Solution	Marks
1. (a) $z_1\overline{z_1} + z_2\overline{z_2} - z_1 z_2 - \overline{z_1 z_2}$	
$= z_1(\overline{z_1} - z_2) - \overline{z_2}(\overline{z_1} - z_2)$	
$= (z_1 - \overline{z_2})(\overline{z_1} - z_2)$	1A
$= (z_1 - \overline{z_2})\overline{(z_1 - \overline{z_2})}$	
$= \lvert z_1 - \overline{z_2}\rvert^2$	
≥ 0	1
<u>Alternatively,</u>	
Let $z_r = x_r + iy_r, \quad r = 1, 2$	
$z_1\overline{z_1} + z_2\overline{z_2} - (z_1 z_2 + \overline{z_1 z_2})$	
$= \lvert z_1\rvert^2 + \lvert z_2\rvert^2 - 2\,\mathrm{Re}(z_1 z_2)$	
$= x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2(x_1 x_2 - y_1 y_2)$	1A
$= (x_1 - x_2)^2 + (y_1 + y_2)^2$	
≥ 0	1
$\therefore \quad z_1\overline{z_1} + z_2\overline{z_2} \geq z_1 z_2 + \overline{z_1 z_2}$	
(b) By (a), $\lvert z_1\rvert^2 + \lvert z_2\rvert^2 \geq 2\,\mathrm{Re}(z_1 z_2)$	
$\therefore \quad \lvert z_1\rvert^2 + \lvert z_2\rvert^2 + \cdots + \lvert z_n\rvert^2$	1
$= \frac{1}{2}\left\{\left(\lvert z_1\rvert^2 + \lvert z_2\rvert^2\right) + \left(\lvert z_2\rvert^2 + \lvert z_3\rvert^2\right) + \cdots + \left(\lvert z_{n-1}\rvert^2 + \lvert z_n\rvert^2\right) + \left(\lvert z_n\rvert^2 + \lvert z_1\rvert^2\right)\right\}$	
$\geq \frac{1}{2}\left\{2\,\mathrm{Re}(z_1 z_2) + 2\,\mathrm{Re}(z_2 z_3) + \cdots + 2\,\mathrm{Re}(z_{n-1} z_n) + 2\,\mathrm{Re}(z_n z_1)\right\}$	
$= \mathrm{Re}(z_1 z_2 + z_2 z_3 + \cdots + z_{n-1} z_n + z_n z_1)$	1
	(4)

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Solution	Marks
2. For $n=0$, $3^0(0^2+1)=1=a_0$.	
For $n=1$, $3^1(1^2+1)=6=a_1$.	
For $n=2$, $3^2(2^2+1)=45=a_2$.	
$\therefore$ The statement holds for $n = 0, 1, 2$.	1
Assume it is true for $n = k, k+1, k+2$, where $k \geq 0$, then	1M
$\frac{1}{27}a_{k+3} = a_k - a_{k+1} + \frac{1}{3}a_{k+2}$	
$= 3^k(k^2+1) - 3^{k+1}\left[(k+1)^2+1\right] + \frac{1}{3}\cdot 3^{k+2}\left[(k+2)^2+1\right]$	
$a_{k+3} = 3^{k+3}\left\{k^2+1-3(k^2+2k+2)+3(k^2+4k+5)\right\}$	1A
$= 3^{k+3}(k^2+6k+10)$	
$= 3^{k+3}\left[(k+3)^2+1\right]$	1
By the principle of mathematical induction,	
$a_n = 3^n(n^2+1)$ for $n = 0, 1, 2, \ldots$.	(4)
3. (a) $\because$ (*) has non-trivial solutions	
$\therefore \begin{vmatrix} \lambda & k & 0 \\ 0 & -\lambda & 1 \\ 1 & k & 1 \end{vmatrix} = 0$	
$\lambda^2 + k\lambda - k = 0$	1
(b) If $\lambda^2 + k\lambda - k = 0$ has equal roots, then	
$k^2 + 4k = 0$	
$k = -4$ or 0.	1A
(i) For $k=-4$ and $\lambda = 2$	1A for λ
$\begin{pmatrix} 2 & -4 & 0 \\ 0 & -2 & 1 \\ 1 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	
$x = z = 2y$	
S.S. $= \{(2t, t, 2t) : t \in \mathbf{R}\}$	1A
(ii) For $k = 0$ and $\lambda = 0$	1A for λ
$x = z = 0$	
S.S. $= \{(0, t, 0) : t \in \mathbf{R}\}$	1A
	(6)

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Solution	Marks
4. (a) $\because$ $r+\sqrt{r}$ is a root of $x^3+ax+b=0$	
$\therefore$ $(r+\sqrt{r})^3+a(r+\sqrt{r})+b=0$	1M
$r^3+3r^2\sqrt{r}+3r^2+r\sqrt{r}+ar+a\sqrt{r}+b=0$	1A
$(r^3+3r^2+ar+b)+\sqrt{r}(3r^2+r+a)=0$	
$\because$ $\sqrt{r}$ is not a rational	
$\therefore$ $r^3+3r^2+ar+b=0$ and $3r^2+r+a=0$	1
(b) (i) Sub. $r-\sqrt{r}$ into $x^3+ax+b=0$	
L.H.S. $=(r-\sqrt{r})^3+a(r-\sqrt{r})+b$	
$=(r^3+3r^2+ar+b)-\sqrt{r}(3r^2+r+a)$	1
$=0$ (by (a))	
$\therefore$ $r-\sqrt{r}$ is also a root of the equation.	
(ii) From (a), $3r^2+r+a=0 \Rightarrow r^2=-\frac{r+a}{3}$	
Sub. into $r^3+3r^2+ar+b=0$	
$-\frac{r^2+ar}{3}+3r^2+ar+b=0$	1M+1A
$8r^2+2ar+3b=0$	
$8(-\frac{r+a}{3})+2ar+3b=0$	
$(6a-8)r+(9b-8a)=0$	
$r=\frac{8a-9b}{2(3a-4)}$ for $a\neq\frac{4}{3}$.	1
Alternatively,	
From (a), $\begin{cases}-r^3=3r^2+ar+b & \ldots(1)\\ -3r^2=r+a & \ldots(2)\end{cases}$	
$\frac{(1)}{(2)}$: $\frac{r}{3}=\frac{3r^2+ar+b}{r+a}$	1M+1A
$8r^2+2ar+3b=0$	
Hence $\frac{r+a}{3}=\frac{2ar+3b}{8}$	
$8r+8a=6ar+9b$	
$r=\frac{8a-9b}{2(3a-4)}$ for $a\neq\frac{4}{3}$.	1
	(7)

Solution	Marks	
5. (a) $(1+a)(1+b)(1+c)$		
$= 1+(a+b+c)+(ab+bc+ca)+abc$		
$\geq 1+3\sqrt[3]{abc}+3\sqrt[3]{(abc)^2}+abc$ ($\because$ A.M. $\geq$ G.M.)	1M	
$= (1+\sqrt[3]{abc})^3$	1	
The equality holds when $a=b=c$ and $ab=bc=ca$		can be omitted
i.e. $a=b=c$	1	
(b) $\left[\det(I+P)\right]^{\frac{1}{3}} = \left[\det Q^{-1}(I+P)Q\right]^{\frac{1}{3}}$	1M	
$= \det\left[I+\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\right]^{\frac{1}{3}}$	1A	
$= \left[(1+a)(1+b)(1+c)\right]^{\frac{1}{3}}$	1A	
$\geq 1+\sqrt[3]{abc}$		
$= 1+\left[\det(Q^{-1}PQ)\right]^{\frac{1}{3}}$		
$= 1+\left[\det P\right]^{\frac{1}{3}}$	1	
	(7)	

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Solution	Marks
6. (a) If $\exists\ \lambda, \mu \in \mathbf{R}$ such that $\lambda\mathbf{b} + \mu\mathbf{c} = \mathbf{0}$	
then $\lambda(1-t, 2, 3) + \mu(0, 4, 2-t) = (0,0,0)$	
$\Rightarrow \begin{cases} (1-t)\lambda = 0 & \ldots(1) \\ 2\lambda + 4\mu = 0 & \ldots(2) \\ 3\lambda + (2-t)\mu = 0 & \ldots(3) \end{cases}$	1M
If $t \neq 1$, then $\lambda = 0$ by (1) and $\mu = 0$ by (2)	
If $t = 1$, then $\begin{cases} \lambda + 2\mu = 0 \\ 3\lambda + \mu = 0 \end{cases}$ by (2) & (3)	
$\Rightarrow \lambda = \mu = 0$.	1
$\therefore$ b and c are linearly independent for all real values of t .	
(b) For a, b, c to be linearly dependent	
$\begin{vmatrix} 2 & 3-t & 1 \\ 1-t & 2 & 3 \\ 0 & 4 & 2-t \end{vmatrix} = 0$	1M
$t^3 - 6t^2 + 3t - 18 = 0$	
$(t-6)(t^2+3) = 0$	
6 is the only real root of the equation.	
$\therefore$ There is only one real number t so that a, b, c are linearly dependent.	1
For $t = 6$,	
$\mathbf{a} = (2, -3, 1)$	
$= -\frac{2}{5}(-5, 2, 3) - \frac{11}{20}(0, 4, -4)$	
$= -\frac{2}{5}\mathbf{b} - \frac{11}{20}\mathbf{c}$	1A
	(5)

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Solution	Marks

7. (a) $\det A \det(A^{-1} - xI) = \det A(A^{-1} - xI)$ — **1M**

$= \det(I - xA)$

$= x^3 \det(x^{-1}I - A) \quad (\because \det(\lambda P) = \lambda^3 \det P$ for 3x3 matrix $P)$

$= -x^3 \det(A - x^{-1}I)$

$\therefore \quad \det(A^{-1} - xI) = -\dfrac{x^3}{\det A}\det(A - x^{-1}I)$ — **1**

(b) (i) $\det(A - xI) = \begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 4 & -17 & 8-x \end{vmatrix}$

$= -x^3 + 8x^2 - 17x + 4$ — **1A**

$\det(A - 4I) = -(4^3) + 8(4^2) - 17(4) + 4 = 0$ — **1**

$\therefore$ 4 is a root of $\det(A - xI) = 0$

$\det(A - xI) = -(x-4)(x^2 - 4x + 1)$

$\therefore$ The other roots of $\det(A - xI) = 0$ are

$x = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$ — **1A**

(ii) Clearly $x \neq 0$, otherwise $\det A^{-1} = 0$.

By (a), $\det(A^{-1} - xI) = 0 \Leftrightarrow \det(A - x^{-1}I) = 0$ — **1M**

Using (b)(i), $\quad x = \dfrac{1}{4}, \dfrac{1}{2 \pm \sqrt{3}}$

$x = \dfrac{1}{4}, 2 \pm \sqrt{3}$ — **1A**

Alternatively,

$A^{-1} = \dfrac{1}{4}\begin{pmatrix} 17 & -8 & 1 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$

$A^{-1} - xI = \dfrac{1}{4}\begin{pmatrix} 17-4x & -8 & 1 \\ 4 & -4x & 0 \\ 0 & 4 & -4x \end{pmatrix}$ — **1A**

If $\quad \det(A^{-1} - xI) = 0$, then

$16x^2(17 - 4x) + 16 - 128x = 0$

$4x^3 - 17x^2 + 8x - 1 = 0$

$(4x - 1)(x^2 - 4x + 1) = 0$

$x = \dfrac{1}{4} \quad$ or $\quad 2 \pm \sqrt{3}$ — **1A**

(7)

Solution	Marks
8. (a) If (S) has infinitely many solutions, then	
$\begin{vmatrix} a+1 & 2 & -2 \\ 1 & a & 2 \\ 3 & -1 & a-7 \end{vmatrix} = 0$	1M
$a^3 - 6a^2 - a + 30 = 0$	1A
$(a+2)(a-3)(a-5) = 0$	1A
$a = -2, 3$ or 5	1A
For $a = -2$, the augmented matrix of (S) is	
$\begin{pmatrix} -1 & 2 & -2 \\ 1 & -2 & 2 \\ 3 & -1 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -15 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$	
S.S. = $\{(4t, 3t, t) : t \in \mathbf{R}\}$	1A
For $a = 3$, the augmented matrix of (S) is	
$\begin{pmatrix} 4 & 2 & -2 \\ 1 & 3 & 2 \\ 3 & -1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & -10 & -10 \\ 0 & -10 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	
S.S. = $\{(t, -t, t) : t \in \mathbf{R}\}$	1A
For $a = 5$, the augmented matrix of (S) is	
$\begin{pmatrix} 6 & 2 & -2 \\ 1 & 5 & 2 \\ 3 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & 2 \\ 0 & -16 & -8 \\ 0 & -28 & -14 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	
S.S. = $\{(t, -t, 2t) : t \in \mathbf{R}\}$	1A
(b) For $a = -2$, the augmented matrix of (T) is	
$\left(\begin{array}{ccc\|c} -1 & 2 & -2 & 6 \\ 1 & -2 & 2 & 5b-1 \\ 3 & -1 & -9 & 1-b \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -2 & 2 & 5b-1 \\ 0 & 5 & -15 & 19-b \\ 0 & 0 & 0 & 5b+5 \end{array}\right)$	1M
Hence (T) is consistent when $b = -1$.	1A
For $a = -2$ and $b = -1$, the augmented matrix of (T) is	
$\left(\begin{array}{ccc\|c} -1 & 2 & -2 & 6 \\ 1 & -2 & 2 & -6 \\ 3 & -1 & -9 & 2 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -2 & 2 & -6 \\ 0 & 5 & -15 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & -4 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$	1M
S.S. = $\{(2 + 4t, 4 + 3t, t) : t \in \mathbf{R}\}$	1A
(c) From (b), the first three equations give	
$x = 2 + 4t$, $y = 4 + 3t$, $\sqrt{z} = t$.	1A
Substitute into the 4th equation,	
$3(2+4t) - 4(4+3t) - t^2 = -11$	
$t^2 - 1 = 0$	
$t = \pm 1$	1A
$\because \sqrt{z} = t > 0$	
$\therefore t = 1$	1M
Hence S.S. = $\{(6, 7, 1)\}$	1A

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Solution	Marks
9. (a) (i) $\frac{y_2}{y_1}D_1,\ \frac{y_1}{y_2}D_2>0$	
$\frac{y_2}{y_1}D_1+\frac{y_1}{y_2}D_2 \geq 2\sqrt{\frac{y_2}{y_1}D_1\cdot\frac{y_1}{y_2}D_2}$ (by A.M. ≥ G.M.)	1M
$= 2\sqrt{D_1D_2}$	1
(ii) $\frac{y_2}{y_1}D_1+\frac{y_1}{y_2}D_2$	
$=\frac{y_2}{y_1}(x_1y_1-z_1^2)+\frac{y_1}{y_2}(x_2y_2-z_2^2)$	
$=x_1y_2+x_2y_1-\left(\frac{y_2}{y_1}\cdot z_1^2+\frac{y_1}{y_2}\cdot z_2^2\right)$	1A
$\leq x_1y_2+x_2y_1-2\sqrt{\frac{y_2}{y_1}\cdot z_1^2\cdot\frac{y_1}{y_2}\cdot z_2^2}$ (by A.M. ≥ G.M.)	1A
$=x_1y_2+x_2y_1-2z_1z_2$	1
(iii) $(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2$	
$=(x_1y_1+x_2y_2+x_2y_1+x_1y_2)-(z_1^2+2z_1z_2+z_2^2)$	
$=(x_1y_2+x_2y_1-2z_1z_2)+D_1+D_2$	1A
$\geq\frac{y_2}{y_1}D_1+\frac{y_1}{y_2}D_2+D_1+D_2$ (by (a)(ii))	1A
$\geq 2\sqrt{D_1D_2}+D_1+D_2$ (by (a)(i))	1A
$\geq 2\sqrt{D_1D_2}+2\sqrt{D_1D_2}$ (by A.M. ≥ G.M.)	
$=4\sqrt{D_1D_2}$	1
(b) $\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2}\leq\frac{8}{4\sqrt{D_1D_2}}$	1A
$=2\sqrt{\frac{1}{D_1}\cdot\frac{1}{D_2}}$	
$\leq\frac{1}{D_1}+\frac{1}{D_2}$	1
$=\frac{1}{x_1y_1-z_1^2}+\frac{1}{x_2y_2-z_2^2}$	
If the equality holds, then	
$\begin{cases} D_1=D_2 & \ldots(1)\\ \frac{y_2}{y_1}D_1=\frac{y_1}{y_2}D_2 & \ldots(2)\\ \frac{y_2}{y_1}z_1^2=\frac{y_1}{y_2}z_2^2 & \ldots(3)\end{cases}$	2M
By (1) & (2): $\frac{y_2}{y_1}=\frac{y_1}{y_2}$	
$\Rightarrow y_1^2=y_2^2$	
$\Rightarrow y_1=y_2$ (as $y_1, y_2>0$)	1
From (3): $z_1=z_2$	
Sub. into (1): $x_1=x_2$	1

Solution	Marks
10. (a) (i) For any $A, B \in S$, let $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$, then	1M
$AB = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$	
$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \in S$	1A
(ii) The statement is clearly true for $n = 1$.	
Assume it is true for $n = k$, where $k \geq 1$, then	1M
$[\mathrm{T}(\theta)]^{k+1} = [\mathrm{T}(\theta)]^k \cdot \mathrm{T}(\theta)$	
$= \mathrm{T}(k\theta) \cdot \mathrm{T}(\theta)$	
$= \mathrm{T}[(k+1)\theta]$. (by (a))	1A
By the principle of mathematical induction,	
$[\mathrm{T}(\theta)]^n = \mathrm{T}(n\theta)$ for all positive integers n.	
(b) (i) $M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	
$= \sqrt{a^2+b^2} \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$	1M
$\because \left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$	
$\therefore$ There exists real θ such that $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$.	1
Let $k = \sqrt{a^2+b^2}$,	
then $M = k\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = k\mathrm{T}(\theta)$.	1
Alternatively,	
$M = -\sqrt{a^2+b^2} \begin{pmatrix} \frac{-a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ \frac{-b}{\sqrt{a^2+b^2}} & \frac{-a}{\sqrt{a^2+b^2}} \end{pmatrix}$.	1M
$\because \left(\frac{-a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{-b}{\sqrt{a^2+b^2}}\right)^2 = 1$	
$\therefore$ There exists real θ such that $\cos\theta = \frac{-a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{-b}{\sqrt{a^2+b^2}}$.	1
Let $k = -\sqrt{a^2+b^2}$,	
then $M = k\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = k\mathrm{T}(\theta)$.	1

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Solution	Marks
(ii) By (i), $M^n = k^n[\mathrm{T}(\theta)]^n$	
$= k^n \mathrm{T}(n\theta)$ (by (a)(ii))	
$= k^n\begin{pmatrix}\cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta\end{pmatrix}$	1A
If $\exists\, n \in \mathbf{Z}^+$ such that M^n is diagonal, then	
$\sin n\theta = 0$	
$n\theta = r\pi$ for some $r \in \mathbf{Z}$	1M
$\tan\theta = \tan\frac{r}{n}\pi$	
$\frac{b}{a} = \tan\frac{r}{n}\pi$ ($\because\ a \neq 0$)	
$\tan^{-1}\frac{b}{a} = \frac{r}{n}\pi + m\pi$ for some $m \in \mathbf{Z}$	
$\frac{1}{\pi}\tan^{-1}\frac{b}{a} = \frac{r}{n} + m \in \mathbf{Q}$	1
Conversely, if $\frac{1}{\pi}\tan^{-1}\frac{b}{a}$ is rational, then	
$\frac{1}{\pi}\tan^{-1}\frac{b}{a} = \frac{r}{n}$ for some $r \in \mathbf{Z}$ and $n \in \mathbf{Z}^+$.	1M
$\tan^{-1}\frac{b}{a} = \frac{r}{n}\pi$	
$\theta = \frac{r}{n}\pi + m\pi$ for some $m \in \mathbf{Z}$	
$n\theta = r\pi + nm\pi$	
$\sin n\theta = 0$	
$\Rightarrow$ M^n is diagonal.	1
(iii) If $a = 0$, then $M = \begin{pmatrix}0 & -b \\ b & 0\end{pmatrix}$	
$M^2 = \begin{pmatrix}0 & -b \\ b & 0\end{pmatrix}\begin{pmatrix}0 & -b \\ b & 0\end{pmatrix}$	
$= \begin{pmatrix}-b^2 & 0 \\ 0 & -b\end{pmatrix}$	
$= -b^2 I$ which is diagonal	1A
For any $m \in \mathbf{Z}^+$,	
$M^{2m} = (-b^2 I)^m = (-1)^m b^{2m} I$ which is diagonal	1
and $M^{2m+1} = (-1)^m b^{2m} I \cdot M = (-1)^m b^{2m} M$ which is not diagonal as $b \neq 0$	1
Hence if $a = 0$ and $n \in \mathbf{Z}^+$, M^n is diagonal iff n is even.	

Solution	Marks
11. (a) (i) Let the angle between $\mathbf{m}$ and $\mathbf{n}$ be θ.	
$\det\begin{pmatrix}\mathbf{m}\cdot\mathbf{m} & \mathbf{m}\cdot\mathbf{n}\\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}\cdot\mathbf{n}\end{pmatrix} = \lvert\mathbf{m}\rvert^2\lvert\mathbf{n}\rvert^2 - \lvert\mathbf{m}\rvert^2\lvert\mathbf{n}\rvert^2\cos^2\theta$	1M+1A
$= \lvert\mathbf{m}\rvert^2\lvert\mathbf{n}\rvert^2(1-\cos^2\theta)$	
$= \lvert\mathbf{m}\rvert^2\lvert\mathbf{n}\rvert^2\sin^2\theta$	
$= \lvert\mathbf{m}\times\mathbf{n}\rvert^2$	1
Alternatively, Let $\mathbf{m} = m_1\mathbf{i}+m_2\mathbf{j}+m_3\mathbf{k}$ and $\mathbf{n} = n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$.	
$\mathbf{m}\times\mathbf{n} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix} = \begin{vmatrix}m_2 & m_3\\ n_2 & n_3\end{vmatrix}\mathbf{i} + \begin{vmatrix}m_3 & m_1\\ n_3 & n_1\end{vmatrix}\mathbf{j} + \begin{vmatrix}m_1 & m_2\\ n_1 & n_2\end{vmatrix}\mathbf{k}$	
$\lvert\mathbf{m}\times\mathbf{n}\rvert^2 = (m_2n_3-m_3n_2)^2+(m_3n_1-m_1n_3)^2+(m_1n_2-m_2n_1)^2$	} 1M+1A For either
$\det\begin{pmatrix}\mathbf{m}\cdot\mathbf{m} & \mathbf{m}\cdot\mathbf{n}\\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}\cdot\mathbf{n}\end{pmatrix} = (m_1^2+m_2^2+m_3^2)(n_1^2+n_2^2+n_3^2)-(m_1n_1+m_2n_2+m_3n_3)^2$	}
$= (m_1n_2)^2+(m_1n_3)^2+(m_2n_1)^2+(m_2n_3)^2+(m_3n_1)^2+(m_3n_2)^2 +2(m_1m_2n_1n_2+m_2m_3n_2n_3+m_1m_3n_1n_3)$	
$= \lvert\mathbf{m}\times\mathbf{n}\rvert^2$	1
(ii) Let $\mathbf{m} = m_1\mathbf{i}+m_2\mathbf{j}+m_3\mathbf{k}$ and $\mathbf{n} = n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$.	
Then $(\mathbf{n}\cdot\mathbf{n})\mathbf{m}-(\mathbf{m}\cdot\mathbf{n})\cdot\mathbf{n}$	
$= (n_1^2+n_2^2+n_3^2)(m_1\mathbf{i}+m_2\mathbf{j}+m_3\mathbf{k})-(m_1n_1+m_2n_2+m_3n_3)(n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k})$	1M+1A (1M for using unit base vectors)
$= (m_1n_2^2+m_1n_3^2-m_2n_1n_2-m_3n_1n_3)\mathbf{i}+(m_2n_1^2+m_2n_3^2-m_1n_1n_2-m_3n_2n_3)\mathbf{j} +(m_3n_1^2+m_3n_2^2-m_1n_1n_3-m_2n_2n_3)\mathbf{k}$	
and $\mathbf{n}\times(\mathbf{m}\times\mathbf{n})$	
$= (n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k})\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}$	1A
$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ n_1 & n_2 & n_3\\ m_2n_3-m_3n_2 & m_3n_1-m_1n_3 & m_1n_2-m_2n_1\end{vmatrix}$	1A
$= (m_1n_2^2-m_2n_1n_2-m_3n_1n_3+m_1n_3^2)\mathbf{i}+(m_2n_3^2-m_3n_2n_3-m_1n_1n_2+m_2n_1^2)\mathbf{j} +(m_3n_1^2-m_1n_1n_3-m_2n_2n_3+m_3n_2^2)\mathbf{k}$	1A
$\therefore\ (\mathbf{n}\cdot\mathbf{n})\mathbf{m}-(\mathbf{m}\cdot\mathbf{n})\cdot\mathbf{n} = \mathbf{n}\times(\mathbf{m}\times\mathbf{n})$	

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Solution	Marks

Alternatively,

If $\mathbf{m}$, $\mathbf{n}$ are dependent, i.e. $\mathbf{m} = r\mathbf{n}$ for some $r \in \mathbf{R}$,

then $\mathbf{m} \times \mathbf{n} = \mathbf{0}$ and $(\mathbf{m} \cdot \mathbf{n})\mathbf{m} - (\mathbf{m} \cdot \mathbf{n})\mathbf{n} = 0$. — **2**

If $\mathbf{m}$, $\mathbf{n}$ are independent,

then $\mathbf{n} \times (\mathbf{m} \times \mathbf{n}) = a\mathbf{m} + b\mathbf{n}$ for some $a, b \in \mathbf{R}$.

$0 = \mathbf{n} \cdot [\mathbf{n} \times (\mathbf{m} \times \mathbf{n})] = a(\mathbf{m} \cdot \mathbf{n}) + b(\mathbf{n} \cdot \mathbf{n})$

Let $a = k(\mathbf{n} \cdot \mathbf{n})$ where $k \in \mathbf{R}$,

then $b = -k(\mathbf{m} \cdot \mathbf{n})$.

Let $\mathbf{n} \times (\mathbf{m} \times \mathbf{n}) = \mathbf{e} = (e_1, e_2, e_3)$,

then
$$\begin{aligned} e_1 &= n_2(m_1n_2 - m_2n_1) - n_3(m_3n_1 - m_1n_3) \\ &= m_1(n_2^2 + n_3^2) - n_1(n_2m_2 + n_3n_3) \\ &= m_1(\mathbf{n} \cdot \mathbf{n} - n_1^2) - n_1(\mathbf{m} \cdot \mathbf{n} - n_1m_1) \\ &= (\mathbf{n} \cdot \mathbf{n})m_1 - (\mathbf{m} \cdot \mathbf{n})n_1 \end{aligned}$$

$\therefore \quad k = 1$

$\Rightarrow \quad \mathbf{n} \times (\mathbf{m} \times \mathbf{n}) = (\mathbf{n} \cdot \mathbf{n})\mathbf{m} - (\mathbf{m} \cdot \mathbf{n})\mathbf{n}$ — **3**

(b) $\mathbf{p} = \lambda\mathbf{m} + \mu\mathbf{n}$ lies on L

$$\Rightarrow \begin{cases} (\lambda\mathbf{m} + \mu\mathbf{n} - \mathbf{a}) \cdot \mathbf{m} = 0 \\ (\lambda\mathbf{m} + \mu\mathbf{n} - \mathbf{b}) \cdot \mathbf{n} = 0 \end{cases}$$ — **1M**

$$\Rightarrow \begin{cases} (\mathbf{m} \cdot \mathbf{m})\lambda + (\mathbf{m} \cdot \mathbf{n})\mu = \mathbf{a} \cdot \mathbf{m} \\ (\mathbf{m} \cdot \mathbf{n})\lambda + (\mathbf{n} \cdot \mathbf{n})\mu = \mathbf{b} \cdot \mathbf{n} \end{cases}$$ — **1A**

$$\because \quad \begin{vmatrix} \mathbf{m} \cdot \mathbf{m} & \mathbf{m} \cdot \mathbf{m} \\ \mathbf{m} \cdot \mathbf{n} & \mathbf{n} \cdot \mathbf{n} \end{vmatrix} = |\mathbf{m} \times \mathbf{n}|^2 \neq 0$$ — **1A**

$$\therefore \quad \lambda = \frac{\begin{vmatrix} \mathbf{a} \cdot \mathbf{m} & \mathbf{m} \cdot \mathbf{n} \\ \mathbf{b} \cdot \mathbf{n} & \mathbf{n} \cdot \mathbf{n} \end{vmatrix}}{|\mathbf{m} \times \mathbf{n}|^2}, \quad \mu = \frac{\begin{vmatrix} \mathbf{m} \cdot \mathbf{m} & \mathbf{a} \cdot \mathbf{m} \\ \mathbf{m} \cdot \mathbf{n} & \mathbf{b} \cdot \mathbf{n} \end{vmatrix}}{|\mathbf{m} \times \mathbf{n}|^2}$$ — **1A**

$$\mathbf{p} = \frac{(\mathbf{a} \cdot \mathbf{m})(\mathbf{n} \cdot \mathbf{n}) - (\mathbf{m} \cdot \mathbf{n})(\mathbf{b} \cdot \mathbf{n})}{|\mathbf{m} \times \mathbf{n}|^2}\mathbf{m} + \frac{(\mathbf{m} \cdot \mathbf{m})(\mathbf{b} \cdot \mathbf{n}) - (\mathbf{m} \cdot \mathbf{n})(\mathbf{a} \cdot \mathbf{m})}{|\mathbf{m} \times \mathbf{n}|^2}\mathbf{n}$$ — **1A**

$$= \frac{1}{|\mathbf{m} \times \mathbf{n}|^2}\left\{(\mathbf{a} \cdot \mathbf{m})\left[(\mathbf{n} \cdot \mathbf{n})\mathbf{m} - (\mathbf{m} \cdot \mathbf{n})\mathbf{n}\right] + (\mathbf{b} \cdot \mathbf{n})\left[(\mathbf{m} \cdot \mathbf{m})\mathbf{n} - (\mathbf{m} \cdot \mathbf{n})\mathbf{m}\right]\right\}$$ — **1A**

$$= (\mathbf{a} \cdot \mathbf{m})\frac{\mathbf{n} \times (\mathbf{m} \times \mathbf{n})}{|\mathbf{m} \times \mathbf{n}|^2} + (\mathbf{b} \cdot \mathbf{n})\frac{\mathbf{m} \times (\mathbf{n} \times \mathbf{m})}{|\mathbf{m} \times \mathbf{n}|^2}$$ (by (a)(ii)) — **1**

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Solution	Marks
12. (a) [Existence]	
For $n = 1$,	
$(1+\sqrt{3})^2 = 4+2\sqrt{3}$	
Put $a_1 = 4$, $b_1 = 2$, then	1A
(i) $a_1^2 - 3b_1^2 = 16-12 = 4 = 2^2$	
(ii) a_1, b_1 are both divisible by 2.	1
Assume the result holds for $n = k$, where $k \ge 1$.	
For $n = k+1$,	
$(1+\sqrt{3})^{2(k+1)} = (1+\sqrt{3})^{2k}(1+\sqrt{3})^2$	
$= (a_k + b_k\sqrt{3})(4+2\sqrt{3})$	2M
$= (4a_k + 6b_k) + (2a_k + 4b_k)\sqrt{3}$	
Put $a_{k+1} = 4a_k + 6b_k$, $b_{k+1} = 2a_k + 4b_k$, then	
(i) $a_{k+1}{}^2 - 3b_{k+1}{}^2$	
$= (4a_k + 6b_k)^2 - 3(2a_k + 4b_k)^2$	
$= 16a_k{}^2 + 48a_kb_k + 36b_k{}^2 - 3(4a_k{}^2 + 16a_kb_k + 16b_k{}^2)$	
$= 4a_k{}^2 - 12b_k{}^2$	
$= 4(a_k{}^2 - 3b_k{}^2)$	
$= 2^2 \cdot 2^{2k}$	
$= 2^{2(k+1)}$	1
(ii) $\because$ a_k , b_k are divisible by 2^k	
$\therefore$ $a_{k+1} = 2(2a_k + 3b_k)$ and $b_{k+1} = 2(a_k + 2b_k)$ are divisible by 2^{k+1}.	1
Hence the result also holds for $n = k+1$.	
By the principle of mathematical induction, the result holds for all positive integers n	
[Uniqueness]	
Suppose $a + b\sqrt{3} = c + d\sqrt{3}$ for some $a, b, c, d \in \mathbf{Z}^+$, then	
$(a-c) = (b-d)\sqrt{3}$	
$\because$ $\sqrt{3}$ is irrational,	
$\therefore$ $a-c = b-d = 0 \Rightarrow a = c$ and $b = d$	2

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Solution	Marks
(b) By (a), $a_n^2 - 3b_n^2 = 2^{2n}$	
$\therefore \quad a_n - b_n\sqrt{3} = \dfrac{2^{2n}}{a_n + b_n\sqrt{3}}$	1M
$= \dfrac{2^{2n}}{(1+\sqrt{3})^{2n}} \cdot \dfrac{(1-\sqrt{3})^{2n}}{(1-\sqrt{3})^{2n}}$	
$= \dfrac{2^{2n}}{(-2)^{2n}}(1-\sqrt{3})^{2n}$	
$= (1-\sqrt{3})^{2n}$	1
Alternatively,	
For $n = 1$, $(1-\sqrt{3})^2 = 4 - 2\sqrt{3} = a_1 - b_1\sqrt{3}$	1M
Assume $(1-\sqrt{3})^{2k} = a_k - b_k\sqrt{3}$, where $k \geq 1$.	
For $n = k+1$,	
$(1-\sqrt{3})^{2(k+1)} = (1-\sqrt{3})^{2k}(1-\sqrt{3})^2$	
$= (a_k - b_k\sqrt{3})(4 - 2\sqrt{3})$	
$= (4a_k + 6b_k) - (2a_k + 4b_k)\sqrt{3}$	
$= a_{k+1} - b_{k+1}\sqrt{3}$	
By the principle of mathematical induction, the result follows.	1
Alternatively,	
$(1+\sqrt{3})^{2n} = \sum_{r=0}^{2n} C_r^{2n}(\sqrt{3})^r$	
$= \sum_{r=0}^{n} C_{2r}^{2n}(\sqrt{3})^{2r} + \sqrt{3}\left(\sum_{r=0}^{n-1} C_{2r+1}^{2n}(\sqrt{3})^{2r}\right)$	
$\therefore \quad a_n = \sum_{r=0}^{n} C_{2r}^{2n}(\sqrt{3})^{2r}$, $\quad b_n = \sum_{r=0}^{n-1} C_{2r+1}^{2n}(\sqrt{3})^{2r}$	1A
$(1-\sqrt{3})^{2n} = \sum_{r=0}^{2n} C_r^{2n}(-\sqrt{3})^r$	
$= \sum_{r=0}^{n} C_{2r}^{2n}(\sqrt{3})^{2r} - \sqrt{3}\left(\sum_{r=0}^{n-1} C_{2r+1}^{2n}(\sqrt{3})^{2r}\right)$	1
$= a_n - b_n\sqrt{3}$	
(c) From (b), $\dfrac{a_n}{b_n} - \sqrt{3} = \dfrac{(1-\sqrt{3})^{2n}}{b_n}$	1M
$\because \quad b_n$ is divisible by 2^n $\quad \therefore \quad \lim_{n\to\infty} \dfrac{1}{b_n} = 0$.	1 For either
$\because \quad \lvert 1-\sqrt{3} \rvert < 1$ $\quad \therefore \quad \lim_{n\to\infty}(1-\sqrt{3})^{2n} = 0$	
Hence $\lim_{n\to\infty}\left(\dfrac{a_n}{b_n} - \sqrt{3}\right) = \lim_{n\to\infty}(1-\sqrt{3})^{2n} \cdot \lim_{n\to\infty}\dfrac{1}{b_n} = 0$.	
$\Rightarrow \quad \lim_{n\to\infty}\dfrac{a_n}{b_n} = \sqrt{3}$	1
(d) $\because \quad (1+\sqrt{3})^{2n} + (1-\sqrt{3})^{2n} = 2a_n \in \mathbf{Z}^+$ and $0 < (1-\sqrt{3})^{2n} < 1$	1M
$\therefore \quad 2a_n$ is the smallest integer greater than $(1+\sqrt{3})^{2n}$.	
$\because \quad a_n$ is divisible by 2^n	
$\therefore \quad 2a_n$ is divisible by 2^{n+1} .	1

Solution	Marks
13. (a) $T_{n,r} = C_r^n (\frac{1}{n})^r$	1A
$\therefore \quad T_{n,0} = T_{n,1} = 1$ and	
$T_{n,r} = \frac{n(n-1)\cdots(n-r+1)}{r!}\frac{1}{n^r}$	
$= \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n})$ for $r \ge 2$.	1
(b) For any fixed r ,	
(i) $T_{n,r} \le 1$ as each factor $\frac{1}{r!}, 1-\frac{1}{n}, \cdots, 1-\frac{r-1}{n} \le 1$	
$\therefore \quad \{T_{n,r}\}$ is bounded above by 1 .	1
(ii) $T_{n+1,r} = \frac{1}{r!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{r-1}{n+1})$	
$T_{n,r} = \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n})$	
$\because \quad 1-\frac{k}{n+1} > 1-\frac{k}{n} > 0$ for $k = 1, 2, \ldots, r-1$	1M
$\therefore \quad T_{n+1,r} \ge T_{n,r}$	1
Hence $\{T_{n,r}\}$ is monotonic increasing for $n \ge r$.	
$\lim_{n\to\infty} T_{n,r} = \begin{cases} 1 & \text{for } r = 0, 1 \\ \lim_{n\to\infty} \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n}) & \text{for } r \ge 2 \end{cases}$	
$= \frac{1}{r!}$	1
(c) (i) $\frac{T_{n,r-1}}{T_{n,r}} = \frac{C_{r-1}^n \frac{1}{n^{r-1}}}{C_r^n \frac{1}{n^r}}$	
$= \frac{n!}{(n-r+1)!(r-1)!}\frac{(n-r)!r!}{n!}\frac{n^r}{n^{r-1}}$	
$= \frac{nr}{n-r+1}$	1A
$\ge r$ $(\because \ n \ge n-r+1$ for $r \ge 1)$	1
Alternatively,	
Using (a), $\frac{T_{n,0}}{T_{n,1}} = 1$ and $\frac{T_{n,1}}{T_{n,2}} = \frac{1}{\frac{1}{2!}(1-\frac{1}{n})} = \frac{2}{1-\frac{1}{n}} > 2$	1A
For $r \ge 3$, $\frac{T_{n,r-1}}{T_{n,r}} = \frac{r}{1-\frac{r-1}{n}} = \frac{nr}{n-r+1} > r$	1

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Solution	Marks
(ii) For $n \ge k \ge r \ge 1$, $T_{n,k} \le \frac{1}{k} T_{n,k-1}$	
$\le \frac{1}{k} \cdot \frac{1}{k-1} T_{n,k-2}$	1M
$\le \frac{1}{k} \frac{1}{k-1} \cdots \frac{1}{r} T_{n,r-1}$	
$\le \frac{1}{r^{k-r+1}} T_{n,r-1}$	1
(iii) If $r > 1$, $\sum_{k=r}^{n} T_{n,k} \le \sum_{k=r}^{n} \frac{1}{r^{k-r+1}} T_{n,r-1}$	
$= \sum_{k=1}^{n-r+1} \frac{1}{r^k} T_{n,r-1}$	
$< \dfrac{\frac{1}{r}}{1-\frac{1}{r}} T_{n,r-1}$	1M
$= \frac{1}{r-1} T_{n,r-1}$	1
$\therefore \quad T_{n,r-1} > (r-1)\sum_{k=r}^{n} T_{n,k}$ for $r > 1$.	
If $r = 1$, L.H.S. $= T_{n,0} = 1$ and R.H.S. $= 0$	
Hence the inequality holds for $r \ge 1$.	1
(d) For $r \ge 2$, $\sum_{k=r}^{n} T_{n,k} < \frac{1}{r-1} T_{n,r-1}$ (by (c))	1A
$= \frac{1}{r-1} \cdot \frac{1}{(r-1)!} (1-\frac{1}{n})(1-\frac{2}{n}) \cdots (1-\frac{r-2}{n})$ (by (a))	
$\le \frac{1}{(r-1)(r-1)!}$	1

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Solution	Marks
1. (a) $\frac{d}{dx}\tan\frac{x}{2} = \frac{1}{2}\sec^2\frac{x}{2}$	1A
$= \frac{1}{2\cos^2\frac{x}{2}}$	
$= \frac{1}{1+\cos x}$	1
(b) $\int \frac{x+\sin x}{1+\cos x}dx$	
$= \int (x+\sin x)\,d(\tan\frac{x}{2})$ (by (a))	
$= (x+\sin x)\tan\frac{x}{2} - \int \tan\frac{x}{2}(1+\cos x)dx$	1M
$= (x+\sin x)\tan\frac{x}{2} - 2\int \tan\frac{x}{2}\cos^2\frac{x}{2}dx$	
$= (x+\sin x)\tan\frac{x}{2} - 2\int \sin\frac{x}{2}\cos\frac{x}{2}dx$	
$= (x+\sin x)\tan\frac{x}{2} - 4\int \sin\frac{x}{2}d(\sin\frac{x}{2})$	1A
$= (x+\sin x)\tan\frac{x}{2} - 2\sin^2\frac{x}{2} + c$	
$= (x+2\sin\frac{x}{2}\cos\frac{x}{2})\tan\frac{x}{2} - 2\sin^2\frac{x}{2} + c$	
$= x\tan\frac{x}{2} + c$	1A
[or $(x+\sin x)\tan\frac{x}{2} - 2\sin^2\frac{x}{2} + c$,	
$(x+\sin x)\tan\frac{x}{2} + 2\cos^2\frac{x}{2} + c$,	
$(x+\sin x)\tan\frac{x}{2} + \cos x + c$,	
$x\tan\frac{x}{2} + 2\sin^2\frac{x}{2} + \cos x + c$ etc.]	
	(5)
2. $f(x) = xe^{-x}$	
$f'(x) = e^{-x} - xe^{-x}$	1A
$= e^{-x}(1-x)$	
$\therefore$ $f'(x) < 0$ for $x > 1$	1M+1A
$f(x)$ is strictly decreasing for $x > 1$.	
If $1 \le a < b$, then $f(a) > f(b)$	
$ae^{-a} > be^{-b}$	1A
$ae^{b} > be^{a}$	
	(4)

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Solution	Marks	
3. (a) $\mathcal{P}: y^2 = kx$		
$2y\frac{dy}{dx} = k$		
$\frac{dy}{dx} = \frac{k}{2y}$	1A	For slope
Slope of tangent to $\mathcal{P}$ at $A = \frac{k}{2(ks)} = \frac{1}{2s}$		
Slope of tangent to $\mathcal{P}$ at $B = \frac{k}{2(kt)} = \frac{1}{2t}$		
$\because$ The tangents to $\mathcal{P}$ at A and B are pendicular		
$\therefore$ $\frac{1}{2s}\cdot\frac{1}{2t} = -1$		
$st = -\frac{1}{4}$	1	
(b) Let $M = (x, y)$, then $\begin{cases} x = \frac{k}{2}(s^2+t^2) \\ y = \frac{k}{2}(s+t) \end{cases}$	1A	
$\because$ $(s+t)^2 = (s^2+t^2)+2st$		
$\therefore$ $\left(\frac{2y}{k}\right)^2 = \frac{2x}{k} + 2(-\frac{1}{4})$	1M+1A	
$y^2 = \frac{k}{2}x - \frac{k^2}{8}$ which is an equation of a parabola.		
Hence M lies on the parabola $y^2 = \frac{k}{2}x - \frac{k^2}{8}$.	(5)	

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Solution	Marks
4. $\because$ $2\sin A\sin B=\cos(A-B)-\cos(A+B)$ and	
$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$	
$\therefore$ $(\sin 2x+\sin 4x+\cdots+\sin 2nx)\sin x$	
$=\sin 2x\sin x+\sin 4x\sin x+\cdots+\sin 2nx\sin x$	
$=\frac{1}{2}[\cos x-\cos 3x+\cos 3x-\cos 5x+\cdots+\cos(2n-1)x-\cos(2n+1)x]$	1M+1A
$=\frac{1}{2}[\cos x-\cos(2n+1)x]$	
$=\sin nx\sin(n+1)x$	1
$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin 6x\sin 7x}{\sin x}dx$	
$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin 2x+\sin 4x+\cdots+\sin 12x)dx$	1A
$=\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(\sin y+\sin 2y+\cdots+\sin 6y)dy$	
$=-\frac{1}{2}\left[\cos y+\frac{1}{2}\cos 2y+\frac{1}{3}\cos 3y+\cdots+\frac{1}{6}\cos 6y\right]_{\frac{\pi}{2}}^{\pi}$	1A
$=-\frac{1}{2}\left[\left(-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}\right)-\left(-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}\right)\right]$	
$=\frac{1}{10}$	1A
Alternatively,	
$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin 6x\sin 7x}{\sin x}dx$	
$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin 2x+\sin 4x+\cdots+\sin 12x)dx$	1A
$=-\left[\frac{1}{2}\cos 2x+\frac{1}{4}\cos 4x+\frac{1}{6}\cos 6x+\cdots+\frac{1}{12}\cos 12x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$	1A
$=-\left[\left(-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{10}+\frac{1}{12}\right)-\left(-\frac{1}{4}+\frac{1}{8}-\frac{1}{12}\right)\right]$	
$=\frac{1}{10}$	1A
	(6)

Solution	Marks
5. (a) Let $y=(\sin x)^x$, then $\ln y = x\ln(\sin x)$	1M
$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}}$	
$= \lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}}$ (by L'Hospital rule)	1M
$= \lim_{x\to 0^+}\frac{-x^2\cos x}{\sin x}$ $\left[\text{or } \lim_{x\to 0^+}\frac{-x^2}{\tan x}\right]$	
$= -\left(\lim_{x\to 0^+}\frac{x}{\sin x}\right)\left(\lim_{x\to 0^+} x\cos x\right)$ $\left[\text{or } \lim_{x\to 0^+}\frac{-2x}{\sec^2 x}\right]$	
$=0$	1A
$\because$ Exponential functions are continuous	
$\therefore$ $\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+}\ln y} = 1$	1A
(b) $\lim_{x\to 0^+}\left(\frac{1}{x^3}\int_0^x e^{t^2}\,dt - \frac{1}{x^2}\right) = \lim_{x\to 0^+}\frac{\int_0^x e^{t^2}\,dt - x}{x^3}$	
$= \lim_{x\to 0^+}\frac{e^{x^2}-1}{3x^2}$ (by L'Hospital rule)	1A+1A
$= \lim_{x\to 0^+}\frac{2xe^{x^2}}{6x}$ (by L'Hospital rule)	
$= \lim_{x\to 0^+}\frac{e^{x^2}}{3}$	
$=\frac{1}{3}$	1A
	(7)
6. (a) For $x>0$, $f(x)=x^2 \Rightarrow f'(x)=2x$.	1A
For $x<0$, $f(x)=-x^2 \Rightarrow f'(x)=-2x$.	1A
(b) $\because$ $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x^2}{x} = 0$	1M
$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{-x^2}{x} = 0$	
$\therefore$ $f'(0)$ exists and $f'(0)=0$.	1
(c) $\because$ $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} 2x = 0$ and	1M
$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} -2x = 0$	
Hence $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 0 = f'(0)$.	
$f'(x)$ is continuous at $x=0$.	1
	(6)

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Solution	Marks
7. (a) $h = r\sin\theta$	
$= a\sin\theta(1+\cos\theta)$	1A
(b) $\frac{dh}{d\theta} = a\left[-\sin^2\theta + \cos\theta(1+\cos\theta)\right]$	1A
$= a(2\cos^2\theta + \cos\theta - 1)$	
$= a(\cos\theta+1)(2\cos\theta - 1)$	
$\frac{dh}{d\theta}\begin{cases} >0 & \text{for } 0<\theta<\frac{\pi}{3} \\ =0 & \text{for } \theta=\frac{\pi}{3} \\ <0 & \text{for } \frac{\pi}{3}<\theta<\pi \end{cases}$	1A
$\therefore$ h attains its greatest value when $\theta = \frac{\pi}{3}$	
$\Rightarrow$ $P_0 = (\frac{3}{2}a, \frac{\pi}{3})$.	1A
Length of arc OP_0 of $\mathscr{C}$	
$= \int_{\frac{\pi}{3}}^{\pi}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\,d\theta$	1M
$= \int_{\frac{\pi}{3}}^{\pi}\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$	
$= a\int_{\frac{\pi}{3}}^{\pi}\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta$	
$= a\int_{\frac{\pi}{3}}^{\pi}\sqrt{2(1+\cos\theta)}\,d\theta$	
$= 2a\int_{\frac{\pi}{3}}^{\pi}\cos\frac{\theta}{2}\,d\theta$	1A
$= 4a\left[\sin\frac{\theta}{2}\right]_{\frac{\pi}{3}}^{\pi}$	
$= 4a(1-\frac{1}{2})$	
$= 2a$	1A
	(7)

Solution	Marks

8. (a) (i) For $x \neq -1, 0$, $f'(x) = \dfrac{2-x}{3x^{\frac{1}{3}}(x+1)^2}$ — 1A

$\because \dfrac{f(0+h)-f(0)}{h} = \dfrac{h^{\frac{2}{3}}}{(h+1)h}$ — 1M

$= \dfrac{1}{h^{\frac{1}{3}}(h+1)}$

$\begin{cases} \to +\infty & \text{if } h \to 0^+ \\ \to -\infty & \text{if } h \to 0^- \end{cases}$

$\therefore$ $f'(0)$ does not exist. — 1A

(ii) For $x \neq -1, 0$, $f''(x) = \dfrac{-3x^{\frac{1}{3}}(x+1)^2 - 3(2-x)[\frac{1}{3}x^{-\frac{2}{3}}(x+1)^2 + 2(x+1)x^{\frac{1}{3}}]}{9x^{\frac{2}{3}}(x+1)^4}$

$= \dfrac{2(2x^2-8x-1)}{9x^{\frac{4}{3}}(x+1)^3}$ — 1

(b) (i) $f'(x) > 0 \iff 0 < x < 2$ — 1A
$f'(x) > 0$ on $(0, 2)$

(ii) $f'(x) < 0 \iff x < -1$ or $-1 < x < 0$ or $x > 2$ — 1A
$f'(x) < 0$ on $(-\infty, -1) \cup (-1, 0) \cup (2, \infty)$

Let α, β $(\alpha < \beta)$ be the roots of $2x^2 - 8x - 1 = 0$, then

$\alpha = \dfrac{4-3\sqrt{2}}{2} \approx -0.1213$

$\beta = \dfrac{4+3\sqrt{2}}{2} \approx 4.1213$

(iii) $f''(x) > 0 \iff -1 < x < \alpha$ or $x > \beta$ — 1A
$f''(x) > 0$ on $(-1, \alpha) \cup (\beta, \infty)$

(iv) $f''(x) < 0 \iff x < -1$ or $\alpha < x < 0$ or $0 < x < \beta$ — 1A
$f''(x) < 0$ on $(-\infty, -1) \cup (\alpha, 0) \cup (0, \beta)$

(c)

x	$(-\infty,-1)$	-1	$(-1,\alpha)$	α	$(\alpha,0)$	0	(0,2)	2	$(2,\beta)$	β	(β,∞)
$f(x)$	↓	U	↓	0.2789	↓	0	↑	0.5291	↓	0.5019	↓
$f'(x)$	−	U	−	−	−	U	+	0	−	−	−
$f''(x)$	−	U	+	0	−	U	−	−	−	0	+

* In the table, "U" stands for undefined.

$\therefore$ (0, 0) is a relative minimum point ; — 1A
(2, f(2)) is a relative maximum point where $f(2) \approx 0.5291$; — 1A
$(\alpha, f(\alpha))$ and $(\beta, f(\beta))$ are points of inflexion — 1A
where $\alpha \approx -0.1213$, $f(\alpha) \approx 0.2789$,
$\beta \approx 4.1213$, $f(\beta) \approx 0.5019$.

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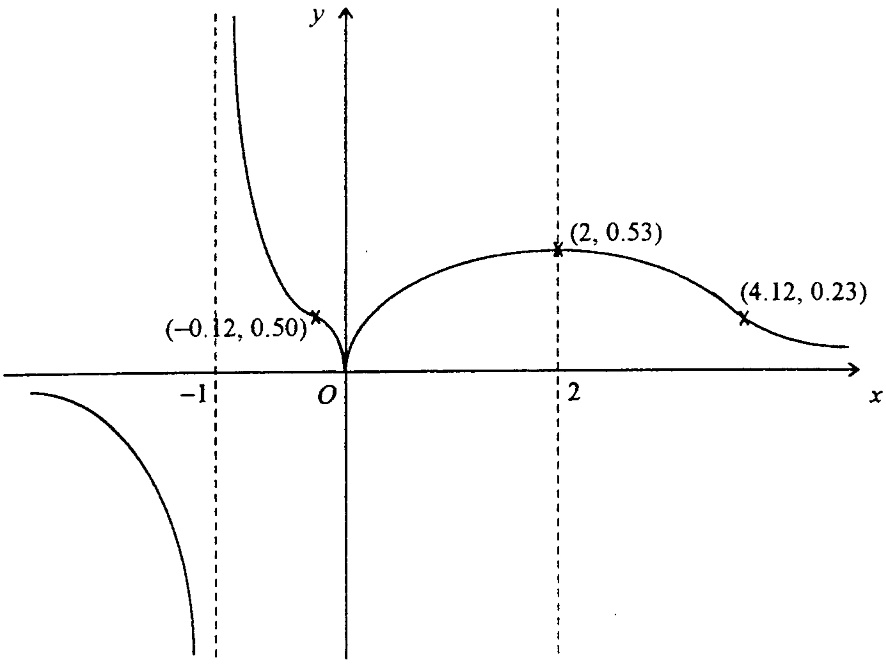
Solution	Marks
(d) $x = -1$ is the vertical asymptote.	
As $f(x) \to 0$ when $x \to \pm\infty$, the x-axis ($y = 0$) is the horizontal asymptote.	1A
There is no oblique asymptote.	
(e) 	1A+1A+1A

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Solution	Marks
9. (a) For $m \ge 0$ and $n \ge 1$,	
$I_{m,n} = \int_0^1 x^m(1-x)^n dx$	
$= \frac{1}{m+1}\int_0^1 (1-x)^n d(x^{m+1})$	1M
$= \frac{1}{m+1}\left[x^{m+1}(1-x)^n\right]_0^1 + \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1}dx$	1A
$= \frac{n}{m+1} I_{m+1,n-1}$	1
$\therefore \quad (m+1)I_{m,n} = nI_{m+1,n-1}$	
For $n \ge 1$,	
$I_{m,n} = \frac{n}{m+1} I_{m+1,n-1}$	
$= \frac{n}{m+1}\cdot\frac{n-1}{m+2} I_{m+2,n-2}$	1M
$\vdots$	
$= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\cdots\frac{1}{m+n} I_{m+n,0}$	1A
$\because \quad I_{m+n,0} = \int_0^1 x^{m+n}dx = \frac{1}{m+n+1}[x^{m+n+1}]_0^1 = \frac{1}{m+n+1}$	
$\therefore \quad I_{m,n} = \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\cdots\frac{1}{m+n}\cdot\frac{1}{m+n+1}$	
$= \frac{(m!)(n!)}{(m+n+1)!}$	1
For $n = 0$, $I_{m,0} = \frac{1}{m+1} = \frac{(m!)(0!)}{(m+0+1)!}$	1
Hence the result follows.	
Alternatively,	
For any nonnegative integer m,	
$\because \quad I_{m,0} = \int_0^1 x^m dx = \frac{1}{m+1}[x^{m+1}]_0^1 = \frac{1}{m+1} = \frac{(m!)(0!)}{(m+0+1)!}$	1A
$\therefore$ The statements holds for $n = 0$.	
Assume $I_{m,k} = \frac{(m!)(k!)}{(m+k+1)!}$ for some $k > 0$ and any nonnegative integer m,	1M
then $I_{m,k+1} = \frac{k+1}{m+1} I_{m+1,k}$	
$= \frac{k+1}{m+1}\cdot\frac{(m+1)!(k!)}{(m+1+k+1)!}$	1
$= \frac{(m!)((k+1)!)}{(m+(k+1)+1)!}$	1
By the principle of mathematical induction, the result follows.	

Solution	Marks
(b) (i) $\int_0^1 f^{(2n)}(x)\,g(x)\,dx$	
$= \int_0^1 g(x)\frac{d}{dx}[f^{(2n-1)}(x)]\,dx$	1M
$= \left[g(x)\,f^{(2n-1)}(x)\right]_0^1 - \int_0^1 f^{(2n-1)}(x)\,g'(x)\,dx$	
$= -\int_0^1 f^{(2n-1)}(x)\,g'(x)\,dx$	1A
$= (-1)^2\int_0^1 f^{(2n-2)}(x)\,g''(x)\,dx$	
$\vdots$	
$= (-1)^{2n}\int_0^1 f(x)\,g^{(2n)}(x)\,dx$	1A
$= \int_0^1 f(x)\,g^{(2n)}(x)\,dx$	
$\because$ $g(x) = x^n(1-x)^n$	
$= (-1)^n x^{2n}$ + terms of lower powers of x	
$\therefore$ $g^{(2n)}(x) = (-1)^n(2n)!$	1A
Hence $\int_0^1 f^{(2n)}(x)\,g(x)\,dx = (-1)^n(2n)!\int_0^1 f(x)\,dx$	1
(ii) From (i),	
$\left\lvert\int_0^1 f^{(2n)}(x)\,g(x)\,dx\right\rvert = \left\lvert(-1)^n(2n)!\int_0^1 f(x)\,dx\right\rvert$	
$\Rightarrow$ $\left\lvert\int_0^1 f(x)\,dx\right\rvert = \frac{1}{(2n)!}\left\lvert\int_0^1 f^{(2n)}(x)\,g(x)\,dx\right\rvert$	
$\leq \frac{1}{(2n)!}\int_0^1 \lvert f^{(2n)}(x)\rvert\lvert g(x)\rvert\,dx$	1A
$\leq \frac{M}{(2n)!}\int_0^1 \lvert g(x)\rvert\,dx$	
$= \frac{M}{(2n)!}\int_0^1 g(x)\,dx$ ($\because$ $g(x)\geq 0$ $\forall\, x\in[0, 1]$)	1A
$= \frac{M}{(2n)!}\cdot\frac{n!\,n!}{(2n+1)!}$ (by(a))	
$= \frac{(n!)^2 M}{(2n)!\,(2n+1)!}$	1

Solution	Marks
10. (a) (i) $\because$ $f(1) = f(1\times1) = f(1) + f(1) = 2f(1)$	
$\therefore$ $f(1) = 0$	1
(ii) $\because$ $f(x^{-1}) + f(x) = f(x^{-1}\cdot x) = f(1) = 0$	
$\therefore$ $f(x^{-1}) = -f(x)$	1
(iii) Clearly $f(x^0) = f(1) = 0 = 0\cdot f(x)$.	1
Assume $f(x^k) = kf(x)$ for some $k \in \mathbf{Z}_0^+$,	
then $f(x^{k+1}) = f(x^k) + f(x) = kf(x) + f(x) = (k+1)f(x)$	
By the principle of mathematical induction,	
$f(x^n) = nf(x)$ for all nonnegative integers n.	1
$\forall n \in \mathbf{Z}^-$, $-n \in \mathbf{Z}^+$	
$f(x^n) = f[(x^{-1})^{-n}]$	1
$= (-n)f(x^{-1})$	
$= (-n)(-f(x))$	
$= nf(x)$	1
Hence $f(x^n) = nf(x)$ $\forall n \in \mathbf{Z}$ and $x \in \mathbf{R}^+$.	
(b) $\forall r \in \mathbf{Q}$, let $r = \frac{p}{q}$ where $p \in \mathbf{Z}$ and $q \in \mathbf{Z}^+$.	1M
$rf(x) = \frac{p}{q}f(x)$	
$= \frac{p}{q}f[(x^{\frac{1}{q}})^q]$	1A
$= q(\frac{p}{q})f(x^{\frac{1}{q}})$	1A
$= pf(x^{\frac{1}{q}})$	
$= f(x^{\frac{1}{q}p})$	
$= f(x^{\frac{p}{q}})$	
$= f(x^r)$	1

Solution	Marks
(c) $\forall \alpha \in \mathbf{R}$, let $\{r_n\}$ be a sequence in $\mathbf{Q}$ such that $\lim_{n\to\infty} r_n = \alpha$.	
Then	
$\alpha f(x) = \lim_{n\to\infty} r_n f(x)$	
$= \lim_{n\to\infty} f(x^{r_n})$	1A
$= f(\lim_{n\to\infty} x^{r_n})$ (∵ f is continuous)	1A
$= f(x^{\lim_{n\to\infty} r_n})$ (∵ exponential functions are continuous)	
$= f(x^{\alpha})$	1
(d) If $f(2) = 1$,	
$f(x) = f(2^{\log_2 x})$	1A
$= (\log_2 x) f(2)$	
$= \log_2 x$.	1

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Solution	Marks
11. (a) (i) $F'(x)=\dfrac{1}{\sqrt{1+x^3}}>0 \quad \forall\ x\geq 1$	
$\therefore$ F(x) is strictly increasing for $x\geq 1$.	1
(ii) $F(x)\leq \int_1^x \dfrac{1}{\sqrt{t^3}}\,dt$	1M
$=-2\left[\dfrac{1}{\sqrt{t}}\right]_1^x$	
$=2(1-\dfrac{1}{\sqrt{x}})$	
$<2 \quad \forall\ x>1$	1
From (i), $F(x)>F(1)=0 \quad \forall\ x>1$	1
<u>Alternatively,</u> $F(x)\geq \int_1^x \dfrac{1}{\sqrt{2t^3}}\,dt=\sqrt{2}(1-\dfrac{1}{\sqrt{x}})>0 \quad \forall\ x>1$	1
$\therefore\ 0<F(x)<2 \quad \forall\ x>1$	
(iii) $\because\ F(x)\geq \int_1^x \dfrac{1}{\sqrt{2t^3}}\,dt=\sqrt{2}(1-\dfrac{1}{\sqrt{x}})$	1M
$\therefore$ Let $x_0=16$, then $F(x_0)\geq\sqrt{2}(1-\dfrac{1}{4})=\dfrac{3\sqrt{2}}{4}>1$	1
(b) (i) $\because\ F(G(u))=u$	
$\therefore\ F'(G(u))G'(u)=1$	1A
$G'(u)=\dfrac{1}{F'(G(u))}$	
$=\sqrt{1+[G(u)]^3}$	1
$G''(u)=\dfrac{3[G(u)]^2\,G'(u)}{2\sqrt{1+[G(u)]^3}}$	1M
$=\dfrac{3}{2}[G(u)]^2$	1

Solution	Marks
(ii) $\because$ $F(G(u)) = u$	
$\therefore$ $G(u) \geq 1$ for all u.	1A
Suppose on the contrary that $G''(u) \leq G'(u)$, then	
$\frac{9}{4}[G(u)]^4 \leq 1 + [G(u)]^3$	
$\leq 1 + [G(u)]^4$	1M
$\Rightarrow$ $\frac{5}{4}[G(u)]^4 \leq 1$ which is impossible as $G(u) \geq 1$.	1
Alternatively,	
$G''(u) - G'(u) = \frac{3}{2}[G(u)]^2 - \sqrt{1+[G(u)]^3}$	
$\geq \frac{3}{2}[G(u)]^2 - \sqrt{2[G(u)]^3}$ $(\because G(u) \geq 1)$	1M
$\geq \frac{3}{2}[G(u)]^2 - \sqrt{2[G(u)]^4}$ $(\because G(u) \geq 1)$	
$= (\frac{3}{2} - 2)[G(u)]^2$	
> 0	1
Hence $G''(u) > G'(u)$ for all u.	
(iii) $\because$ $G''(u) > G'(u) = \sqrt{1+[G(u)]^3} \geq 0$ for all u.	1
$\therefore$ The graph of $G(u)$ does not have any points of inflexion.	1

Solution	Marks
12. (a) $PQ=\sqrt{(a+\frac{1}{a})^2+(\frac{1}{a}-a)^2}=\sqrt{a^2+(\frac{1}{a})^2}$	
Similarly, $QR=RS=SP=\sqrt{a^2+(\frac{1}{a})^2}$	1A
$\therefore$ The lengths of the 4 sides of $PQRS$ are equal.	
$\because$ (slope of PQ)$\cdot$(slope of PS) $=\dfrac{a-\frac{1}{a}}{-\frac{1}{a}-a}\cdot\dfrac{\frac{1}{a}+a}{a-\frac{1}{a}}=-1$	1A
$\therefore PQ\perp PS$	
Hence $PQRS$ is a square.	1
(b) Equation of PQ:	
$\dfrac{y-\frac{1}{a}}{x-a}=\dfrac{a-\frac{1}{a}}{-\frac{1}{a}-a}$	1M
$\dfrac{ay-1}{a}=\dfrac{1-a^2}{1+a^2}(x-a)$	
$a(1-a^2)(x-a)-(1+a^2)(ay-1)=0$	
$a(1-a^2)x-a(1+a^2)y+a^4+1=0$	1A
Solve $\begin{cases}xy=-1 & (x<0)\\ a(1-a^2)x-a(1+a^2)y+a^4+1=0 & (0<a\le 1)\end{cases}$	
$a(1-a^2)x-a(1+a^2)(-\frac{1}{x})+a^4+1=0$	
$a(1-a^2)x^2+(a^4+1)x+a(1+a^2)=0$	
$(ax+1)[(1-a^2)x+a(1+a^2)]$	1M+1A
$\because -\frac{1}{a}$ is the x-coordinate of Q	
$\therefore Q'=\left(-\dfrac{a(1+a^2)}{1-a^2},\dfrac{1-a^2}{a(1+a^2)}\right)$	1A
If PQ lies inside the region $\mathcal{D}$, then	
$-\dfrac{a(a^2+1)}{1-a^2}\le-\dfrac{1}{a}$ with $0<a\le 1$	1M
$a^2(a^2+1)\ge 1-a^2$	
$a^4+2a^2-1\ge 0$	
$(a^2+1)^2\ge 2$	
$a^2\ge\sqrt{2}-1$ $(\because a^2+1>0)$	1A
$a\le-\sqrt{\sqrt{2}-1}$ or $a\ge\sqrt{\sqrt{2}-1}$	
$\because 0<a\le 1$	
$\therefore \sqrt{\sqrt{2}-1}\le a\le 1$	1A

Solution	Marks
(c) Let A be the area of $PQRS$, then	
$A = PQ^2 = a^2 + (\frac{1}{a})^2$	
$\frac{\mathrm{d}A}{\mathrm{d}a} = 2a - \frac{2}{a^3}$	1A
$= \frac{2(a^2+1)(a+1)(a-1)}{a^3}$	
By symmetry, $PQRS$ lies within the region $\mathcal{D}$ iff PQ lies inside $\mathcal{D}$.	
i.e. $\sqrt{\sqrt{2}-1} \le a \le 1$	1M
$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}a} < 0$	1
A is strictly decreasing for $a \in [\sqrt{\sqrt{2}-1}, 1]$.	
The area of $PQRS$ is maximized within the region $\mathcal{D}$ when $a = \sqrt{\sqrt{2}-1}$.	1A

Solution	Marks
13. (a) (i) $\because$ f is decreasing and $f(x) > 0$	
$\therefore$ $f(n) \ge f(x) \ge f(n+1)$ $\quad \forall\ x \in (n, n+1)$	
$\Rightarrow$ $f(n) \ge \int_n^{n+1} f(x)dx \ge f(n+1)$	1
Thus $f(n) - f(n) \le f(n) - \int_n^{n+1} f(x)\,dx \le f(n) - f(n+1)$	
$0 \le a_n \le f(n) - f(n+1)$	1
Hence $c_n = a_1 + a_2 + \cdots + a_n$	
$\le f(1) - f(2) + f(2) - f(3) + \cdots + f(n) - f(n+1)$	
$= f(1) - f(n+1)$	
$\le f(1)$ $\quad$ (as $f(n+1) > 0$)	1
On the other hand, $c_{n+1} - c_n = a_{n+1} \ge 0$ $\quad \forall\ n \in \mathbf{N}$	
$\therefore$ $\{c_n\}$ is increasing and bounded above by f(1).	
$\Rightarrow$ $\lim_{n\to\infty} c_n$ exists.	1
(ii) $\because$ $c_k - c_n = a_{n+1} + a_{n+2} + \cdots + a_k$	
$\therefore$ $0 \le c_k - c_n \le \sum_{r=n+1}^{k} [f(r) - f(r+1)]$	
$= f(n+1) - f(k+1)$	
$\le f(n+1)$ $\quad$ for $k > n$	1
(b) (i) Using (a)(ii) and putting $k \to \infty$, we have $0 \le c - c_n \le f(n+1)$.	1
(ii) $f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)\,dx$	
$= \sum_{r=1}^{n} \left(f(r) - \int_r^{r+1} f(x)\,dx \right) + \int_n^{n+1} f(x)\,dx$	1M
$= a_1 + a_2 + \cdots + a_n + \int_n^{n+1} f(x)\,dx$	
$= c_n + \int_n^{n+1} f(x)\,dx$	1A
$\because$ $f(n) \ge \int_n^{n+1} f(x)\,dx \ge f(n+1)$ and	
$0 \le c - c_n \le f(n+1)$	
$\therefore$ $f(n) - 0 \ge \int_n^{n+1} f(x)\,dx - c + c_n \ge f(n+1) - f(n+1)$	1M
$f(n) \ge c_n + \int_n^{n+1} f(x)\,dx - c \ge 0$	
$\Rightarrow$ $c_n + \int_n^{n+1} f(x)\,dx - c = \theta_n f(n)$ $\quad$ for some $\theta_n \in [0,1]$	1
i.e. $f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)\,dx = c + \theta_n f(n)$ $\quad$ for some $\theta_n \in [0,1]$	

Solution	Marks
(c) (i) Let $f(x)=\frac{1}{x}$, then $f(x)$ is decreasing, continuous and positive for $x \geq 1$.	1M
$\int_1^n f(x)\,dx = [\ln x]_1^n = \ln n$	
By (b)(ii), $S_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n$	
$= f(1)+f(2)+\cdots+f(n)-\int_1^n f(x)\,dx$	
$= c+\theta_n f(n)$ for some $\theta_n \in [0,1]$	
$= c+\frac{\theta_n}{n}$	1A
$\therefore\ c \leq S_n = c+\frac{\theta_n}{n} \leq c+\frac{1}{n}$	
As $\lim_{n\to\infty}\frac{1}{n}=0$	
$\therefore\ \lim_{n\to\infty} S_n$ exists and $\lim_{n\to\infty} S_n = c$.	1
(ii) $T_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$	
$= (1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n})-2(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n})$	
$= (1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n})-(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})$	
$= (S_{2n}+\ln 2n)-(S_n+\ln n)$	1A
$= S_{2n}-S_n+\ln 2$	
$\because\ \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_n = c$	
$\therefore\ \lim_{n\to\infty} T_n = c-c+\ln 2$	
$= \ln 2$	1A