97-AL
P MATHS
PAPER I

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1997

PURE MATHEMATICS A-LEVEL PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

SECTION A (40 marks)

Answer **ALL** questions in this section.

Write your answers in the AL(C1) answer book.

1. Let $z_1, z_2, \ldots, z_n$ be arbitrary complex numbers.

   (a) Prove that $z_1\overline{z_1} + z_2\overline{z_2} \geq z_1 z_2 + \overline{z_1 z_2}$.

   (b) Using (a), or otherwise, show that
   $|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 \geq \mathrm{Re}(z_1 z_2 + z_2 z_3 + \cdots + z_{n-1} z_n + z_n z_1)$.

   (4 marks)

2. Let $\{a_n\}$ be a sequence of real numbers, where
   $a_0 = 1$, $a_1 = 6$, $a_2 = 45$ and
   $a_n - a_{n+1} + \frac{1}{3} a_{n+2} - \frac{1}{27} a_{n+3} = 0$ for $n = 0, 1, 2, \ldots$.

   Using mathematical induction, or otherwise, show that
   $a_n = 3^n (n^2 + 1)$ for $n = 0, 1, 2, \ldots$.

   (4 marks)

3. Suppose the system of linear equations

$$(*) : \begin{cases} \lambda x + ky \phantom{{}+z} = 0 \\ -\lambda y + z = 0 \\ x + ky + z = 0 \end{cases}$$

   has nontrivial solutions.

   (a) Show that λ satisfies the equation $\lambda^2 + k\lambda - k = 0$.

   (b) If the quadratic equation in λ in (a) has equal roots, find k .
   Solve (*) for each of these values of k .

   (6 marks)

4. Suppose $r+\sqrt{r}$ is a root of the cubic equation $x^3+ax+b=0$, where a, b, r are rational numbers and $\sqrt{r}$ is not a rational number.

(a) Show that $r^3+3r^2+ar+b=0$ and $3r^2+r+a=0$.

(b) Using (a), or otherwise, show that

(i) $r-\sqrt{r}$ is also a root of the equation, and

(ii) $r=\dfrac{8a-9b}{2(3a-4)}$ if $a \neq \dfrac{4}{3}$.

(7 marks)

5. Let a, b and c be positive numbers.

(a) Using A.M. $\geq$ G.M., show that $(1+a)(1+b)(1+c) \geq (1+\sqrt[3]{abc})^3$.

Under what condition on a, b and c will the equality hold?

(b) Let P be a 3×3 real matrix such that $Q^{-1}PQ=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ for some 3×3 real matrix Q.

Using (a), show that $[\det(I+P)]^{\frac{1}{3}} \geq 1+[\det P]^{\frac{1}{3}}$.

(7 marks)

6. Let $\mathbf{a}=(2,3-t,1)$, $\mathbf{b}=(1-t,2,3)$ and $\mathbf{c}=(0,4,2-t)$.

(a) Show that $\mathbf{b}$ and $\mathbf{c}$ are linearly independent for all real values of t.

(b) Show that there is only one real number t so that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent.

For this value of t, express $\mathbf{a}$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$.

(5 marks)

7. (a) Let A be a 3×3 non-singular matrix. Show that

$$\det(A^{-1}-xI)=-\frac{x^3}{\det A}\det(A-x^{-1}I).$$

(b) Let $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$.

(i) Show that 4 is a root of $\det(A-xI)=0$ and hence find the other roots in surd form.

(ii) Solve $\det(A^{-1}-xI)=0$.

(7 marks)

SECTION B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Write your answers in the AL(C2) answer book.

8. Consider the following two systems of linear equations:

$$(S): \begin{cases} (a+1)x + 2y - 2z = 0 \\ x + ay + 2z = 0 \\ 3x - y + (a-7)z = 0 \end{cases} \quad \text{and}$$

$$(T): \begin{cases} (a+1)x + 2y - 2z = 6 \\ x + ay + 2z = 5b-1 \\ 3x - y + (a-7)z = 1-b \end{cases} .$$

(a) If (S) has infinitely many solutions, find all the values of a.

Solve (S) for each of these values of a.

(7 marks)

(b) For the smallest value of a found in (a), find the value of b so that (T) is consistent.

Solve (T) for these values of a and b.

(4 marks)

(c) Solve the system of equations

$$\begin{cases} -x + 2y - 2\sqrt{z} = 6 \\ x - 2y + 2\sqrt{z} = -6 \\ 3x - y - 9\sqrt{z} = 2 \\ 3x - 4y - z = -11 \end{cases} .$$

(4 marks)

9. Let x_1, x_2, y_1, y_2, z_1 and z_2 be positive numbers such that $x_1y_1 - z_1^2 > 0$ and $x_2y_2 - z_2^2 > 0$.

(a) Let $D_1 = x_1y_1 - z_1^2$ and $D_2 = x_2y_2 - z_2^2$. Using A.M. ≥ G.M., show that

(i) $\frac{y_2}{y_1}D_1 + \frac{y_1}{y_2}D_2 \ge 2\sqrt{D_1D_2}$,

(ii) $\frac{y_2}{y_1}D_1 + \frac{y_1}{y_2}D_2 \le x_1y_2 + x_2y_1 - 2z_1z_2$,

(iii) $(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2 \ge 4\sqrt{D_1D_2}$.

(9 marks)

(b) Show that $\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \le \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}$,
and if the equality holds, then $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

(6 marks)

10. (a) (i) Let $\mathcal{S} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbf{R} \right\}$.

Show that for any matrices A and B in $\mathcal{S}$, AB is also in $\mathcal{S}$.

(ii) Let $\mathrm{T}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Prove that $[\mathrm{T}(\theta)]^n = \mathrm{T}(n\theta)$ for any positive integer n.

(4 marks)

(b) Let $M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where $a, b \in \mathbf{R}$ and $a^2 + b^2 \neq 0$.

(i) Show that $M = k\mathrm{T}(\theta)$ for some real numbers k and θ.

Express k, $\cos\theta$ and $\sin\theta$ in terms of a and b.

(ii) If $a \neq 0$, prove that there exists a positive integer n such that M^n is diagonal (i.e. of the form $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$) if and only if $\frac{1}{\pi}\tan^{-1}\frac{b}{a}$ is rational.

(iii) If $a = 0$, find all positive integers n such that M^n is diagonal.

(11 marks)

11. (a) Let $\mathbf{m}$, $\mathbf{n}$ be vectors in $\mathbf{R}^3$. Show that

(i) $\det\begin{pmatrix} \mathbf{m}\cdot\mathbf{m} & \mathbf{m}\cdot\mathbf{n} \\ \mathbf{m}\cdot\mathbf{n} & \mathbf{n}\cdot\mathbf{n} \end{pmatrix} = |\mathbf{m}\times\mathbf{n}|^2$,

(ii) $(\mathbf{n}\cdot\mathbf{n})\mathbf{m} - (\mathbf{m}\cdot\mathbf{n})\mathbf{n} = \mathbf{n}\times(\mathbf{m}\times\mathbf{n})$.

(8 marks)

(b) Two planes $(\mathbf{r}-\mathbf{a})\cdot\mathbf{m} = 0$ and $(\mathbf{r}-\mathbf{b})\cdot\mathbf{n} = 0$ intersect in a line L, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{m}$, $\mathbf{n}$ are constant vectors and $\mathbf{r}$ is a position vector in $\mathbf{R}^3$. Express the real numbers λ and μ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{m}$ and $\mathbf{n}$ such that the point represented by the position vector $\mathbf{p} = \lambda\mathbf{m} + \mu\mathbf{n}$ lies on the line L.

Show that $\mathbf{p} = (\mathbf{a}\cdot\mathbf{m})\dfrac{\mathbf{n}\times(\mathbf{m}\times\mathbf{n})}{|\mathbf{m}\times\mathbf{n}|^2} + (\mathbf{b}\cdot\mathbf{n})\dfrac{\mathbf{m}\times(\mathbf{n}\times\mathbf{m})}{|\mathbf{m}\times\mathbf{n}|^2}$.

(7 marks)

12. (a) Show that for any positive integer n, there exist unique positive integers a_n and b_n such that

$$(1+\sqrt{3})^{2n} = a_n + b_n\sqrt{3}$$

and that (i) $a_n^2 - 3b_n^2 = 2^{2n}$,

(ii) a_n and b_n are both divisible by 2^n.

(8 marks)

(b) For a_n and b_n as determined in (a), show that

$$(1-\sqrt{3})^{2n} = a_n - b_n\sqrt{3} .$$

(2 marks)

(c) Using (b), or otherwise, prove that $\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{3}$.

(3 marks)

(d) Using (a) and (b), or otherwise, prove that for any positive integer n, the smallest integer greater than $(1+\sqrt{3})^{2n}$ is divisible by 2^{n+1}.

(2 marks)

13. Let $(1+\frac{1}{n})^n = \sum_{r=0}^{n} T_{n,r}$, where $T_{n,r}$ is the $(r+1)$-th term in the binomial expansion of $(1+\frac{1}{n})^n$ in ascending powers of $\frac{1}{n}$.

(a) Show that $T_{n,0} = T_{n,1} = 1$ and

$$T_{n,r} = \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n}) \quad \text{for } r \ge 2 .$$

(2 marks)

(b) For any fixed r, show that the sequence $T_{r,r}$, $T_{r+1,r}$, $T_{r+2,r}$, … is bounded above and monotonic increasing.

Find $\lim_{n\to\infty} T_{n,r}$ in terms of r.

(4 marks)

(c) Show that

(i) $\frac{T_{n,r-1}}{T_{n,r}} \ge r$ for $r \ge 1$,

(ii) $T_{n,k} \le \frac{1}{r^{k-r+1}} T_{n,r-1}$ for $n \ge k \ge r \ge 1$,

(iii) $T_{n,r-1} > (r-1)\sum_{k=r}^{n} T_{n,k}$ for $r \ge 1$.

(7 marks)

(d) Show that $\sum_{k=r}^{n} T_{n,k} < \frac{1}{(r-1)(r-1)!}$ for $r \ge 2$.

(2 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1997

PURE MATHEMATICS A-LEVEL PAPER II

2.00 pm–5.00 pm (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

SECTION A (40 marks)

Answer **ALL** questions in this section.

Write your answers in the AL(C1) answer book.

1. (a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\tan\frac{x}{2}=\frac{1}{1+\cos x}$.

   (b) Using (a), or otherwise, find $\int\frac{x+\sin x}{1+\cos x}\,\mathrm{d}x$.

(5 marks)

2. By considering the function $f(x)=xe^{-x}$, or otherwise, show that if $1\le a<b$, then $ae^{b}>be^{a}$.

(4 marks)

3. Let $\mathcal{P}$ be the parabola $y^2=kx$, where k is a non-zero constant. $A(ks^2, ks)$ and $B(kt^2, kt)$ are two distinct points on $\mathcal{P}$ moving in such a way that the tangents drawn to $\mathcal{P}$ at A and B are perpendicular to each other.

   (a) Show that $st=-\frac{1}{4}$.

   (b) If M is the mid-point of AB, show that M lies on a parabola and find the equation of this parabola.

(5 marks)

4. Show that $(\sin 2x + \sin 4x + \cdots + \sin 2nx)\sin x = \sin nx \sin(n+1)x$.

Hence, or otherwise, evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 6x \sin 7x}{\sin x} \mathrm{d}x$.

(6 marks)

5. Evaluate

(a) $\lim_{x \to 0^+} (\sin x)^x$,

(b) $\lim_{x \to 0^+} \left(\frac{1}{x^3} \int_0^x e^{t^2} \mathrm{d}t - \frac{1}{x^2} \right)$.

(7 marks)

6. Let $f(x) = x|x|$.

(a) Find $f'(x)$ for $x > 0$ and $x < 0$ respectively.

(b) Prove that $f'(0)$ exists.

(c) Prove that $f'(x)$ is continuous at $x = 0$.

(6 marks)

7. Figure 1 shows the graph of the curve $\mathscr{C}$ with polar equation $r = a(1+\cos\theta)$, where $0 \le \theta \le \pi$ and $a > 0$. Denote the pole by O . For any point $P(r, \theta)$ on $\mathscr{C}$, let h be the distance from P to the initial line.

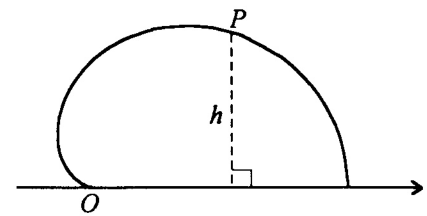

Figure 1

(a) Express h in terms of a and θ.

(b) Let P_0 be the point on $\mathscr{C}$ where h attains its greatest value. Find the polar coordinates of P_0 and the length of the arc OP_0 of $\mathscr{C}$.

(7 marks)

SECTION B (60 marks)
Answer any **FOUR** questions in this section. Each question carries 15 marks.
Write your answers in the AL(C2) answer book.

8. Let $f(x) = \dfrac{x^{\frac{2}{3}}}{x+1}$ $(x \neq -1)$.

(a) (i) Find $f'(x)$ for $x \neq -1, 0$.

Does $f'(0)$ exist? Explain.

(ii) Show that $f''(x) = \dfrac{2(2x^2 - 8x - 1)}{9x^{\frac{4}{3}}(x+1)^3}$ for $x \neq -1, 0$.

(4 marks)

(b) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$, (ii) $f'(x) < 0$,

(iii) $f''(x) > 0$, (iv) $f''(x) < 0$.

(4 marks)

(c) Find all relative extrema and points of inflexion of $f(x)$.

(3 marks)

(d) Find the asymptote(s) of the graph of $f(x)$.

(1 mark)

(e) Sketch the graph of $f(x)$.

(3 marks)

9. Let m , n be non-negative integers.

(a) Define $I_{m,n} = \int_0^1 x^m (1-x)^n \, dx$.

Show that $(m+1) I_{m,n} = n I_{m+1,n-1}$ for $n \geq 1$.

Hence, or otherwise, show that $I_{m,n} = \dfrac{m!\,n!}{(m+n+1)!}$.

(7 marks)

(b) Let f be a real-valued function with continuous derivatives on [0, 1] up to order $2n$, where $n \geq 1$, and $f^{(k)}(0) = f^{(k)}(1) = 0$ for $k = 0, 1, \dots, 2n-1$.

(i) Show that $\int_0^1 f^{(2n)}(x)\, g(x)\, dx = (-1)^n (2n)! \int_0^1 f(x)\, dx$, where $g(x) = x^n (1-x)^n$.

(ii) Suppose there is a constant M such that $|f^{(2n)}(x)| \leq M$ for all $x \in [0, 1]$. Using (i), or otherwise, deduce that $\left| \int_0^1 f(x)\, dx \right| \leq \dfrac{(n!)^2 M}{(2n)!\,(2n+1)!}$.

(8 marks)

10. Denote the open interval $(0, \infty)$ by $\mathbf{R}^+$. Let $f : \mathbf{R}^+ \to \mathbf{R}$ be a continuous function such that $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbf{R}^+$.

(a) Show that

(i) $f(1) = 0$,

(ii) $f(x^{-1}) = -f(x)$ for all $x \in \mathbf{R}^+$,

(iii) $f(x^n) = n\,f(x)$ for all $n \in \mathbf{Z}$ and $x \in \mathbf{R}^+$.

(6 marks)

(b) Show that $f(x^r) = r\,f(x)$ for all $r \in \mathbf{Q}$ and $x \in \mathbf{R}^+$.

(4 marks)

(c) Show that $f(x^\alpha) = \alpha\,f(x)$ for all $\alpha \in \mathbf{R}$ and $x \in \mathbf{R}^+$.

(You may use the fact that for any $\alpha \in \mathbf{R}$, there exists a sequence $\{r_n\}$ in $\mathbf{Q}$ such that $\lim_{n\to\infty} r_n = \alpha$.)

(3 marks)

(d) If $f(2) = 1$, show that $f(x) = \log_2 x$.

(2 marks)

11. Define $F(x) = \int_1^x \frac{1}{\sqrt{1+t^3}}\,dt$ for any $x \geq 1$.

(a) (i) Show that $F(x)$ is strictly increasing.

(ii) Show that $0 < F(x) < 2$ for any $x > 1$.

(iii) Find an x_0 such that $F(x_0) > 1$.

(6 marks)

(b) Suppose $G(u)$ is a function on $(0, 1)$ such that $F(G(u)) = u$.

(i) Show that $G'(u) = \sqrt{1+[G(u)]^3}$ and $G''(u) = \frac{3}{2}[G(u)]^2$.

(ii) Show that $G''(u) > G'(u)$.

(iii) Does the graph of $G(u)$ have any points of inflexion? Explain.

(9 marks)

12. Consider the curves

$\mathscr{C}_1$: $xy = 1$ $(x > 0)$,
$\mathscr{C}_2$: $xy = -1$ $(x < 0)$,
$\mathscr{C}_3$: $xy = 1$ $(x < 0)$,
$\mathscr{C}_4$: $xy = -1$ $(x > 0)$

and the region $\mathscr{D}$ determined by $|xy| \le 1$ in the rectangular coordinate plane as shown in Figure 2.

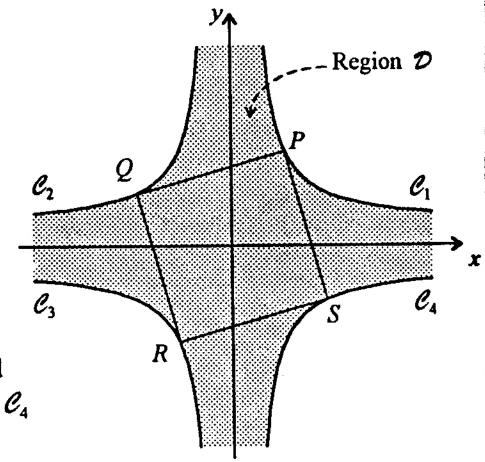

Figure 2

For $0 < a \le 1$, let P, Q, R and S be points on $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_4$ respectively, where

$$P = (a, \frac{1}{a}),\ Q = (-\frac{1}{a}, a),$$
$$R = (-a, -\frac{1}{a}) \text{ and } S = (\frac{1}{a}, -a).$$

(a) Show that $PQRS$ is a square.

(3 marks)

(b) If the straight line PQ intersects $\mathscr{C}_2$ at another point Q', find the coordinates of Q' in terms of a.

Hence find the range of values of a such that the line segment PQ lies inside the region $\mathscr{D}$.

(8 marks)

(c) If $PQRS$ lies within the region $\mathscr{D}$, determine a such that its area is maximized.

(You may use the fact that $PQRS$ lies within the region $\mathscr{D}$ if and only if PQ lies inside $\mathscr{D}$.)

(4 marks)

13. Let $f(x)$ be a decreasing continuous function on $[1, \infty)$, and $f(x) > 0$ for all x. For any positive integer n, define

$$a_n = f(n) - \int_n^{n+1} f(x)dx \quad \text{and} \quad c_n = a_1 + a_2 + \cdots + a_n .$$

(a) (i) Show that $0 \le a_n \le f(n) - f(n+1)$ and $c_n \le f(1)$.

Hence deduce that $\lim_{n\to\infty} c_n$ exists.

(ii) Prove that $0 \le c_k - c_n \le f(n+1)$ for $k > n$.

(5 marks)

(b) Let $c = \lim_{n\to\infty} c_n$. Show that

(i) $0 \le c - c_n \le f(n+1)$, and

(ii) $f(1) + f(2) + \cdots + f(n) - \int_1^n f(x)dx = c + \theta_n f(n)$

for some $\theta_n \in [0, 1]$.

(5 marks)

(c) Let $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$ and

$T_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$.

(i) Using (b)(ii), or otherwise, show that $\lim_{n\to\infty} S_n$ exists.

(ii) Express T_n in terms of S_{2n} and S_n.

Hence find $\lim_{n\to\infty} T_n$.

(5 marks)

END OF PAPER