1M
$= S^{-1}\begin{pmatrix}17^n & 0\\ 0 & 0\end{pmatrix}S$	
$= \begin{pmatrix}1 & -\frac{7}{17}\\ 2 & \frac{3}{17}\end{pmatrix}\begin{pmatrix}17^n & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{3}{17} & \frac{7}{17}\\ -2 & 1\end{pmatrix}$	1A
$= \begin{pmatrix}17^n & 0\\ 2\cdot 17^n & 0\end{pmatrix}\begin{pmatrix}\frac{3}{17} & \frac{7}{17}\\ -2 & 1\end{pmatrix}$	
$= \begin{pmatrix}3\cdot 17^{n-1} & 7\cdot 17^{n-1}\\ 6\cdot 17^{n-1} & 14\cdot 17^{n-1}\end{pmatrix}$	1A

Solution	Marks
10. (a) Let x, y, z be real numbers such that $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$.	
The equation is equivalent to $\begin{pmatrix}-1 & 4 & 0\\ 3 & 1 & 1\\ 2 & -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$.	1M
Since $\begin{vmatrix}-1 & 4 & 0\\ 3 & 1 & 1\\ 2 & -1 & 3\end{vmatrix}=-32\neq 0$	1A
∴ The system has the trivial solution only, i.e. $x=y=z=0$ Hence a, b and c are linearly independent.	1
(b) (i) $\overline{OG}=\overline{OA}+\overline{OB}+\overline{OC}=3\mathbf{i}+5\mathbf{j}+4\mathbf{k}$	1A
(ii) Let (x,y,z) be any point on the plane containing $AOBD$ and $\mathbf{v}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $\mathbf{v}\cdot(\mathbf{a}\times\mathbf{b})=0$.	1M
Since $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 3 & 2\\ 4 & 1 & -1\end{vmatrix}=-5\mathbf{i}+7\mathbf{j}-13\mathbf{k}$	
∴ The required equation is $5x-7y+13z=0$.	1A
(iii) Let (x,y,z) be any point on the plane containing $AEGD$ and $\mathbf{v}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $(\mathbf{v}-\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c})=0$.	
Since $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 1 & -1\\ 0 & 1 & 3\end{vmatrix}=4\mathbf{i}-12\mathbf{j}+4\mathbf{k}$	1A
∴ The required equation is $4(x+1)-12(y-3)+4(z-2)=0$	1M
$x-3y+z+8=0$.	1A
(iv) Let the required angle be θ, then $\cos\theta=\dfrac{(5)(1)+(-7)(-3)+(13)(1)}{\sqrt{5^2+(-7)^2+13^2}\sqrt{1^2+(-3)^2+1^2}}$	2M
$\theta=\cos^{-1}\dfrac{39}{\sqrt{243}\cdot\sqrt{11}}$	
$=\cos^{-1}\dfrac{13\sqrt{33}}{99}$ (0.716 or 41.03°)	1A accept 0.7 or 41°
(v) Volume $=\lvert\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\rvert$	1M
$=\lvert(\mathbf{j}+3\mathbf{k})\cdot(-5\mathbf{i}+7\mathbf{j}-13\mathbf{k})\rvert$	1M for substitution
$=\lvert 7-39\rvert$	
$=32$	1A

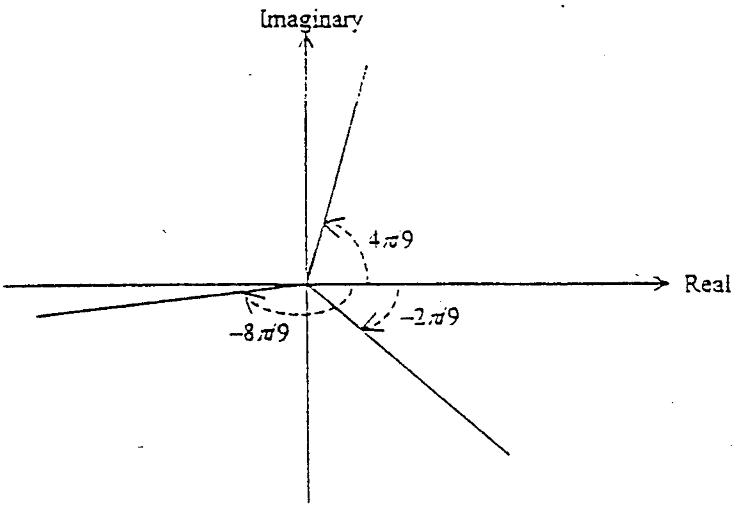
Solution	Marks
11. (a) (i) Let $f(x) = x^3 - 3px + 2q$.	
If (*) has a repeated root, then $f(x) = 0$ and $f'(x) = 0$ has a common root.	1M
$f'(x) = 3x^2 - 3p$	
$f'(x) = 0 \Rightarrow x^2 = p$	1A
$x = \pm\sqrt{p}$	
If $x = \sqrt{p}$ is the common root, then	
$\left(\sqrt{p}\right)^3 - 3p\left(\sqrt{p}\right) + 2q = 0$	
$\left(\sqrt{p}\right)^3 = q$	
$p^3 = q^2$	
If $x = -\sqrt{p}$ is the common root, then	
$\left(-\sqrt{p}\right)^3 - 3p\left(-\sqrt{p}\right) + 2q = 0$	
$\left(\sqrt{p}\right)^3 = -q$	
$p^3 = q^2$	1
<u>Alternatively</u>,	
$f'(x) = 0 \Rightarrow x^2 = p$ ....(1)	1A
$f(x) = 0 \Rightarrow x^2(x^2 - 3p)^2 = 4q^2$ ....(2)	
(*) has a repeated root iff $f(x) = 0$ and $f'(x) = 0$ has a common root.	1M
Sub. (1) into (2), $p(p - 3p)^2 = 4q^2$	
$p^3 = q^2$	1
(ii) If $q = \sqrt{p^3}$, $f\left(\sqrt{p}\right) = \left(\sqrt{p}\right)^3 - 3p\left(\sqrt{p}\right) + 2\left(\sqrt{p^3}\right) = 0$	1A
$\sqrt{p}$ satisfies both $f(x) = 0$ and $f'(x) = 0$	
$\Rightarrow$ $\sqrt{p}$ is a repeated root of (*).	1
(iii) If $q = -\sqrt{p^3}$, $f\left(-\sqrt{p}\right) = \left(-\sqrt{p}\right)^3 - 3p\left(-\sqrt{p}\right) - 2\left(\sqrt{p^3}\right) = 0$	1A+1A
$-\sqrt{p}$ satisfies both $f(x) = 0$ and $f'(x) = 0$	
$\Rightarrow$ (*) has a repeated root and the root is $-\sqrt{p}$.	1
(b) (i) Let $x = y - h$, then (**) becomes	
$2(y - h)^3 + 3(y - h)^2 + (y - h) + c = 0$	
$2y^3 + (-6h + 3)y^2 + (6h^2 - 6h + 1)y + (-2h^3 + 3h^2 - h + c) = 0$	
Put $h = \frac{1}{2}$, (**) becomes	1A
$2y^3 - \frac{1}{2}y + c = 0$	
$y^3 - 3\left(\frac{1}{12}\right)y + 2\left(\frac{c}{4}\right) = 0$ ....(3)	1A

Solution	Marks
(ii) Using the results of (a)(ii) and (iii), (3) and hence (**) has a repeated root if	
$\left(\frac{1}{12}\right)^3 = \left(\frac{c}{4}\right)^2$	1M
$c^2 = \frac{1}{108}$	
$c = \frac{1}{6\sqrt{3}}$ $\quad (\because\ c > 0)$	1A
Consider (3) when $c = \frac{1}{6\sqrt{3}}$,	
$y^3 - 3\left(\frac{1}{12}\right)y + 2\left(\frac{1}{24\sqrt{3}}\right) = 0$ ....(4)	
Taking $p = \frac{1}{12}$ and $q = \frac{1}{24\sqrt{3}}$ in (a),	
$\because\ q = \sqrt{p^3}$, $\sqrt{p} = \sqrt{\frac{1}{12}}$ is a repeated root of (4).	1M
$\therefore$ (4) becomes $\left(y + \frac{1}{\sqrt{3}}\right)\left(y - \frac{1}{2\sqrt{3}}\right)^2 = 0$	
$y = -\frac{1}{\sqrt{3}}$ or $\frac{1}{2\sqrt{3}}$ (repeated)	1A
For the roots of (**),	
$x = y - \frac{1}{2}$	
$x = -\frac{1}{2} - \frac{1}{\sqrt{3}}$ or $-\frac{1}{2} + \frac{1}{2\sqrt{3}}$ (repeated)	1A

Solution	Marks
12. (a) (i) $p = \frac{\alpha+\beta}{2}$, $q = \alpha\beta$	1A
Since $\frac{\alpha+\beta}{2} \geq \sqrt{\alpha\beta}$	
$\therefore$ $p^2 \geq q$ or $p^2 - q = \frac{1}{4}(\alpha-\beta)^2 \geq 0$	1
Alternatively, $\because$ $x^2 - 2px + q = 0$ has two real roots $\therefore$ $\Delta = 4p^2 - 4q \geq 0$ $\Rightarrow$ $p^2 \geq q$	2
(ii) $b = \frac{\alpha+\beta+\gamma}{3} = \frac{1}{3}(2p+\gamma)$	1A
$c = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{3} = \frac{1}{3}(q+2p\gamma)$	1A
$d = \alpha\beta\gamma = q\gamma$	1A
$b^2 - c = \frac{1}{9}(2p+\gamma)^2 - \frac{1}{3}(q+2p\gamma)$ $= \frac{1}{9}(4p^2 + 4p\gamma + \gamma^2 - 3q - 6p\gamma)$ $= \frac{1}{9}[(p-\gamma)^2 + 3(p^2-q)] \geq 0$	1A
$c^2 - bd = \frac{1}{9}(q+2p\gamma)^2 - \frac{1}{3}(2p+\gamma)q\gamma$ $= \frac{1}{9}(q^2 + 4pq\gamma + 4p^2\gamma^2 - 6pq\gamma - 3q\gamma^2)$ $= \frac{1}{9}[(q-p\gamma)^2 + 3\gamma(p^2-q)] \geq 0$	1A
$\because$ $\alpha, \beta, \gamma > 0$, $\therefore$ $c > 0$.	1
$\because$ $b^2 \geq c$, $\therefore$ $b \geq \sqrt{c}$ $\because$ $c^2 \geq bd$, $\therefore$ $c^2 \geq \sqrt{c}d$	1
Hence $\left(\sqrt{c}\right)^3 \geq d$ $\sqrt{c} \geq \sqrt[3]{d}$	1

Solution	Marks
(b) $\tan\frac{C}{2} = \tan\left[\frac{\pi}{2}-\left(\frac{A}{2}+\frac{B}{2}\right)\right]$	1M
$= \dfrac{1}{\tan\left(\frac{A}{2}+\frac{B}{2}\right)}$	
$= \dfrac{1-\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$	
$\therefore\quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$	1
Since $0 < A, B, C < \pi$.	
$\therefore\quad \tan\frac{A}{2},\ \tan\frac{B}{2},\ \tan\frac{C}{2} > 0$	
Putting $\alpha=\tan\frac{A}{2}$, $\beta=\tan\frac{B}{2}$, $\gamma=\tan\frac{C}{2}$ in (a),	1M
$\because\quad b \ge \sqrt{c} \ge \sqrt[3]{d}$	
$\therefore\quad \dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}{3} \ge \sqrt{\dfrac{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}}{3}}$	
$\ge \sqrt[3]{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$	1
$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2} \ge 3\sqrt{\frac{1}{3}} = \sqrt{3}$	
and $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \le \left(\sqrt{\frac{1}{3}}\right)^3 = \frac{\sqrt{3}}{9}$	1

Solution	Marks
13. (a) If $r = 0$, then $\dfrac{r + \cos\theta + i\sin\theta}{1 + r\cos\theta - ir\sin\theta} = \cos\theta + i\sin\theta$.	1A
If $r \neq 0$, then $\dfrac{r + \cos\theta + i\sin\theta}{1 + r\cos\theta - ir\sin\theta} = \dfrac{1}{r}\left(\dfrac{r^2 + r\cos\theta + ir\sin\theta}{1 + r\cos\theta - ir\sin\theta}\right)$	1A
$= \dfrac{1}{r}\left(\dfrac{z\bar{z} + z}{1 + \bar{z}}\right)$	1A
$= \dfrac{1}{r} \cdot \dfrac{z(1 + \bar{z})}{1 + \bar{z}}$	
$= \dfrac{z}{r}$	
$= \cos\theta + i\sin\theta$	1A
Alternatively,	
$\dfrac{r + \cos\theta + i\sin\theta}{1 + r\cos\theta - ir\sin\theta}$	
$= \dfrac{(r + \cos\theta + i\sin\theta)(1 + r\cos\theta + ir\sin\theta)}{(1 + r\cos\theta)^2 + r^2\sin^2\theta}$	1A+1A
$= \dfrac{(r + \cos\theta + r^2\cos\theta + r\cos^2\theta - r\sin^2\theta) + i\sin\theta(1 + 2r\cos\theta + r^2)}{1 + 2r\cos\theta + r^2}$	1A
$= \dfrac{\cos\theta(1 + 2r\cos\theta + r^2) + i\sin\theta(1 + 2r\cos\theta + r^2)}{1 + 2r\cos\theta + r^2}$	
$= \cos\theta + i\sin\theta$	1A
(b) $\left(\dfrac{r + \sin\theta + i\cos\theta}{1 + r\sin\theta - ir\cos\theta}\right)^n = \left(\dfrac{r + \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)}{1 + r\cos\left(\frac{\pi}{2} - \theta\right) - ir\sin\left(\frac{\pi}{2} - \theta\right)}\right)^n$	1M
$= \left[\cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)\right]^n$	1A
$= \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$	1

Solution	Marks
(c) $\left(\dfrac{r+\sin\theta+i\cos\theta}{1+r\sin\theta-ir\cos\theta}\right)^3=\dfrac{\sqrt{3}+i}{2}$	
$\cos\left(\dfrac{3\pi}{2}-3\theta\right)+i\sin\left(\dfrac{3\pi}{2}-3\theta\right)=\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}$ (where $r\geq0$)	1M+1A
$\dfrac{3\pi}{2}-3\theta=2k\pi+\dfrac{\pi}{6}$ $(k\in\mathbb{Z})$	1M
$3\theta=-2k\pi+\dfrac{4}{3}\pi$ $(k\in\mathbb{Z})$	
$\therefore\ \theta=\dfrac{2}{3}h\pi+\dfrac{4}{9}\pi$ $(h\in\mathbb{Z})$	1A
and $r\geq0$	1A or for the directed lines below
The points representing $z=r(\cos\theta+i\sin\theta)$ on an Argand diagram are shown below:	
	1A
(d) $\because\ \left\|\left(\dfrac{r+\sin\theta+i\cos\theta}{1+r\sin\theta-ir\cos\theta}\right)^3\right\|=1$ and $\left\|\sqrt{3}+i\right\|=2$	1
$\therefore\ \left(\dfrac{r+\sin\theta+i\cos\theta}{1+r\sin\theta-ir\cos\theta}\right)^3=\sqrt{3}+i$ has no solution.	1
Alternatively,	
$\cos\left(\dfrac{3\pi}{2}-3\theta\right)+i\sin\left(\dfrac{3\pi}{2}-3\theta\right)=\sqrt{3}+i$	
$\cos\left(\dfrac{3\pi}{2}-3\theta\right)=\sqrt{3}>1$	1
$\therefore\ \left(\dfrac{r+\sin\theta+i\cos\theta}{1+r\sin\theta-ir\cos\theta}\right)^3=\sqrt{3}+i$ has no solution.	1

Solution	Marks
1. (a) $\lim\limits_{x\to 0}\dfrac{e^x-1-\sin x}{x^2} = \lim\limits_{x\to 0}\dfrac{e^x-\cos x}{2x}$ (By L'Hospital Rule)	1M+1A
$= \lim\limits_{x\to 0}\dfrac{e^x+\sin x}{2}$ (By L'Hospital Rule)	
$= \dfrac{1}{2}$	1A
(b) Let $y=\left(\dfrac{3e^x+2}{5}\right)^{\frac{1}{x}}$, then $\ln y = \dfrac{1}{x}\ln\left(\dfrac{3e^x+2}{5}\right)$	1A
$\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}\dfrac{5}{3e^x+2}\cdot\dfrac{3}{5}e^x$ (By L'Hospital Rule)	1A
$= \dfrac{3}{5}$	
$\therefore \lim\limits_{x\to 0} y = e^{\frac{3}{5}}$	1A
	(6)
2. (a) $\dfrac{d}{dx}\left(\int_0^{x+T} f(t)dt - \int_0^x f(t)dt\right) = f(x+T) - f(x)$	1M
$= f(x) - f(x)$	
$= 0$	1A
(b) Let $F(x) = \int_0^{x+T} f(t)dt - \int_0^x f(t)dt = \int_x^{x+T} f(t)dt$	
By (a), $F(x) = F(0)$ for all x.	1M for $F(x)$=constant
Hence $\int_x^{x+T} f(t)dt = \int_0^T f(t)dt$ for all x.	1
	(4)

Solution	Marks	
3. $\int \ln(1+x)\,dx$		
$= x\ln(1+x) - \int \frac{x}{1+x}\,dx$	1A	
$= x\ln(1+x) - \int (x - \frac{1}{1+x})\,dx$	1A	
$= x\ln(1+x) - x + \ln(1+x) + c$	1A	
$= (1+x)\ln(1+x) - x + c$		
<u>Alternatively,</u>		
$\int \ln(1-x)\,dx$		
$= \int \ln(1-x)\frac{d(1+x)}{dx}dx$		
$= (1+x)\ln(1+x) - \int (1+x)\frac{d\ln(1+x)}{dx}dx$	1A	
$= (1+x)\ln(1+x) - x + c$	2A	
$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln(1+\frac{k}{n})$		
$= \int_0^1 \ln(1+x)\,dx$	1A	
$= \left[(1+x)\ln(1+x) - x\right]_0^1$		
$= 2\ln 2 - 1$	1A	
	(5)	
4. (a) Let $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z}{2} = t$ and sub. $(2t+1, -t+2, 2t)$ into the equation of π,		
then $(2t+1)+(-t+2)+(2t) = 0$	1M+1A	
$t = -1$		
$\therefore$ The intersection point is $(-1, 3, -2)$.	1A	
<u>Alternatively,</u>		
L can be written as $\begin{cases} x+2y=5 \\ 2y+z=4 \end{cases}$	1A	for writing as 2 linear equations
Sub. into the equation of π, $(5-2y)+y+(4-2y)=0$	1M	
$y = 3$		
$\therefore$ The intersection point is $(-1, 3, -2)$.	1A	
(b) Let the angle between L and π be θ.		
$\cos(\frac{\pi}{2}-\theta) = \frac{(2)(1)+(-1)(1)+(2)(1)}{\sqrt{2^2+(-1)^2+2^2}\cdot\sqrt{1^2+1^2+1^2}}$	1M+1A	1M for identifying the angle
$= \frac{1}{\sqrt{3}}$		
$\therefore \theta = \frac{\pi}{2} - \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (0.6155 or 35.26°)	1A	
	(6)	

Solution	Marks
5. (a) $\because$ Shaded area $= \int_a^b \frac{1}{x}dx$	1A
$\therefore \int_a^b \frac{1}{x}dx < \frac{1}{2}(b-a)(\frac{1}{a}+\frac{1}{b})$	
$\ln b - \ln a < \frac{1}{2}(b-a)(\frac{1}{a}+\frac{1}{b})$	1
(b) From (a), $\ln 2 - \ln 1 < \frac{1}{2}(1+\frac{1}{2})$ $\ln 3 - \ln 2 < \frac{1}{2}(\frac{1}{2}+\frac{1}{3})$ $\vdots$ $\ln n - \ln(n-1) < \frac{1}{2}(\frac{1}{n-1}+\frac{1}{n})$	1M
$\therefore \sum_{k=2}^{n}[\ln k - \ln(k-1)] < \sum_{k=2}^{n}\frac{1}{2}(\frac{1}{k-1}+\frac{1}{k})$	
$\ln n - \ln 1 < \frac{1}{2}+(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1})+\frac{1}{2n}$	
$= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{2}-\frac{1}{n}+\frac{1}{2n}$	
$= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\frac{n+1}{2n}$	
$\ln n < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\frac{n+1}{2n}$	1
Hence $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > \ln n+\frac{n+1}{2n} > \ln n$	
$\because \ln n \to \infty$ as $n \to \infty$	
$\therefore 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \to \infty$ as $n \to \infty$	
$\Rightarrow \lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})$ does not exist	2
	(6)

Solution	Marks
6. (a) Clearly, a_n, $b_n > 0$ for $n = 1, 2, 3, \ldots$	
and $a_1 > b_1$, $a_1 b_1 = 3 = 2(1)+1$.	1
Assume $a_{k-1} > b_{k-1}$ and $a_{k-1}b_{k-1} = 2(k-1)+1$ for some $k > 1$.	1
$a_k = \frac{2k}{2k-1}a_{k-1}$	
$> (1+\frac{1}{2k-1})b_{k-1}$	
$> (1+\frac{1}{2k})b_{k-1}$	
$= \frac{2k+1}{2k}b_{k-1}$	
$= b_k$	1
$a_k b_k = \frac{2k}{2k-1}\cdot\frac{2k+1}{2k}a_{k-1}b_{k-1}$	
$= \frac{2k+1}{2k-1}[2(k-1)+1]$	
$= 2k+1$	1
By the principle of mathematical induction, the statement holds for $n = 1, 2, 3, \ldots$.	
Alternatively,	
(i) Clearly $a_1 > b_1$.	
Assume $a_{k-1} > b_{k-1}$ for some $k > 1$.	1
$\frac{a_k}{b_k} = \frac{\frac{2k}{2k-1}a_{k-1}}{\frac{2k+1}{2k}b_{k-1}} = \frac{(2k)^2}{(2k)^2-1}\cdot\frac{a_{k-1}}{b_{k-1}} > 1$	1
(or $a_k - b_k = \frac{2k}{2k-1}a_{k-1} - \frac{2k+1}{2k}b_{k-1} > \frac{2k+1}{2k}(a_{k-1} - b_{k-1}) > 0$)	
By the principle of mathematical induction, $a_n > b_n$ for $n = 1, 2, 3, \ldots$.	
(ii) $a_1 b_1 = 3 = 2(1)+1$ and assume $a_{k-1}b_{k-1} = 2(k-1)+1$ for some $k > 1$.	1
$a_k b_k = \frac{2k}{2k-1}\cdot\frac{2k+1}{2k}a_{k-1}b_{k-1} = \frac{2k+1}{2k-1}[2(k-1)+1] = 2k+1$	1
(b) $a_n^2 > a_n b_n = 2n+1$	1
$0 < \frac{1}{a_n} < \frac{1}{\sqrt{2n+1}}$	1A no mark for one side only, accept $\frac{1}{a^2} < \frac{1}{2n+1}$
$\because \lim_{n\to\infty}\frac{1}{\sqrt{2n+1}} = 0$	
$\therefore \lim_{n\to\infty}\frac{1}{a_n} = 0$.	1A
	(7)

Solution	Marks
7. (a) $r = \cos^3 \frac{\theta}{3}$	
$\frac{dr}{d\theta} = 3\left(\cos^2 \frac{\theta}{3}\right)\left(-\sin \frac{\theta}{3}\right)\frac{1}{3}$	
$= -\cos^2 \frac{\theta}{3} \sin \frac{\theta}{3}$	1A independent of the A marks below
$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$	
$= \int \sqrt{\cos^6 \frac{\theta}{3} + \cos^4 \frac{\theta}{3} \sin^2 \frac{\theta}{3}}\, d\theta$	
$= \int \cos^2 \frac{\theta}{3}\, d\theta$	1A
$= \frac{1}{2} \int \left(1 + \cos \frac{2\theta}{3}\right) d\theta$	
$= \frac{1}{2}\left(\theta + \frac{3}{2} \sin \frac{2\theta}{3}\right) + C$	1
(b) $a + b + c = \int_0^{\frac{3\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$	
$= \frac{1}{2}\left[\theta + \frac{3}{2} \sin \frac{2\theta}{3}\right]_0^{\frac{3\pi}{2}}$	
$= \frac{3\pi}{4}$	1A
$b = \frac{1}{2}\left[\theta + \frac{3}{2} \sin \frac{2\theta}{3}\right]_{\frac{\pi}{2}}^{\pi}$	
$= \frac{1}{2}\left[\pi + \frac{3\sqrt{3}}{4} - \frac{\pi}{2} - \frac{3\sqrt{3}}{4}\right]$	
$= \frac{\pi}{4}$	1A
$\therefore \quad a + c = \frac{3\pi}{4} - \frac{\pi}{4} = 2 \cdot \frac{\pi}{4} = 2b$	1
Alternatively,	
$a = \frac{\pi}{4} + \frac{3\sqrt{3}}{8}$	1A
$b = \frac{\pi}{4}$	1A
$c = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}$	1A
$\therefore \quad a + c = 2b$	
	(6)

Solution | Marks

8. (a) (i) $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x+1)^{\frac{2}{3}} + \frac{2}{3}x^{\frac{1}{3}}(x+1)^{-\frac{1}{3}}$

$= \frac{3x+1}{3x^{\frac{2}{3}}(x+1)^{\frac{1}{3}}}$ for $x \neq -1, 0$ 1A

(ii) $f''(x) = \frac{3x^{\frac{2}{3}}(x+1)^{\frac{1}{3}}(3) - 3(3x+1)\left[\frac{2}{3}x^{-\frac{1}{3}}(x+1)^{\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}}(x+1)^{-\frac{2}{3}}\right]}{9x^{\frac{5}{3}}(x+1)^{\frac{4}{3}}}$ 1

$= \frac{9x(x+1) - (3x+1)[2(x+1)+x]}{9x^{\frac{5}{3}}(x+1)^{\frac{4}{3}}}$

$= \frac{-2}{9x^{\frac{5}{3}}(x+1)^{\frac{4}{3}}}$ for $x = -1, 0$

(b) $\because \frac{f(x)-f(-1)}{x-(-1)} = \frac{x^{\frac{1}{3}}(x+1)^{\frac{2}{3}}}{x+1}$

$= (\frac{x}{x+1})^{\frac{1}{3}} \to \pm\infty$ as $x \to -1$

$\therefore$ $f'(-1)$ does not exist. 1

$\because \frac{f(x)-f(0)}{x} = \frac{x^{\frac{1}{3}}(x+1)^{\frac{2}{3}}}{x}$

$= (1+\frac{1}{x})^{\frac{2}{3}} \to \infty$ as $x \to 0$

$\therefore$ $f'(0)$ does not exist. 1

(c) (i) $f'(x) > 0 \Leftrightarrow x < -1$ or $-\frac{1}{3} < x < 0$ or $x > 0$ 1A

$f'(x) > 0$ on $(-\infty, -1) \cup (-\frac{1}{3}, 0) \cup (0, \infty)$

(ii) $f'(x) < 0 \Leftrightarrow -1 < x < -\frac{1}{3}$ 1A

$f'(x) < 0$ on $(-1, -\frac{1}{3})$

(iii) $f''(x) > 0 \Leftrightarrow x < -1$ or $-1 < x < 0$ 1A

$f''(x) > 0$ on $(-\infty, -1) \cup (-1, 0)$

(iv) $f''(x) < 0 \Leftrightarrow x > 0$ 1A

$f''(x) < 0$ on $(0, \infty)$

A maximum of 3 marks could be awarded

Solution	Marks

(d)

x	$(-\infty,-1)$	-1	$(-1,-\frac{1}{3})$	$-\frac{1}{3}$	$(-\frac{1}{3},0)$	0	$(0,\infty)$
$f(x)$	↑	0	↓	$-\frac{2^{\frac{2}{3}}}{3}$	↑	0	↑
$f'(x)$	+	Undefined	−	0	+	Undefined	+
$f''(x)$	+	Undefined	+	+	+	Undefined	−

$(-1, 0)$ is a relative maximum point. 1A

$(-\frac{1}{3}, -\frac{2^{\frac{2}{3}}}{3})$ is a relative minimum point. 1A

$(0, 0)$ is a point of inflexion. 1A

(e) If $y = ax + b$ is an asymptote, then

$$a = \lim_{x\to\infty}\frac{y}{x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\frac{2}{3}} = 1$$ 1A

$$b = \lim_{x\to\infty}[x^{\frac{1}{3}}(x+1)^{\frac{2}{3}} - x]$$

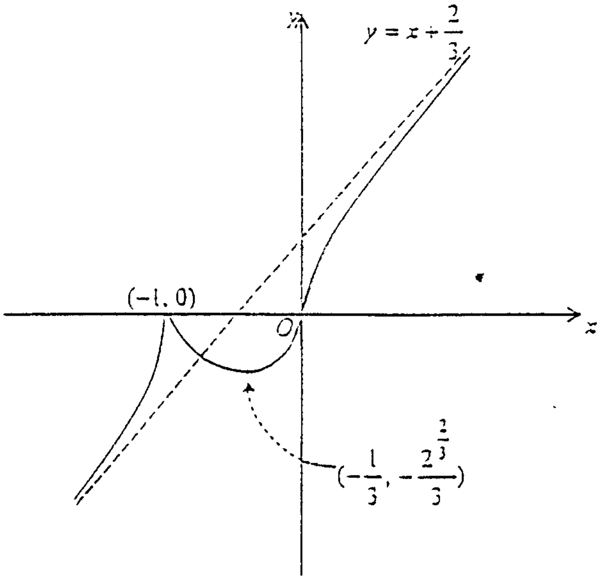
$$= \lim_{x\to\infty}\frac{(1+\frac{1}{x})^{\frac{2}{3}}-1}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{2}{3}(1+\frac{1}{x})^{\frac{1}{3}}$$ (by L'Hospital Rule)

$$= \frac{2}{3}$$ 1A

$\therefore$ $y = x + \frac{3}{2}$ is an asymptote of the graph of $f(x)$.

(f)



1 for max. & min. pts.

1 for the shape of the curve (correct at x=−1 and x=0)

1 for the asymptote

Solution	Marks
9. (a) (i) $I_0 = \int_0^{\frac{\pi}{2}} dt = [t]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$	1A
$I_1 = \int_0^{\frac{\pi}{2}} \cos t \, dt = [\sin t]_0^{\frac{\pi}{2}} = 1$	
(ii) For $m \geq 2$,	
$I_m = \int_0^{\frac{\pi}{2}} \cos^{m-1} t \frac{d \sin t}{dt} dt$	
$= \left[\cos^{m-1} t \sin t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin t \frac{d \cos^{m-1} t}{dt} dt$	1A
$= (m-1) \int_0^{\frac{\pi}{2}} \cos^{m-2} t \sin^2 t \, dt$	
$= (m-1) \int_0^{\frac{\pi}{2}} \cos^{m-2} t (1 - \cos^2 t) \, dt$	1A
$= (m-1)[I_{m-2} - I_m]$	
$\therefore \quad mI_m = (m-1)I_{m-2}$	
$I_m = \frac{m-1}{m} I_{m-2}$	1
For $n \geq 1$,	
$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$	
$\vdots$	
$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \frac{1}{2} I_0$	1M
$= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{\pi}{2}$	1A
$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}$	
$\vdots$	
$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} I_1$	
$= \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$	1A
(b) $0 \leq \cos t \leq 1$ for $0 \leq t \leq \frac{\pi}{2}$	
$\Rightarrow \quad \cos^{2n-1} t \geq \cos^{2n} t \geq \cos^{2n+1} t$ for $n \geq 1$ and $0 \leq t \leq \frac{\pi}{2}$	1
$\Rightarrow \quad \int_0^{\frac{\pi}{2}} \cos^{2n-1} t \, dt \geq \int_0^{\frac{\pi}{2}} \cos^{2n} t \, dt \geq \int_0^{\frac{\pi}{2}} \cos^{2n+1} t \, dt$	
$\Rightarrow \quad I_{2n-1} \geq I_{2n} \geq I_{2n+1}$ for $n \geq 1$	1

Solution	Marks
(c) (i) For $n \ge 1$,	
$I_{2n-1} \ge I_{2n} \ge I_{2n+1}$	
$\Rightarrow \frac{2\cdot4\cdot6\cdots(2n-2)}{3\cdot5\cdot7\cdots(2n-1)} \ge \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\cdot\frac{\pi}{2} \ge \frac{2\cdot4\cdot6\cdots(2n)}{3\cdot5\cdot7\cdots(2n+1)}$	
$\Rightarrow \frac{1}{2n}\left[\frac{2\cdot4\cdot6\cdots(2n)}{1\cdot3\cdot5\cdots(2n-1)}\right]^2 \ge \frac{\pi}{2} \ge \frac{1}{2n+1}\left[\frac{2\cdot4\cdot6\cdots(2n)}{1\cdot3\cdot5\cdots(2n-1)}\right]^2$	
$\Rightarrow \frac{2n+1}{2n}A_n \ge \frac{\pi}{2} \ge A_n$	1
(ii) For $n \ge 1$,	
$\frac{A_n}{A_{n-1}} = \frac{1}{2n+1}\left[\frac{2\cdot4\cdot6\cdots(2n)}{1\cdot3\cdot5\cdots(2n-1)}\right]^2\cdot(2n-1)\left[\frac{1\cdot3\cdot5\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n-2)}\right]^2$	1M
$= \frac{2n-1}{2n+1}\cdot\left[\frac{2n}{2n-1}\right]^2$	
$= \frac{4n^2}{4n^2-1} > 1$	1
$\therefore$ $\{A_n\}$ is monotonic increasing.	
(iii) $\because$ $\{A_n\}$ is monotonic increasing and bounded above by $\frac{\pi}{2}$	
$\therefore$ $\lim\limits_{n\to\infty} A_n$ exists.	1
Let $\lim\limits_{n\to\infty} A_n = \ell$,	
then $\lim\limits_{n\to\infty}\frac{2n+1}{2n}A_n = \lim\limits_{n\to\infty}\frac{2n+1}{2n}\cdot\ell$	
$= \ell$	
Hence $\frac{2n+1}{2n}A_n \ge \frac{\pi}{2} \ge A_n$	
$\Rightarrow \ell \ge \frac{\pi}{2} \ge \ell$	1
$\Rightarrow \ell = \frac{\pi}{2}$	
$\because$ $A_n > 0$, $\therefore$ $\lim\limits_{n\to\infty}\sqrt{A_n}$ exists and	
$\lim\limits_{n\to\infty}\sqrt{A_n} = \sqrt{\frac{\pi}{2}}$.	1A
Alternatively,	
$\because$ $\frac{2n+1}{2n}A_n \ge \frac{\pi}{2} \ge A_n$ for $n \ge 1$	
$\therefore$ $\sqrt{\frac{\pi}{2}}\sqrt{\frac{2n}{2n+1}} \le \sqrt{A_n} \le \sqrt{\frac{\pi}{2}}$ for $n \ge 1$	1
As $\lim\limits_{n\to\infty}\sqrt{\frac{\pi}{2}}\sqrt{\frac{2n}{2n+1}} = \sqrt{\frac{\pi}{2}}$,	1
by the Sandwich theorem,	
$\lim\limits_{n\to\infty}\sqrt{A_n}$ exists and $\lim\limits_{n\to\infty}\sqrt{A_n} = \sqrt{\frac{\pi}{2}}$.	1

Solution	Marks
10. (a) (i) Let $u = t + b$, then	
$\int_0^a f(t+b)\,dt = \int_b^{a+b} f(u)\,du$	1A
$= \int_0^{a+b} f(t)\,dt - \int_0^b f(t)\,dt$ for $a, b \in R$	1
(ii) $\int_0^x f(t+1)\,dt = \int_0^x [f(1) + f(t)]\,dt$	1A
$= f(1)\int_0^x dt + \int_0^x f(t)\,dt$	
$= x\,f(1) + \int_0^x f(t)\,dt$ for $x \in R$	1
$x\,f(1) = \int_0^x f(t+1)\,dt - \int_0^x f(t)\,dt$	
$= \int_0^{x+1} f(t)\,dt - \int_0^1 f(t)\,dt - \int_0^x f(t)\,dt$ [putting $a = x$, $b = 1$ in (i)]	1A
$= \int_0^1 f(t+x)\,dt - \int_0^1 f(t)\,dt$ [putting $a = 1$, $b = x$ in (i)]	1M+1A
$= \int_0^1 [f(t+x) - f(t)]\,dt$	
$= f(x)\int_0^1 dt$	
$= f(x)$ for $x \in R$	1
(b) Let $f(t) = g(e^t)$ for $t \in R$, then f is a continuous function on R and	1
$f(x+y) = g(e^{x+y}) = g(e^x e^y) = g(e^x) + g(e^y) = f(x) + f(y)$	1
For any $x > 0$, $g(x) = g(e^{\ln x})$	1A
$= f(\ln x)$	
$= f(1)\ln x$ by (a)	1
$= g(e)\ln x$	1A
If $g(e) = 0$, then $g(x) = 0$ for $x > 0$ which contradicts that $g(x)$ is non-constant.	
$\therefore$ $g(e) = 0$	1
Putting $a = e^{\frac{1}{g(e)}} > 0$, then $\ln a = \frac{1}{g(e)}$	
$\Rightarrow g(x) = \frac{\ln x}{\ln a}$	
$= \log_a x$ for $x > 0$	1

Solution	Marks
11. (a) $F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ for $x \in [a, b]$	1M for considering the derivative of F
$\therefore$ $F'(x) \ge 0$ on $[a, b]$	
$\Rightarrow$ $F(x)$ is increasing on $[a, b]$	1 for $F'(x) \ge 0$ or $f(x) \ge 0$
If $\int_a^b f(t)\,dt = 0$, then $F(a) = F(b) = 0$.	1A
Since F is increasing on $[a, b]$, $0 = F(a) \le F(x) \le F(b) = 0$ $\Rightarrow$ $F(x) = 0$ for all $x \in [a, b]$	1 for F is constant
$\therefore$ $f(x) = F'(x) = 0$ for all $x \in [a, b]$	1 for $F' = f$
(b) Let $u(x) = g(x)$ for all $x \in [a, b]$, then	1M
$\int_a^b g^2(t)\,dt = \int_a^b g(t)u(t)\,dt = 0$ and g^2 is non-negative	1
From (a), $g^2(x) = 0$ for all $x \in [a, b]$	
$\Rightarrow$ $g(x) = 0$ for all $x \in [a, b]$	1
(c) (i) $\int_a^b v(t)\,dt = \int_a^b [h(x) - A]\,dx$	
$= \int_a^b h(x)\,dx - (b-a)A$	1A integration of a constant function
$= \int_a^b h(x)\,dx - (b-a)\frac{1}{b-a}\int_a^b h(t)\,dt$	
$= 0$	1
(ii) Choosing $w(x) = h(x) - A$ for all $x \in [a, b]$, then	
$\int_a^b w(x)\,dx = 0$ (by c(i))	1 for letting $w(x)=h(x)-A=v(x)$
Hence $\int_a^b h(x)w(x)\,dx = 0$	
$\Rightarrow$ $\int_a^b (w(x) + A)\,w(x)\,dx = 0$	1M
$\Rightarrow$ $\int_a^b w^2(x)\,dx + A\int_a^b w(x)\,dx = 0$	
$\Rightarrow$ $\int_a^b w^2(x)\,dx = 0$ ($\because$ $\int_a^b w(x)\,dx = 0$)	1A
$\Rightarrow$ $w^2(x) = 0$ for all $x \in [a, b]$ (by (a))	can be omitted
$\Rightarrow$ $w(x) = 0$ for all $x \in [a, b]$	1A
$\Rightarrow$ $h(x) - A = 0$ for all $x \in [a, b]$	
$\Rightarrow$ $h(x) = \frac{1}{b-a}\int_a^b h(t)\,dt$ for all $x \in [a, b]$	1

Solution | Marks

12. (a) In Figure 3, draw a horizontal line passing thru' Q and a vertical line passing thru' R. Let the lines intersect at S and α be the inclination of L to the x-axis. 1M

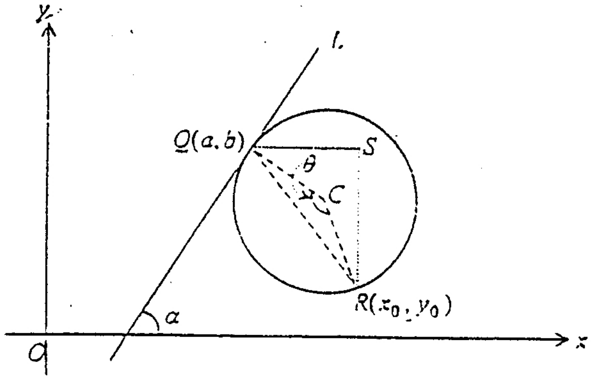


Figure 3

$$\angle SQR = \angle SQC + \angle CQR$$
$$= (\frac{\pi}{2}-\alpha)+(\frac{\pi}{2}-\frac{\theta}{2})$$
$$= \pi-\alpha-\frac{\theta}{2}$$ 1A

$$QR = 2\sin\frac{\theta}{2}$$

$\because \tan\alpha = m,\quad \therefore \sin\alpha = \frac{m}{\sqrt{1+m^2}}$ and $\cos\alpha = \frac{1}{\sqrt{1+m^2}}$ 1A

Hence
$$x_0 = a+QS$$
$$= a+QR\cos\angle SQR$$ 1M
$$= a+2\sin\frac{\theta}{2}\cos(\pi-\alpha-\frac{\theta}{2})$$
$$= a-2\sin\frac{\theta}{2}\cos(\alpha+\frac{\theta}{2})$$
$$= a-2\sin\frac{\theta}{2}[\cos\alpha\cos\frac{\theta}{2}-\sin\alpha\sin\frac{\theta}{2}]$$
$$= a-\frac{2}{\sqrt{1+m^2}}\sin\frac{\theta}{2}(\cos\frac{\theta}{2}-m\sin\frac{\theta}{2})$$ 1

$$y_0 = b-RS$$
$$= b-QR\sin\angle SQR$$ 1M
$$= b-2\sin\frac{\theta}{2}\sin(\pi-\alpha-\frac{\theta}{2})$$
$$= b-2\sin\frac{\theta}{2}\sin(\alpha+\frac{\theta}{2})$$
$$= b-2\sin\frac{\theta}{2}[\sin\alpha\cos\frac{\theta}{2}+\cos\alpha\sin\frac{\theta}{2}]$$
$$= b-\frac{2}{\sqrt{1+m^2}}\sin\frac{\theta}{2}(m\cos\frac{\theta}{2}+\sin\frac{\theta}{2})$$ 1

Solution	Marks
(b) (i) $y=\frac{2}{3}(x-1)^{\frac{3}{2}}$ ($x \geq 1$)	
$\frac{dy}{dx}=(x-1)^{\frac{1}{2}}$	
∴ The slope of tangent of Γ at $x=a$ is $(a-1)^{\frac{1}{2}}$.	1A
(ii) The length of arc of Γ from $x=1$ to $x=a$	
$=\int_1^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$	1M
$=\int_1^a \sqrt{x}\,dx$	1A
$=\frac{2}{3}[x^{\frac{3}{2}}]_1^a$	
$=\frac{2}{3}(a^{\frac{3}{2}}-1)$	1A
(iii) Using the results in (a) and let L be the tangent to Γ at $x=4$, then	
$m=(4-1)^{\frac{1}{2}}=\sqrt{3}$ by (i)	1A
$\theta=\frac{2}{3}(4^{\frac{3}{2}}-1)$ by (ii)	1M
$=\frac{14}{3}$	1A
∴ x-coordinate of P	
$=4-\sin\frac{7}{3}(\cos\frac{7}{3}-\sqrt{3}\sin\frac{7}{3})$	1A

Solution	Marks
13. (a) Let $f(x) = x - g(x)$, then f is continuous on I.	
$\because$ $f(0) = -\frac{2}{3} < 0$ and $f(\frac{\pi}{3}) = \frac{\pi}{3} - \frac{11}{24} > 0$	
$\therefore$ $x = g(x)$ has a root in I.	1
$f'(x) = 1 - g'(x)$	
$= 1 - [-\sin x + \cos^2 x \sin x]$	
$= 1 + \sin^3 x$	1A
> 0 for $x \in I$	
$\therefore$ $f(x)$ is strictly increasing on I.	
Hence $x = g(x)$ has exactly one root in I.	1
(b) $g'(x) = -\sin^3 x \le 0$ for $x \in I$	
$\therefore$ $g(x)$ is decreasing on I.	1
= $g(0) \ge g(x) \ge g(\frac{\pi}{3})$ for $x \in I$	
$\frac{2}{3} \ge g(x) \ge \frac{11}{24}$ for $x \in I$	
$g(x) \in \left[\frac{11}{24}, \frac{2}{3}\right] \subset I$ for $x \in I$	1
$\because$ $x_0 \in I$ and $x_{k+1} = g(x_k) \in I$ if $x_k \in I$	
$\therefore$ $x_n \in I$ for all n by Mathematical Induction.	1
(c) For $x \in I$,	
$\lvert g'(x) \rvert = \lvert -\sin^3 x \rvert$	
$\le \sin^3\left(\frac{\pi}{3}\right)$	1A
$= \frac{3\sqrt{3}}{8}$	
$\le \frac{3}{4}$	1

Solution	Marks
(d) (i) $\because$ α is the root of $x = g(x)$	
$\therefore$ $g(\alpha) = \alpha$	1A
$\lvert x_n - \alpha \rvert = \lvert g(x_{n-1}) - g(\alpha) \rvert$	
$= \lvert g'(\xi) \rvert \lvert x_{n-1} - \alpha \rvert$ for some ξ lies between x_{n-1} and α by the Mean Value Theorem	1A with reason
$\le \frac{3}{4} \lvert x_{n-1} - \alpha \rvert$	1
(ii) $\lvert x_n - \alpha \rvert \le \frac{3}{4} \lvert x_{n-1} - \alpha \rvert$	
$\le \left(\frac{3}{4}\right)^2 \lvert x_{n-2} - \alpha \rvert$	1M can be awarded in (iii) for the same method
$\vdots$	
$\le \left(\frac{3}{4}\right)^n \lvert x_0 - \alpha \rvert$	
$\because$ $\lim_{n \to \infty} \left(\frac{3}{4}\right)^n \lvert x_0 - \alpha \rvert = 0$	1
$\therefore$ $\{x_n\}$ converges and $\lim_{n \to \infty} x_n = \alpha$.	1
(iii) $\lvert x_n - \alpha \rvert \le \left(\frac{3}{4}\right)^n \left\lvert \frac{\pi}{6} - \alpha \right\rvert < \left(\frac{3}{4}\right)^n \frac{\pi}{6}$	
Take $n = 14$ (or above), then	
$\lvert x_{14} - \alpha \rvert \le \left(\frac{3}{4}\right)^{14} \frac{\pi}{6} < \frac{1}{100}$	1