FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin A + \sin B = 2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\sin A - \sin B = 2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$\cos A + \cos B = 2 \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$\cos A - \cos B = -2 \sin \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C)1 answer book.

1. Consider the system of linear equations

$$(*):\begin{cases} 2x + y + 2z = 0 \\ x + (k+1)z = 0 \\ kx - y + 4z = 0 \end{cases}.$$

Suppose (*) has infinitely many solutions.

(a) Find k.

(b) Solve (*).

(6 marks)

2. If $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ for any $(x, y) \in \mathbf{R}^2$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be the matrix representation of the transformation which transforms (x, y) to (x', y').
Find the matrix representation of

(a) the transformation which transforms any point (x, y) to $(-x, y)$,

(b) the transformation which transforms any point (x, y) to (y, x).

(4 marks)

Go on to the next page

3. Let $\mathbf{m}$, $\mathbf{n}$ be two vectors, where $|\mathbf{m}| = 2$, $|\mathbf{n}| = 1$ and the angle between them is $\frac{2\pi}{3}$. If $\mathbf{p} = 3\mathbf{m} + 4\mathbf{n}$ and $\mathbf{q} = 2\mathbf{m} - \mathbf{n}$, find

(a) $\mathbf{m} \cdot \mathbf{n}$,

(b) $|\mathbf{p}|$ and $|\mathbf{q}|$,

(c) the area of the parallelogram with $\mathbf{p}$ and $\mathbf{q}$ as two adjacent sides.

(7 marks)

4. Let z be a complex number satisfying $2|z - 2i| = |z + i|$.

(a) Show that the locus of z on an Argand diagram is a circle. Find its centre and radius.

(b) Let $S = \{z \in \mathbf{C} : 2|z - 2i| \le |z + i|\}$. Draw and shade the region which represents S on an Argand diagram.

Hence find $z_0 \in S$ such that $|z_0| \le |z|$ for all $z \in S$.

(6 marks)

5. Let α, β be the roots of $x^2 - 14x + 36 = 0$. Show that $\alpha^n + \beta^n$ is divisible by 2^n for $n = 1, 2, 3, \dots$.

(5 marks)

6. Suppose $0 < p < 1$.

(a) Let $f(x) = x^p - px + p - 1$ for $x > 0$. Find the absolute maximum value of $f(x)$.

(b) Show that for all positive numbers a and b, $a^p b^{1-p} \le pa + (1-p)b$.

(6 marks)

7. It is given that $f(x) = 2x^4 + x^3 + 10x^2 + 2x + 15$ and $g(x) = x^3 + 2x - 3$. Let $d(x)$ be the H.C.F. of $f(x)$ and $g(x)$.

(a) Using Euclidean Algorithm, or otherwise, find $d(x)$.

(b) Find polynomials $u(x)$ and $v(x)$ of degree ≤ 1 such that $u(x)f(x) + v(x)g(x) = d(x)$ for all x.

(6 marks)

Go on to the next page

SECTION B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Write your answers in the AL(C)2 answer book.

8. Consider the system of linear equations

$$(E):\begin{cases} ax + y + bz = 1 \\ x + ay + bz = 1 \\ x + y + abz = b \end{cases}.$$

(a) Show that (E) has a unique solution if and only if $a \neq -2$, $a \neq 1$ and $b \neq 0$. Solve (E) in this case.

(7 marks)

(b) For each of the following cases, determine the value(s) of b for which (E) is consistent. Solve (E) in each case.

(i) $a = -2$,

(ii) $a = 1$.

(6 marks)

(c) Determine whether (E) is consistent or not for $b = 0$.

(2 marks)

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbf{R}$, $a \neq 0$ and $\det A = 0$.

(a) Show that $A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$ for some $k \in \mathbf{R}$.

(3 marks)

(b) Find P in the form of $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ such that $PA = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}$ for some $\alpha, \beta \in \mathbf{R}$.

If $a + d \neq 0$, find Q in the form of $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ such that $PAP^{-1}Q = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ for some $\gamma \in \mathbf{R}$.

(5 marks)

(c) Find S such that $S\begin{pmatrix} 3 & 7 \\ 6 & 14 \end{pmatrix}S^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ for some $\lambda \in \mathbf{R}$.

Hence, or otherwise, evaluate $\begin{pmatrix} 3 & 7 \\ 6 & 14 \end{pmatrix}^n$ where n is a positive integer.

(7 marks)

Go on to the next page

10. Let $\mathbf{a} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{j} + 3\mathbf{k}$.

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.

(3 marks)

(b) Figure 1 shows the parallelepiped $AOBDECFG$ formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Find

(i) $\overrightarrow{OG}$,

(ii) the Cartesian equation of the plane containing $AOBD$,

(iii) the Cartesian equation of the plane containing $AEGD$,

(iv) the acute angle between the planes mentioned in (ii) and (iii), and

(v) the volume of the parallelepiped $AOBDECFG$.

(12 marks)

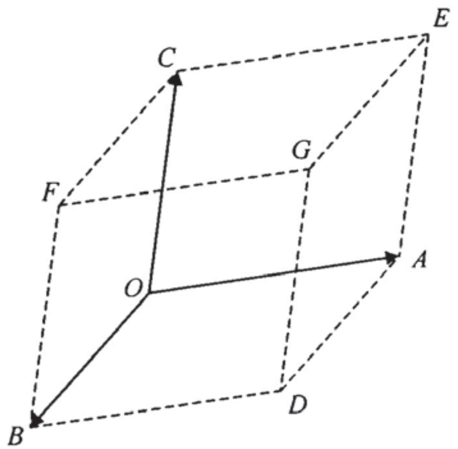

Figure 1

11. Consider the equation

$$x^3 - 3px + 2q = 0 \quad \text{.........} \quad (*) ,$$

where p , q are real numbers.

(a) (i) If (*) has a repeated root, show that $p^3 = q^2$.

(ii) If $q = \sqrt{p^3}$, show that $\sqrt{p}$ is a repeated root of (*) .

(iii) If $q = -\sqrt{p^3}$, show that (*) has a repeated root.

(8 marks)

(b) Consider the equation

$$2x^3 + 3x^2 + x + c = 0 \quad \text{.........} \quad (**) ,$$

where c is a real number.

(i) Transform (**) into the form $y^3 - 3py + 2q = 0$ by using the substitution $x = y - h$ for some constant h .

(ii) Find $c > 0$ such that (**) has a repeated root.

Solve (**) for this value of c .

(7 marks)

Go on to the next page

12. (a) Let α, β and γ be positive numbers. Suppose

$(x-\alpha)(x-\beta)=x^2-2px+q$ and

$(x-\alpha)(x-\beta)(x-\gamma)=x^3-3bx^2+3cx-d$ for all x.

(i) Show that $p^2 \geq q$.

(ii) By expressing b, c and d in terms of γ, p and q, or otherwise, show that $b^2 \geq c > 0$ and $c^2 \geq bd$.

Hence, or otherwise, show that $b \geq \sqrt{c} \geq \sqrt[3]{d}$.

(10 marks)

(b) Let A, B, C be the angles of a triangle. Show that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1 .$$

Using (a), or otherwise, show that

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geq\sqrt{3} \quad \text{and} \quad \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\leq\frac{\sqrt{3}}{9} .$$

(5 marks)

13. Let r and θ be real numbers.

(a) By considering $z=r(\cos\theta+i\sin\theta)$, or otherwise, simplify

$$\frac{r+\cos\theta+i\sin\theta}{1+r\cos\theta-ir\sin\theta} .$$

(4 marks)

(b) For any positive integer n, show that

$$\left(\frac{r+\sin\theta+i\cos\theta}{1+r\sin\theta-ir\cos\theta}\right)^n=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right).$$

(3 marks)

(c) Find r and θ, with $r \geq 0$, such that

$\left(\frac{r+\sin\theta+i\cos\theta}{1+r\sin\theta-ir\cos\theta}\right)^3=\frac{\sqrt{3}+i}{2}$. For such r and θ, sketch the points representing $z=r(\cos\theta+i\sin\theta)$ on an Argand diagram.

(6 marks)

(d) Determine with reasons whether there exist r and θ, with $r \geq 0$, such that $\left(\frac{r+\sin\theta+i\cos\theta}{1+r\sin\theta-ir\cos\theta}\right)^3=\sqrt{3}+i$.

(2 marks)

END OF PAPER

FORMULAS FOR REFERENCE

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$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \pm \tan A \tan B}$

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$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C)1 answer book.

1. Evaluate

   (a) $\lim_{x \to 0} \dfrac{e^x - 1 - \sin x}{x^2}$,

   (b) $\lim_{x \to 0} \left(\dfrac{3e^x + 2}{5} \right)^{\frac{1}{x}}$.

(6 marks)

2. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous periodic function with period T.

   (a) Evaluate $\dfrac{d}{dx}\left(\int_0^{x+T} f(t)dt - \int_0^x f(t)dt \right)$.

   (b) Using (a), or otherwise, show that $\int_x^{x+T} f(t)dt = \int_0^T f(t)dt$ for all x .

(4 marks)

3. Evaluate $\int \ln(1+x)\, dx$.

   Hence, or otherwise, find $\lim_{n \to \infty} \sum_{k=1}^{n} \dfrac{1}{n} \ln(1 + \dfrac{k}{n})$.

(5 marks)

Go on to the next page

4. Consider the line

$$L: \frac{x-1}{2} = \frac{y-2}{-1} = \frac{z}{2}$$

and the plane

$$\pi: x+y+z=0 .$$

(a) Find the coordinates of the point where L intersects π.

(b) Find the angle between L and π.

(6 marks)

5.

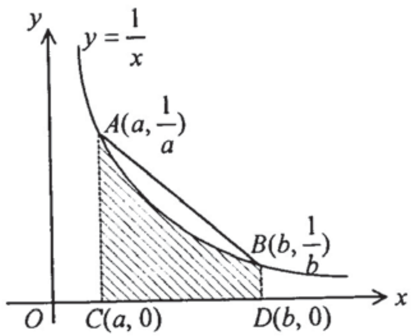

Figure 1

(a) In Figure 1, using the fact that the shaded area is less than the area of the trapezium $ACDB$, or otherwise, show that

$$\ln b - \ln a < \frac{1}{2}(b-a)\left(\frac{1}{a}+\frac{1}{b}\right) .$$

(b) Using (a), or otherwise, show that $\ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{n+1}{2n}$ for any positive integer n.

State with reasons whether $\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$ exists.

(6 marks)

6. Let $a_1 = 2$, $b_1 = \frac{3}{2}$ and $a_n = \frac{2n}{2n-1}a_{n-1}$, $b_n = \frac{2n+1}{2n}b_{n-1}$ for $n \ge 2$.

(a) Prove that $a_n > b_n$ and $a_n b_n = 2n+1$ for $n \ge 1$.

(b) Using (a), or otherwise, show that $a_n{}^2 > 2n+1$ for $n \ge 1$.

Hence find $\lim_{n\to\infty} \frac{1}{a_n}$.

(7 marks)

7.

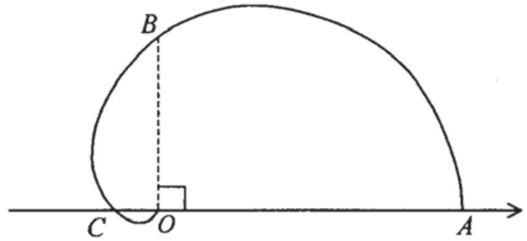

Figure 2

Figure 2 shows the graph of $r = \cos^3\frac{\theta}{3}$ $\left(0 \le \theta \le \frac{3\pi}{2}\right)$.

(a) Evaluate $\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

(b) Let the lengths of $\widehat{AB}$, $\widehat{BC}$ and $\widehat{CO}$ be a, b and c respectively. Show that $a + c = 2b$.

(6 marks)

Go on to the next page

SECTION B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Write your answers in the AL(C)2 answer book.

8. Let $f(x) = x^{\frac{1}{3}}(x+1)^{\frac{2}{3}}$.

(a) (i) Find $f'(x)$ for $x \neq -1, 0$.

(ii) Show that $f''(x) = \dfrac{-2}{9x^{\frac{5}{3}}(x+1)^{\frac{4}{3}}}$ for $x \neq -1, 0$.

(2 marks)

(b) Determine with reasons whether $f'(-1)$ and $f'(0)$ exist or not.

(2 marks)

(c) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$, (ii) $f'(x) < 0$,

(iii) $f''(x) > 0$, (iv) $f''(x) < 0$.

(3 marks)

(d) Find all relative extrema and points of inflexion of $f(x)$.

(3 marks)

(e) Find all asymptotes of the graph of $f(x)$.

(2 marks)

(f) Sketch the graph of $f(x)$.

(3 marks)

9. Let $I_m = \int_0^{\frac{\pi}{2}} \cos^m t \, dt$ where $m = 0, 1, 2, \dots$.

(a) (i) Evaluate I_0 and I_1 .

(ii) Show that $I_m = \dfrac{m-1}{m} I_{m-2}$ for $m \geq 2$.

Hence, or otherwise, evaluate I_{2n} and I_{2n+1} for $n \geq 1$.

(7 marks)

(b) Show that $I_{2n-1} \geq I_{2n} \geq I_{2n+1}$ for $n \geq 1$.

(2 marks)

(c) Let $A_n = \dfrac{1}{2n+1}\left[\dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]^2$ where $n = 0, 1, 2, \dots$.

(i) Using (a) and (b), show that $\dfrac{2n+1}{2n} A_n \geq \dfrac{\pi}{2} \geq A_n$.

(ii) Show that $\{A_n\}$ is monotonic increasing.

(iii) Evaluate $\lim\limits_{n \to \infty} \dfrac{1}{\sqrt{2n+1}}\left[\dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right]$.

(6 marks)

Go on to the next page

10. (a) Let $f: \mathbf{R} \to \mathbf{R}$ be a continuous function.

(i) Show that

$$\int_0^a f(t+b)\,dt = \int_0^{a+b} f(t)\,dt - \int_0^b f(t)\,dt \quad \text{for all } a, b \in \mathbf{R}.$$

(ii) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$,

show that $\int_0^x f(t+1)\,dt = f(1)x + \int_0^x f(t)\,dt$ for all $x \in \mathbf{R}$.

Using (i), or otherwise, show that $f(x) = f(1)x$ for all $x \in \mathbf{R}$.

(8 marks)

(b) Suppose g is a non-constant continuous function defined for all positive real numbers and $g(xy) = g(x) + g(y)$ for all $x, y > 0$.

By considering the function $f(t) = g(e^t)$ for $t \in \mathbf{R}$, show that $g(x) = \log_a x$ for some $a > 0$.

(7 marks)

11. (a) Let f be a non-negative continuous function on $[a, b]$. Define

$$F(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a, b].$$

Show that F is an increasing function on $[a, b]$.

Hence deduce that if $\int_a^b f(t)\,dt = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(5 marks)

(b) Let g be a continuous function on $[a, b]$. If $\int_a^b g(x)u(x)\,dx = 0$ for any continuous function u on $[a, b]$, show that $g(x) = 0$ for all $x \in [a, b]$.

(3 marks)

(c) Let h be a continuous function on $[a, b]$. Define

$$A = \frac{1}{b-a}\int_a^b h(t)\,dt.$$

(i) If $v(x) = h(x) - A$ for all $x \in [a, b]$, show that

$$\int_a^b v(x)\,dx = 0.$$

(ii) If $\int_a^b h(x)w(x)\,dx = 0$ for any continuous function w on $[a, b]$ satisfying $\int_a^b w(x)\,dx = 0$, show that $h(x) = A$ for all $x \in [a, b]$.

(7 marks)

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Figure 3 shows a circle with radius 1 and centre C touching a line L with slope $m > 0$ at $Q(a,b)$. $R(x_0, y_0)$ is another point on the circle and $\angle QCR = \theta$.

(a) Show that $\begin{cases} x_0 = a - \dfrac{2}{\sqrt{1+m^2}} \sin\dfrac{\theta}{2}(\cos\dfrac{\theta}{2} - m\sin\dfrac{\theta}{2}) \\ y_0 = b - \dfrac{2}{\sqrt{1+m^2}} \sin\dfrac{\theta}{2}(m\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}) \end{cases}$.

(7 marks)

(b) Consider the curve Γ : $y = \frac{2}{3}(x-1)^{\frac{3}{2}}$ where $x \geq 1$.

(i) Find the slope of tangent of Γ at $x = a$.

(ii) Find the length of arc of Γ from $x = 1$ to $x = a$.

(iii) Figure 4 shows a circle with radius 1 rolling tangentially below the curve Γ without slipping. Let P be a point fixed on the circle with initial position at (1, 0). Find the x-coordinate of P when the circle touches Γ at $x = 4$.

(8 marks)

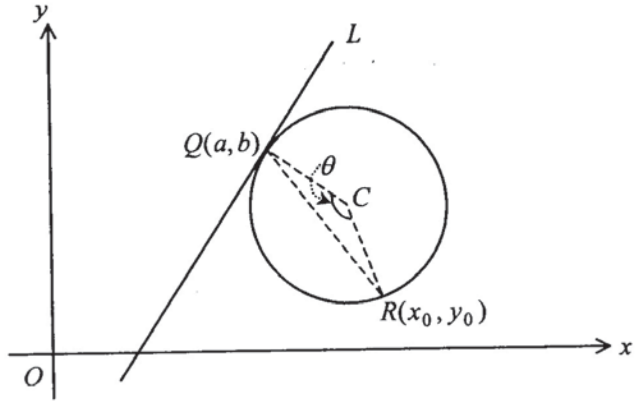

Figure 3

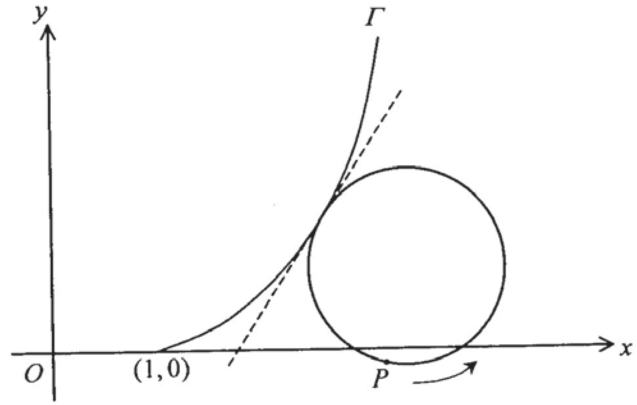

Figure 4

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13. Let $I = \left[0, \frac{\pi}{3}\right]$ and $g(x) = \cos x - \frac{1}{3}\cos^3 x$, where $x \in I$.

Let $x_0 \in I$ and define $x_n = g(x_{n-1})$ for $n = 1, 2, 3, \ldots$.

(a) Show that the equation $x = g(x)$ has exactly one root in I .

(3 marks)

(b) Show that $x_n \in I$ for all n .

(3 marks)

(c) Show that $|g'(x)| \le \frac{3}{4}$ for all $x \in I$.

(2 marks)

(d) Let α be the root of $x = g(x)$ mentioned in (a).

(i) Show that $|x_n - \alpha| \le \frac{3}{4}|x_{n-1} - \alpha|$ for $n = 1, 2, 3, \ldots$.

(ii) Show that $\{x_n\}$ converges and $\lim_{n \to \infty} x_n = \alpha$.

(iii) If $x_0 = \frac{\pi}{6}$, find a positive integer n such that $|x_n - \alpha| < \frac{1}{100}$.

(7 marks)

END OF PAPER