Solution	Marks

1. (a) $A^2 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{pmatrix}$ — 1A

$A^3 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{pmatrix} = \begin{pmatrix} -6 & 0 & -14 \\ 7 & 8 & 7 \\ 7 & 0 & 15 \end{pmatrix}$ — 1A

$$A^3 - 5A^2 + 8A - 4I = \begin{pmatrix} -6 & 0 & -14 \\ 7 & 8 & 7 \\ 7 & 0 & 15 \end{pmatrix} - 5\begin{pmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{pmatrix} + 8\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ — 1A

(b) By (a), $A(A^2 - 5A + 8I) = 4I$,

$\therefore\ A^{-1} = \frac{1}{4}(A^2 - 5A + 8I)$ — 1M+1A

$$= \frac{1}{4}\left[\begin{pmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 10 \\ -5 & -10 & -5 \\ -5 & 0 & -15 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{pmatrix} 6 & 0 & 4 \\ -2 & 2 & -2 \\ -2 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{3}{2} & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$ — 1A

<u>Alternatively,</u>

$\det A = 4 \neq 0$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} \end{pmatrix}^t$$ — 1M+1A　　1A for $\frac{1}{4}$

$$= \frac{1}{4}\begin{pmatrix} 6 & 0 & 4 \\ -2 & 2 & -2 \\ -2 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{3}{2} & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$ — 1A

(6)

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Solution	Marks
2. (a) Since $(1+k)^n = 1 + C_1^n k + C_2^n k^2 + \cdots + C_n^n k^n$	1A
$= 1 + k(C_1^n + C_2^n k + \cdots + C_n^n k^{n-1})$	
$\therefore$ if $(1+k)^n$ is divided by k, the remainder is 1.	1
<u>Alternatively,</u>	
Let $f(x) = (1+x)^n$ for $x \in \mathbf{R}$.	
When $(1+x)^n$ is divided by x, the remainder is $f(0) = 1$.	1A
In particular, when $(1+k)^n$ is divided by k, the remainder is 1.	1
(b) Since $8^{96} = (1+7)^{96}$	
$\therefore$ when 8^{96} is divided by 7, the remainder is 1.	1A
Thus if today is Tuesday, 8^{96} days after is Wednesday.	1A
	(4)
3. (a) 1 is a real root of (*).	1A
(b) (*): $(z-1)[z^2 - (a-1)z + 1] = 0$	1A
For (*) to have non-real roots,	
$(a-1)^2 - 4 < 0$	
$-1 < a < 3$	1A
(c) Let the non-real roots be ω and $\overline{\omega}$.	
$\because$ Product of roots $= 1 \cdot \omega \cdot \overline{\omega} = 1$	1A
$\therefore |\omega| = |\overline{\omega}| = 1$	1
<u>Alternatively,</u>	
$\because \quad \omega = \dfrac{a-1+i\sqrt{4-(a-1)^2}}{2}$ where $-1 < a < 3$,	1A
$\therefore \quad |\omega| = |\overline{\omega}| = \dfrac{1}{2}\sqrt{(a-1)^2 + 4 - (a-1)^2} = 1$	1
	(5)

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Solution	Marks	
4. (a) For $x \in (-1, \infty)$, $f'(x) = \frac{1}{(1+x)^2} > 0$,	1A	1A for f'
$\therefore$ f is increasing on $(-1, \infty)$.	1	for ">0"
Alternatively,		
If $x, y \in (-1, \infty)$ and $y>x$, then $f(y) - f(x) = \frac{y-x}{(1+y)(1+x)} > 0$	1A	
$\therefore$ f is increasing on $(-1, \infty)$.	1	
(b) $\because$ $\lvert r+s\rvert \le \lvert r\rvert + \lvert s\rvert$	1A	
$\therefore$ $\frac{\lvert r+s\rvert}{1+\lvert r+s\rvert} \le \frac{\lvert r\rvert+\lvert s\rvert}{1+\lvert r\rvert+\lvert s\rvert}$ by (a)	1A	Or $f(\lvert r+s\rvert) \le f(\lvert r\rvert + \lvert s\rvert)$
$= \frac{\lvert r\rvert}{1+\lvert r\rvert+\lvert s\rvert} + \frac{\lvert s\rvert}{1+\lvert r\rvert+\lvert s\rvert}$		
$\le \frac{\lvert r\rvert}{1+\lvert r\rvert} + \frac{\lvert s\rvert}{1+\lvert s\rvert}$	1	
	(5)	
5. (a) $X = \frac{1}{2}(b+c-a)$	1A	
$Y = \frac{1}{2}(a+c-b)$		
$Z = \frac{1}{2}(a+b-c)$	1A	
(b) Replacing Z, Y and X in (a) by xy, xz and yz respectively,		
$\because$ $a+b-c>0$, $a+c-b>0$ and $b+c-a>0$		
$\therefore$ $\frac{X}{Y} = \frac{y}{x} = \frac{b+c-a}{a+c-b}$	1A	
Sub. into $xy = \frac{1}{2}(a+b-c)$: $x = \pm\sqrt{\frac{(a+b-c)(a+c-b)}{2(b+c-a)}}$	1A	
Hence $\begin{cases} x = \sqrt{\frac{(a+b-c)(a+c-b)}{2(b+c-a)}} \\ y = \sqrt{\frac{(a+b-c)(b+c-a)}{2(a+c-b)}} \\ z = \sqrt{\frac{(a+c-b)(b+c-a)}{2(a+b-c)}} \end{cases}$ or $\begin{cases} x = -\sqrt{\frac{(a+b-c)(a+c-b)}{2(b+c-a)}} \\ y = -\sqrt{\frac{(a+b-c)(b+c-a)}{2(a+c-b)}} \\ z = -\sqrt{\frac{(a+c-b)(b+c-a)}{2(a+b-c)}} \end{cases}$.	1A+1A	
	(6)	

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Solution	Marks
6. (a) $\because\ x_2 = \dfrac{3}{2}$, $x_3 = \dfrac{7}{4}$ and $x_4 = \dfrac{13}{8}$	1A
$\therefore\ x_2 - x_1 = -\dfrac{1}{2}$, $x_3 - x_2 = \dfrac{1}{4}$ and $x_4 - x_3 = -\dfrac{1}{8}$	1A
(b) By guessing, $x_n - x_{n-1} = (-1)^{n-1}\dfrac{1}{2^{n-1}}$ ....(*)	1A
For $n = 1$, LHS $= x_1 - x_0 = 2 - 1 = 1$,	
RHS $= (-1)^0 \dfrac{1}{2^0} = 1$	
$\therefore$ (*) holds for $n = 1$.	1A
Assume it holds for $n = k$, then	
$x_{k+1} - x_k = \dfrac{x_k + x_{k-1}}{2} - x_k = -\dfrac{1}{2}(x_k - x_{k-1}) = (-1)^k \dfrac{1}{2^k}$	1A
By the principle of M.I., (*) holds for $n = 1, 2, 3, \ldots$.	
$\because\ x_n - x_0 = \sum_{i=1}^{n}(x_i - x_{i-1})$	
$= 1 - \dfrac{1}{2} + \dfrac{1}{4} - \cdots + (-1)^{n-1}\dfrac{1}{2^{n-1}}$	1A
$\therefore\ \lim_{n\to\infty} x_n = 1 + \dfrac{1}{1-(-\frac{1}{2})} = \dfrac{5}{3}$	1A
	(7)
7. (a) If the roots of $x^3 + px^2 + qx + r = 0$ are α, β and γ, then	
$\alpha + \beta + \gamma = -p$ ....(i)	
$\alpha\beta + \beta\gamma + \alpha\gamma = q$ ....(ii)	1A
(ii) $-\alpha$ (i): $\alpha p + q = \alpha\beta + \beta\gamma + \alpha\gamma - \alpha(\alpha + \beta + \gamma) = \beta\gamma - \alpha^2$	1
(b) By symmetry, the roots of $x^3 + Px^2 + Qx + R = 0$ are $\alpha p + q$, $\beta p + q$ and $\gamma p + q$.	
$\Sigma(\alpha p + q) = p(\Sigma\alpha) + 3q = -p^2 + 3q$	
$\Sigma(\alpha p + q)(\beta p + q) = p^2(\Sigma\alpha\beta) + 2pq(\Sigma\alpha) + 3q^2 = -p^2 q + 3q^2$	
$(\alpha p + q)(\beta p + q)(\gamma p + q) = p^3(\alpha\beta\gamma) + p^2 q(\Sigma\alpha\beta) + pq^2(\Sigma\alpha) + q^3 = -p^3 r + q^3$	1M+1M 1M for comparison; 1M for using $\alpha p + q$ etc.
Alternatively, To find the eqtn. whose roots are $\alpha p + q$, $\beta p + q$ and $\gamma p + q$.	
(i) For $p \neq 0$, put $y = px + q$ and sub. $x = \dfrac{y-q}{p}$ into $x^3 + px^2 + qx + r = 0$.	
$\left(\dfrac{y-q}{p}\right)^3 + p\left(\dfrac{y-q}{p}\right)^2 + q\left(\dfrac{y-q}{p}\right) + r = 0$	
$(y^3 - 3qy^2 + 3q^2 y - q^3) + p^2(y^2 - 2qy + q^2) + p^2 q(y - q) + p^3 r = 0$	
$y^3 + (p^2 - 3q)y^2 + (3q^2 - p^2 q)y + (p^3 r - q^3) = 0$ ....(*)	
(ii) For $p = 0$, the required equation is $(y - q)^3 = 0$ which coincides with (*).	1M+1M 1M for dividing into 2 cases; 1M for using $\alpha p + q$ etc.
$\therefore\ P = p^2 - 3q$, $Q = 3q^2 - p^2 q$ and $R = p^3 r - q^3$	1A+1A+1A
	(7)

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Solution	Marks
8. (a) $\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$	1A
$(1-\lambda)(2-\lambda)-6=0$	1A
$\lambda^2-3\lambda-4=0$	
$(\lambda+1)(\lambda-4)=0$	
$\lambda=-1$ or 4	1A
(b) $\lambda_1=-1$, solve $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{0}$.	1M
A possible solution is $(x_1, y_1) = (-3, 2)$.	1A
$\lambda_2=4$, solve $\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{0}$.	
A possible solution is $(x_2, y_2) = (1, 1)$.	1A
If $\alpha(-3, 2)+\beta(1, 1)=0$,	1M
then $\begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underline{0}$ .... (I)	1A
$\because \begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} = -5 \neq 0$	
$\therefore$ (I) has no non-trivial solution	1
$\Rightarrow \alpha=\beta=0$	1
$\therefore$ $(-3, 2)$, $(1, 1)$ are linearly independent	
$P = \begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix}$	
$P^{-1} = \frac{1}{-5}\begin{pmatrix} 1 & -1 \\ -2 & -3 \end{pmatrix}$	1A
$= \begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$	
$P^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}P = -\frac{1}{5}\begin{pmatrix} 1 & -1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 2 & 1 \end{pmatrix}$	
$= -\frac{1}{5}\begin{pmatrix} 1 & -1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix}$	
$= -\frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & -20 \end{pmatrix}$	
$= \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$	1A

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Solution	Marks
(c) $\begin{pmatrix}1 & 3\\ 2 & 2\end{pmatrix}^{1996} = \left[P\begin{pmatrix}-1 & 0\\ 0 & 4\end{pmatrix}P^{-1}\right]^{1996}$ by (b)(iii)	1M
$= P\begin{pmatrix}-1 & 0\\ 0 & 4\end{pmatrix}^{1996}P^{-1}$	1M
$= -\frac{1}{5}\begin{pmatrix}-3 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 4^{1996}\end{pmatrix}\begin{pmatrix}1 & -1\\ -2 & -3\end{pmatrix}$	
$= \frac{1}{5}\begin{pmatrix}-3 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 4^{1996}\end{pmatrix}\begin{pmatrix}-1 & 1\\ 2 & 3\end{pmatrix}$	
$= \frac{1}{5}\begin{pmatrix}-3 & 4^{1996}\\ 2 & 4^{1996}\end{pmatrix}\begin{pmatrix}-1 & 1\\ 2 & 3\end{pmatrix}$	
$= \frac{1}{5}\begin{pmatrix}3+2\cdot 4^{1996} & -3+3\cdot 4^{1996}\\ -2+2\cdot 4^{1996} & 2+3\cdot 4^{1996}\end{pmatrix}$	1A

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Solution	Marks
9 (a) $\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \end{array}\right)$	
$\sim \left(\begin{array}{ccc\|c} 1 & 0 & 5 & 5 \\ 0 & -1 & 3 & 1 \end{array}\right)$	1A
Alternatively, (1st eqtn.) − (2nd eqtn.): $y - 3z = -1$; 2×(2nd eqtn.) − (1st eqtn.): $x + 5z = 5$	1A
∴ S.S. of (*) = $\{(5-5t, 3t-1, t): t \in \mathbf{R}\}$	1M+1A
or $\{(\frac{10-5t}{3}, t, \frac{1+t}{3}): t \in \mathbf{R}\}$	
or $\{(t, \frac{10-3t}{5}, \frac{5-t}{5}): t \in \mathbf{R}\}$	
(b) By (a), substitute $x = 5-5t$, $y = 3t-1$ and $z = t$ into $xy + yz + zx = 2$:	
$(5-5t)(3t-1) + (3t-1)t + (5-5t)t = 2$	1M
$17t^2 - 24t + 7 = 0$	
$t = 1$ or $\frac{7}{17}$	1A
∴ S.S. = $\left\{(0, 2, 1), \left(\frac{50}{17}, \frac{4}{17}, \frac{7}{17}\right)\right\}$.	1A+1A
(c) By (a), substitute $x = 5-5t$, $y = 3t-1$ and $z = t$ into $ax + y + z = \lambda$:	
$a(5-5t) + (3t-1) + t = \lambda$	1M
$(4-5a)t = \lambda - 5a + 1$	
Alternatively, $\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 4 \\ a & 1 & 1 & \lambda \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \\ 0 & 1-a & 1-2a & \lambda-4a \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 4-5a & \lambda-5a+1 \end{array}\right)$	1M
∴ The system of eqtns. is solvable if (i) $4-5a \neq 0$	1A
or (ii) $4-5a = 0$ and $\lambda - 5a + 1 = 0$.	1A
Thus the possible values of a and λ are elements of	
$\left\{(a, \lambda): a \neq \frac{4}{5}, \lambda \in \mathbf{R}\right\} \cup \left\{\left(\frac{4}{5}, 3\right)\right\}$.	1A
(d) By (c), substitute $(0, 2, 1)$ and $\left(\frac{50}{17}, \frac{4}{17}, \frac{7}{17}\right)$ into $ax + y + z = \lambda$ respectively.	
(i) $a(0) + 2 + 1 = \lambda$	
$\lambda = 3$	1A
(ii) $a\left(\frac{50}{17}\right) + \frac{4}{17} + \frac{7}{17} = \lambda$	
$a = \frac{17\lambda - 11}{50}$	1A
Thus the possible values of a and λ are elements of	
$\{(a, \lambda): \lambda = 3, a \in \mathbf{R}\} \cup \left\{(a, \lambda): a = \frac{17\lambda - 11}{50}\right\}$.	1A+1A

Solution	Marks
10. (a) (i) $\overrightarrow{AB} \times \overrightarrow{AC} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} - \mathbf{j} + 2\mathbf{k})$	1A
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 4 & -1 & 2 \end{vmatrix}$	
$= -\mathbf{i} + 2\mathbf{k}$	1A
(ii) Area of $\Delta ABC = \frac{1}{2}\left\lvert \overrightarrow{AB} \times \overrightarrow{AC} \right\rvert$	1M
$= \frac{1}{2}\lvert -\mathbf{i} + 2\mathbf{k} \rvert$	
$= \frac{\sqrt{5}}{2}$	1A
Distance from C to AB $= \dfrac{2(\text{Area of } \Delta ABC)}{\left\lvert \overrightarrow{AB} \right\rvert}$	1M
$= 2 \cdot \frac{\sqrt{5}}{2} \cdot \frac{1}{\sqrt{2^2 + 1 + 1}}$	
$= \sqrt{\frac{5}{6}}$ (or $\frac{\sqrt{30}}{6}$)	1A
(iii) Let the plane passing through A, B and C be π.	
then $\overrightarrow{AB} \times \overrightarrow{AC} = -\mathbf{i} + 2\mathbf{k}$ is a normal vector to π.	1M
Let the equation of π be $-x + 2z + k = 0$ ....(*)	1A
Substitute $A(1, 2, 1)$ into (*), then $k = -1$, $\therefore$ the equation of π is $-x + 2z - 1 = 0$ (or $x - 2z + 1 = 0$).	1A
Altenatively, The equation of π is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ where $\mathbf{n} = -\mathbf{i} + 2\mathbf{k}$	1M
and $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.	1A

Solution	Marks
(b) $\because$ D lies on π,	
$\therefore$ $x - 2z = -1$ ....(1)	1M+1A
$\because$ $\overrightarrow{AD} \cdot \overrightarrow{BC} = 0$	
$\therefore$ $[(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z-1)\mathbf{k}] \cdot [2\mathbf{i} + \mathbf{k}] = 0$	
$\Rightarrow$ $2x + z = 3$ ....(2)	1A
$\because$ $\overrightarrow{BD} \cdot \overrightarrow{AC} = 0$	
$\therefore$ $[(x-3)\mathbf{i} + (y-1)\mathbf{j} + (z-2)\mathbf{k}] \cdot [4\mathbf{i} - \mathbf{j} + 2\mathbf{k}] = 0$	
$\Rightarrow$ $4x - y + 2z = 15$ ....(3)	1A
Note: Candidates use $\overrightarrow{CD} \cdot \overrightarrow{AB} = 0$ will get $2x - y + z = 12$ which is (3) − (2).	
Solving (1), (2) and (3): $\left(\begin{array}{ccc\|c} -1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 3 \\ 4 & -1 & 2 & 15 \end{array}\right) \sim \left(\begin{array}{ccc\|c} -1 & 0 & 2 & 1 \\ 0 & -1 & 10 & 19 \\ 0 & 0 & 5 & 5 \end{array}\right)$	1M
$x = 1, y = -9, z = 1$. i.e. $\overrightarrow{OD} = \mathbf{i} - 9\mathbf{j} + \mathbf{k}$.	1A

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Solution	Marks
11. (a) Let $f(x) = x^4 - 3x^2 + k$	
If $\alpha = \beta$, then α is a root of $f(x) = 0$ with multiplicity ≥ 2	1
$\Rightarrow$ α is a root of $f'(x) = 0$	1
i.e. $4x^3 - 6x = 0$	
$\Rightarrow$ $\alpha = 0$, $\pm\frac{\sqrt{6}}{2}$	1
$\Rightarrow$ $\alpha + \beta = 2\alpha \neq 2$	1
which contradicts that $\alpha + \beta = 2$	1
Hence $\alpha \neq \beta$	
<u>Alternatively,</u>	
If $\alpha = \beta$, then $\alpha = 1$.	1
Sub. into $f(x) = 0$ gives $k = 2$	1
Solving $x^4 - 3x^2 + 2 = 0$, we have $x = \pm 1$ or $\pm\sqrt{2}$	1+1
$\Rightarrow$ Sum of any two roots $\neq 2$ which contradicts that $\alpha + \beta = 2$ **or** No two roots are equal which contradicts that $\alpha = \beta$.	1
Hence $\alpha \neq \beta$	
(b) $\because$ $\alpha^4 - 3\alpha^2 + k = 0$	
$\therefore$ $(\alpha^2)^2 + 3(\alpha^2) + k = 0$	1
$\Rightarrow$ α^2 is a root of $y^2 - 3y + k = 0$	
Similarly, β^2 is a root of $y^2 - 3y + k = 0$	
$\because$ $\alpha \neq \beta$ and $\alpha \neq -\beta$ $(\because \alpha + \beta = 2)$	
$\therefore$ $\alpha^2 \neq \beta^2$	1
$\because$ $(\alpha + \beta)^2 = 4$	1A
$\therefore$ $\alpha^2 + \beta^2 + 2\alpha\beta = 4$	
$3 + 2\alpha\beta = 4$	
$\alpha\beta = \frac{1}{2}$	1A
$k = (\alpha\beta)^2 = \frac{1}{4}$	1A

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Solution	Marks
(c) $x^4 - 3x^2 + \frac{1}{4} = 0$	
$4x^4 - 12x^2 + 1 = 0$	
$x^2 = \frac{12 \pm \sqrt{144-16}}{8}$	1A
$= \frac{3}{2} \pm \sqrt{2}$	1A
$= \frac{6 \pm 4\sqrt{2}}{4}$	
$= \frac{(2 \pm \sqrt{2})^2}{2^2}$	
$\therefore\ x = \pm\left(1 \pm \frac{\sqrt{2}}{2}\right)$	1A+1A
$\because\ \alpha + \beta = 2$	
$\therefore$ The values of α and β are $1 \pm \frac{\sqrt{2}}{2}$.	1A

Solution	Marks
12. (a) $z\bar{z} = a\bar{z} + \bar{a}z + b$	1A
$\Rightarrow (z\bar{z} - a\bar{z} - \bar{a}z + a\bar{a}) = a\bar{a} + b$	1A
$\Rightarrow (z-a)(\overline{z-a}) = \|a\|^2 + b$	1A
$\Rightarrow \|z-a\|^2 = \|a\|^2 + b$	
$\Rightarrow \|z-a\| = \sqrt{\|a\|^2 + b}$ $\quad (\because \|z-a\| \geq 0)$	1
(b) $\because PA = \sqrt{2}PB$	
$\therefore \|z-(2+3i)\| = \sqrt{2}\|z-(1+2i)\|$	1A
$\left[z-(2+3i)\right]\left[\bar{z}-(2-3i)\right] = 2\left[z-(1+2i)\right]\left[\bar{z}-(1-2i)\right]$	1A
$z\bar{z}-(2+3i)\bar{z}-(2-3i)z+13 = 2\left[z\bar{z}-(1+2i)\bar{z}-(1-2i)z+5\right]$	
$z\bar{z} = i\bar{z} - iz + 3$	1
Hence $\mathscr{C}$ has equation $z\bar{z} = i\bar{z} - iz + 3$	
By (a), centre of $\mathscr{C} = i$,	1A
radius of $\mathscr{C} = \sqrt{\|i\|^2 + 3} = 2$.	1A
(c) Denote the circle $\left\|z - \left(\omega + \dfrac{\omega - i}{\|\omega - i\|}\right)\right\| = 1$ by $\mathscr{C}'$.	
Distance between the centres of $\mathscr{C}$ and $\mathscr{C}'$	
$= \left\|i - (\omega + \dfrac{\omega - i}{\|\omega - i\|})\right\|$	1A
$= \left\|\dfrac{(i-\omega)\|\omega - i\| + (i-\omega)}{\|\omega - i\|}\right\|$	
$= \|\omega - i\| + 1$	1A
$=$ radius of $\mathscr{C}$ + radius of $\mathscr{C}'$	1
$\therefore$ $\mathscr{C}'$ touches $\mathscr{C}$ externally.	
As ω satisfies both the equations of $\mathscr{C}$ and $\mathscr{C}'$,	
Q is the point of contact of the two circles.	1
The equations of the two circles with radius r which touch $\mathscr{C}$ at Q are	
$\left\|z - \left(\omega \pm \dfrac{r(\omega - i)}{\|\omega - i\|}\right)\right\| = r$.	1A+1A

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Solution	Marks
13. (a) The inequality clearly holds for $m = 1$.	1A
Assume $\left(\frac{a_1+a_2}{2}\right)^k \le \frac{a_1^k+a_2^k}{2}$ for some positive integer k, then	
$\left(\frac{a_1+a_2}{2}\right)^{k+1} \le \left(\frac{a_1^k+a_2^k}{2}\right)\left(\frac{a_1+a_2}{2}\right)$	1A
$= \frac{a_1^{k+1}+a_2^{k+1}}{4} + \frac{a_1^k a_2 + a_1 a_2^k}{4}$	
$\le \frac{a_1^{k+1}+a_2^{k+1}}{2}$	1
$\left(\because \frac{a_1^{k+1}+a_2^{k+1}}{4} - \frac{a_1^k a_2 + a_1 a_2^k}{4} = \frac{(a_1^k - a_2^k)(a_1 - a_2)}{4} \ge 0 \right)$	1A
$\therefore$ The statement holds for all positive integers m.	
Alternatively,	
The inequality clearly holds for $m = 1$.	1A
Let $f(x) = x^m$ for $m \ge 2$, then	1A
$f''(x) = m(m-1)x^{m-2} \ge 0 \quad \forall\, x > 0$	1A
By convexity, $\frac{a_1^m + a_2^m}{2} \ge \left(\frac{a_1+a_2}{2}\right)^m$	1
Alternatively,	
W.l.g., assume $a_1 \ge a_2 > 0$.	
Let $u = \frac{a_1+a_2}{2}$, $v = \frac{a_1-a_2}{2}$, then $a_1 = u+v$, $a_2 = u-v$	1A
$\frac{a_1^m+a_2^m}{2} = \frac{1}{2}\left[(u+v)^m + (u-v)^m\right]$	1A
$= \left[u^m + C_2^m u^{m-2}v^2 + \ldots\right]$	1A
$\ge u^m$	
$= \left(\frac{a_1+a_2}{2}\right)^m$	1
(b) By (a),	
$\frac{a_1^m+a_2^m+a_3^m+a_4^m}{4} = \frac{1}{2}\left[\frac{a_1^m+a_2^m}{2} + \frac{a_3^m+a_4^m}{2}\right]$	
$\ge \frac{1}{2}\left[\left(\frac{a_1+a_2}{2}\right)^m + \left(\frac{a_3+a_4}{2}\right)^m\right]$	1A
$\ge \left(\frac{\frac{a_1+a_2}{2} + \frac{a_3+a_4}{2}}{2}\right)^m$	1A
$= \left(\frac{a_1+a_2+a_3+a_4}{4}\right)^m$	1

Solution	Marks
(c) (i) If $\dfrac{a_1^m+a_2^m+\dots+a_{2^h}^m}{2^h} \ge \left(\dfrac{a_1+a_2+\dots+a_{2^h}}{2^h}\right)^m$.	
then $\dfrac{a_1^m+a_2^m+\dots+a_{2^{h+1}}^m}{2^{h+1}}$	
$= \dfrac{1}{2}\left[\dfrac{a_1^m+a_2^m+\dots+a_{2^h}^m}{2^h}+\dfrac{a_{2^h+1}^m+a_{2^h+2}^m+\dots+a_{2^{h+1}}^m}{2^h}\right]$	
$\ge \dfrac{1}{2}\left[\left(\dfrac{a_1+a_2+\dots+a_{2^h}}{2^h}\right)^m+\left(\dfrac{a_{2^h+1}+a_{2^h+2}+\dots+a_{2^{h+1}}}{2^h}\right)^m\right]$	1A
$\ge \left(\dfrac{\dfrac{a_1+a_2+\dots+a_{2^h}}{2^h}+\dfrac{a_{2^h+1}+a_{2^h+2}+\dots+a_{2^{h+1}}}{2^h}}{2}\right)^m$	1A
$= \left(\dfrac{a_1+a_2+\dots+a_{2^{h+1}}}{2^{h+1}}\right)^m$	1
$\therefore$ P(2^{h+1}) is true.	
(ii) If $\dfrac{a_1^m+a_2^m+\dots+a_{k+1}^m}{k+1} \ge \left(\dfrac{a_1+a_2+\dots+a_{k+1}}{k+1}\right)^m$.	
then $\dfrac{a_1^m+a_2^m+\dots+a_k^m+\left(\dfrac{a_1+a_2+\dots+a_k}{k}\right)^m}{k+1}$	1A
$\ge \left(\dfrac{a_1+a_2+\dots+a_k+\dfrac{a_1+a_2+\dots+a_k}{k}}{k+1}\right)^m$	
$= \left(\dfrac{a_1+a_2+\dots+a_k}{k}\right)^m$	1A
Hence $a_1^m+a_2^m+\dots+a_k^m+\left(\dfrac{a_1+a_2+\dots+a_k}{k}\right)^m \ge (k+1)\left(\dfrac{a_1+a_2+\dots+a_k}{k}\right)^m$	
$\Rightarrow \dfrac{a_1^m+a_2^m+\dots+a_k^m}{k} \ge \left(\dfrac{a_1+a_2+\dots+a_k}{k}\right)^m$	1
(iii) P(2^0) is clearly true.	
By (c)(i) and the principle of M.I.,	
P(2^k) is true for all non-negative integers k.	
For any natural number n, there exists non-negative integer k	
such that $n \le 2^k$.	1
As P(2^k) is true, by (c)(ii), P(n) must also be true.	1
$\therefore$ P(n) is true for all natural numbers n.	

Solution	Marks	
1. By Leibniz's Theorem,		
$f^{(2n)}(x) = \sum_{k=0}^{2n} C_k^{2n}\left[\frac{d^k}{dx^k}(x^n)\frac{d^{2n-k}}{dx^{2n-k}}e^{ax}\right]$	1M	For $\sum_k^N C_k^N f^{(k)} g^{(N-k)}$
$= C_0^{2n}(x^n)(a^{2n}e^{ax}) + C_1^{2n}(nx^{n-1})(a^{2n-1}e^{ax}) + \cdots + C_n^{2n}(n!)(a^n e^{ax})$	1A+1A	1A for correct derivatives
$f^{(2n)}(0) = C_n^{2n} n! a^n$	1A	
Alternatively,		
Let $g(x) = x^n$, then $g^{(k)}(0) = \begin{cases} n! & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$	1A	
Let $h(x) = e^{ax}$, then $h^{(k)}(0) = a^k$.	1A	
Hence $f^{(2n)}(0) = \sum_{k=0}^{2n} C_k^{2n} g^{(k)}(0)\, h^{(2n-k)}(0)$	1M	
$= C_n^{2n} n! a^n$	1A	
	(4)	
2. (a) For $0 \le x \le 1$ and $n = 1, 2, 3, \ldots$,		
$\because 2 \ge 1 + x^2$, $\therefore \frac{nx^n}{2} \le \frac{nx^n}{1+x^2} \Rightarrow \int_0^1 \frac{nx^n}{2}dx \le a_n$	1	
$\because 1 + x^2 \ge 2x$, $\therefore \frac{nx^n}{1+x^2} \le \frac{nx^{n-1}}{2} \Rightarrow a_n \le \int_0^1 \frac{nx^{n-1}}{2}dx$	1	
(b) For $n = 1, 2, 3, \ldots$,		
$\int_0^1 \frac{nx^n}{2}dx = \frac{n}{2}\left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{n}{2(n+1)}$	1A	
$\int_0^1 \frac{nx^{n-1}}{2}dx = \frac{n}{2}\left[\frac{x^n}{n}\right]_0^1 = \frac{1}{2}$	1A	
As $\lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}$ and $\frac{n}{2(n+1)} \le a_n \le \frac{1}{2}$ by (a),		
$\lim_{n\to\infty} a_n = \frac{1}{2}$ by the Sandwich Theorem.	1A	
	(5)	

Solution	Marks
3. (a) Let $u = a - x$, then	
$\int_0^a f(x)dx = \int_a^0 f(a-u)(-du) = \int_0^a f(a-x)dx$	1
If $f(x) + f(a-x) = K$ for $x \in [0, a]$,	
(i) in particular, $K = f(\frac{a}{2}) + f(a - \frac{a}{2}) = 2f(\frac{a}{2})$.	1
(ii) $\int_0^a f(x)dx = \frac{1}{2}\int_0^a [f(x) + f(a-x)]dx$	1A
$= \frac{1}{2}a \cdot 2f(\frac{a}{2})$	
$= af(\frac{a}{2})$	1
(b) Let $f(x) = \frac{1}{e^{\sin x}+1}$, then $f(x)$ is continuous on $[0, 2\pi]$ and	
$f(x) + f(2\pi - x) = \frac{1}{e^{\sin x}+1} + \frac{1}{e^{-\sin x}+1} = 1$	1
By (a), $\int_0^{2\pi} \frac{1}{e^{\sin x}+1}dx = 2\pi\left(\frac{1}{e^{\sin \pi}+1}\right)$	1A
$= \pi$	
	(6)
4. (a) For $x \neq 0$, $f'(x) = 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}$	1A+1A
(b) $\left\|\frac{f(x)-f(0)}{x-0}\right\| = \left\|\frac{x^3 \sin\frac{1}{x}}{x}\right\| \leq \|x^2\|$ for $x \neq 0$.	1A
As $\lim_{x\to 0}\|x^2\| = 0$, $f'(0)$ exists and $f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0$.	1
(c) $\|f'(x)\| \leq \left\|3x^2 \sin\frac{1}{x}\right\| + \left\|x\cos\frac{1}{x}\right\|$	
$\leq 3\|x\|^2 + \|x\|$	1A
$\to 0$ as $x \to 0$.	
By the Sandwich Theorem, $\lim_{x\to 0} f'(x) = 0 = f'(0)$,	
$f'(x)$ is continuous at $x = 0$.	1
	(6)

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Solution	Marks	
5. (a) $\int x\ln x\,dx = \int \ln x \frac{d}{dx}\left(\frac{x^2}{2}\right)dx$		
$= \left(\frac{x^2}{2}\right)\ln x - \int\left(\frac{x^2}{2}\right)\frac{d(\ln x)}{dx}dx$	1A	
$= \left(\frac{x^2}{2}\right)\ln x - \int\frac{x}{2}dx$		
$= \frac{x^2\ln x}{2} - \frac{x^2}{4} + c$	1A	
$\int\left(\frac{\ln x}{x}\right)dx = \int \ln x\, d(\ln x)$		
$= \frac{(\ln x)^2}{2} + c$	1A	
(b) Surface area $= 2\pi\int_1^e y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$	1A	For the formula
$= 2\pi\int_1^e\left(\frac{x^2}{2}-\frac{\ln x}{4}\right)\sqrt{1+\left(x-\frac{1}{4x}\right)^2}dx$		
$= 2\pi\int_1^e\left(\frac{x^2}{2}-\frac{\ln x}{4}\right)\left(x+\frac{1}{4x}\right)dx$	1A	For simplifying the root
$= 2\pi\int_1^e\left(\frac{x^3}{2}+\frac{x}{8}-\frac{x\ln x}{4}-\frac{\ln x}{16x}\right)dx$		
$= 2\pi\left[\frac{x^4}{8}+\frac{x^2}{16}-\frac{x^2\ln x}{8}+\frac{x^2}{16}-\frac{(\ln x)^2}{32}\right]_1^e$		
$= 2\pi\left[\frac{x^4}{8}+\frac{x^2}{8}-\frac{x^2\ln x}{8}-\frac{(\ln x)^2}{32}\right]_1^e$		
$= 2\pi\left(\frac{e^4}{8}+\frac{e^2}{8}-\frac{e^2}{8}-\frac{1}{32}-\frac{1}{8}-\frac{1}{8}\right)$	1A	
$= \frac{\pi(4e^4-9)}{16}$		
	(6)	

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Solution	Marks	
6. (a) $\lim_{x\to a}\dfrac{\int_a^x f(t)\,dt-\frac{1}{2}(x-a)(f(x)+f(a))}{(x-a)^3}$		
$=\lim_{x\to a}\dfrac{f(x)-\frac{1}{2}(f(x)+f(a))-\frac{1}{2}(x-a)f'(x)}{3(x-a)^2}$	1M+1A	
$=\lim_{x\to a}\dfrac{\frac{1}{2}f'(x)-\frac{1}{2}f'(x)-\frac{1}{2}(x-a)f''(x)}{6(x-a)}$		
$=-\lim_{x\to a}\dfrac{f''(x)}{12}$		Can be omitted
$=-\dfrac{f''(a)}{12}$	1	
(b) If $\left\lvert\int_a^x f(t)\,dt-\frac{1}{2}(x-a)(f(x)+f(a))\right\rvert\le K(x-a)^4$,		
then $0\le\left\lvert\dfrac{\int_a^x f(t)\,dt-\frac{1}{2}(x-a)(f(x)+f(a))}{(x-a)^3}\right\rvert\le K\lvert x-a\rvert$ for $x\neq a$.	1M	For dividing by $(x-a)^3$
Hence $f''(a)=-12\lim_{x\to a}\dfrac{\int_a^x f(t)\,dt-\frac{1}{2}(x-a)(f(x)+f(a))}{(x-a)^3}=0$ for all a.	1A	
$\Rightarrow$ $f'(x)=c_1$ for some constant c_1. $\Rightarrow$ $f(x)=c_1x+c_2$ for some constants c_1 and c_2.	1	Awarded for the argument
	(6)	

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Solution	Marks
7. (a) By the Mean Value Theorem, $\frac{e^x-1}{x}=e^c$ for some $c\in(0,x)$.	1A
For $0<x\le 1$, $e\ge e^x>e^c>e^0=1$.	
$\therefore e>\frac{e^x-1}{x}>1 \Rightarrow 1+ex>e^x>1+x$	1
Alternatively,	
Let $f(x)=1+ex-e^x$ for $0<x\le1$, then $f'(x)=e-e^x>0$ for $0<x<1$.	
Hence $f(x)>f(0)=0$, i.e. $1+ex<e^x$ for $0<x\le1$.	1M
Let $g(x)=e^x-1-x$ for $0<x\le1$, then $g'(x)=e^x-1>0$ for $0<x<1$.	
Hence $g(x)>g(0)=0$, i.e. $e^x<1+x$ for $0<x\le1$.	1A
(b) By (a), the statement is true for $n=0$.	1M
Assume $\sum_{r=0}^{k}\frac{x^r}{r!}+\frac{ex^{k+1}}{(k+1)!}>e^x>\sum_{r=0}^{k+1}\frac{x^r}{r!}$ for some integer $k\ge0$ and $x\in(0,1]$, then	
$\int_0^x\left(\sum_{r=0}^{k}\frac{t^r}{r!}+\frac{et^{k+1}}{(k+1)!}\right)dt>\int_0^x e^t\,dt>\int_0^x\left(\sum_{r=0}^{k+1}\frac{t^r}{r!}\right)dt$	1M
$\left[\sum_{r=0}^{k}\frac{t^{r+1}}{(r+1)!}+\frac{et^{k+2}}{(k+2)!}\right]_0^x>\left[e^t\right]_0^x>\left[\sum_{r=0}^{k+1}\frac{t^{r+1}}{(r+1)!}\right]_0^x$	
$\sum_{r=0}^{k}\frac{x^{r+1}}{(r+1)!}+\frac{ex^{k+2}}{(k+2)!}>e^x-1>\sum_{r=0}^{k+1}\frac{x^{r+1}}{(r+1)!}$	
$1+\sum_{r=0}^{k}\frac{x^{r+1}}{(r+1)!}+\frac{ex^{k+2}}{(k+2)!}>e^x>1+\sum_{r=0}^{k+1}\frac{x^{r+1}}{(r+1)!}$	
$\sum_{r=0}^{k+1}\frac{x^r}{r!}+\frac{ex^{k+2}}{(k+2)!}>e^x>\sum_{r=0}^{k+2}\frac{x^r}{r!}$	1
for $x\in(0,1]$. By the principle of M.I., the statement holds for $n=0,1,2,\ldots$ and $x\in(0,1]$.	
Putting $x=1$,	
$1+1+\frac{1}{2!}+\ldots+\frac{1}{n!}+\frac{e}{(n+1)!}>e>1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{(n+1)!}$	1A
$e-\frac{e}{(n+1)!}<1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}<e-\frac{1}{(n+1)!}$	
As $\lim_{n\to\infty}\frac{e}{(n+1)!}=\lim_{n\to\infty}\frac{1}{(n+1)!}=0$.	
By the Sandwich Theorem, $\lim_{n\to\infty}\left(1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)=e$.	1
	(7)

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Solution	Marks
8. (a) For $x \neq -1$, $f'(x) = \dfrac{(x-1)^2(x+5)}{(x+1)^3}$	1A
$f''(x) = \dfrac{24(x-1)}{(x+1)^4}$	1A
(b) (i) $f'(x) > 0 \Rightarrow x < -5$ or $-1 < x < 1$ or $x > 1$ $f'(x) > 0$ on $(-\infty, -5) \cup (-1, 1) \cup (1, \infty)$	1A
(ii) $f'(x) < 0 \Rightarrow -5 < x < -1$ $f'(x) < 0$ on $(-5, -1)$	1A
(iii) $f''(x) > 0 \Rightarrow x > 1$ $f''(x) > 0$ on $(1, \infty)$	
(iv) $f''(x) < 0 \Rightarrow x < 1$ and $x \neq -1$ $f''(x) < 0$ on $(-\infty, -1) \cup (-1, 1)$	1A For (iii) & (iv)

(c)

x	$(-\infty, -5)$	-5	$(-5, -1)$	-1	$(-1, 1)$	1	$(1, \infty)$
$f(x)$	↑	$-\frac{27}{2}$	↓	undefined	↑	0	↑
$f'(x)$	+	0	−	undefined	+	0	+
$f''(x)$	−	−	−	undefined	−	0	+

Solution	Marks
$\therefore$ $(-5, -\frac{27}{2})$ is a relative maximum point.	1A
$(1, 0)$ is a point of inflexion.	1A
(d) The vertical asymptote is $x = -1$.	1A
$\because$ $f(x) = \dfrac{(x-1)^3}{(x+1)^2} = x - 5 + \dfrac{12x+4}{(x+1)^2}$	
$\therefore$ The oblique asymptote is $y = x - 5$	1A

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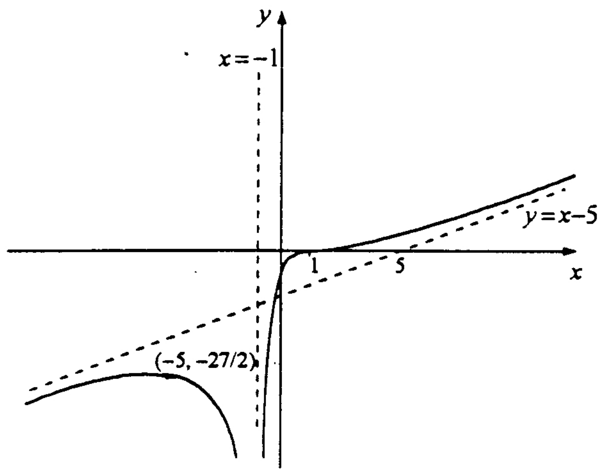
Solution	Marks

(e)



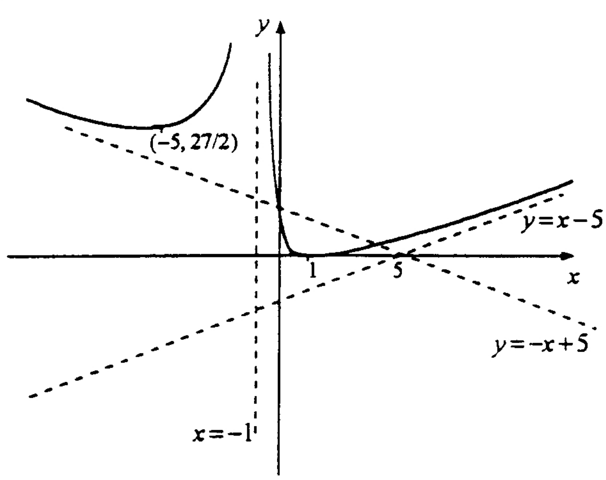
1A+1A

(f) $\because \quad \lim_{x\to 1^-} g(x) = \lim_{h\to 0}\dfrac{g(1-h)-g(1)}{h} = \lim_{h\to 0}\dfrac{h^3}{h(2-h)^2} = 0$

$\lim_{x\to 1^+} g(x) = \lim_{h\to 0}\dfrac{g(1+h)-g(1)}{h} = \lim_{h\to 0}\dfrac{h^3}{h(2+h)^2} = 0$

$\therefore \quad g'(1)$ exists and $g'(1) = 0$ — 1

By symmetry,
the vertical asymptote of $g(x)$ is $x = -1$,
the oblique asymptotes of $g(x)$ are $y = x - 5$ (for $x > 1$) — 1A
and $y = -x + 5$ (for $x < -1$) — 1A



1A

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Solution	Marks
9. (a) (i) L is the tangent to $\mathcal{E}$ at $(-2, 1)$.	
Equation of L: $\frac{-2x}{8}+\frac{y}{2}=1$	
$y=\frac{1}{2}x+2$ (or $x-2y+4=0$)	1A
Sub. $y=\frac{1}{2}x+2$ into $\mathcal{P}$:	
$\frac{1}{2}x+2=kx^2+3$	
$2kx^2-x+2=0$ ....(*)	
$\because$ L touches $\mathcal{P}$	
$\therefore$ $1-16k=0$	1M
$k=\frac{1}{16}$.	1A
(ii) Using (*) to solve $\mathcal{P}$ and L, we have	
$x^2-8x+16=0$	
$x=4$	
$\therefore$ The coordinates of the point at which L touches $\mathcal{P}$ is $(4, 4)$.	1A
(b) Area enclosed by L, $\mathcal{P}$ and the y-axis.	
$=\int_0^4\left[\left(\frac{1}{16}x^2+3\right)-\left(\frac{1}{2}x+2\right)\right]\mathrm{d}x$	1M+1M
$=\int_0^4\left(\frac{1}{16}x^2-\frac{1}{2}x+1\right)\mathrm{d}x$	
$=\left[\frac{1}{48}x^3-\frac{1}{4}x^2+x\right]_0^4$	1A
$=\frac{4}{3}$	1A

Solution	Marks
(c) Let $y = mx + c$ be a common tangent of $\mathcal{E}$ and $\mathcal{P}$.	
Solving $y = mx + c$ and $\frac{x^2}{8} + \frac{y^2}{2} = 1$, we have	
$(4m^2 + 1)x^2 + 8mcx + 4c^2 - 8 = 0$.	1M
Hence $64m^2c^2 - 4(4m^2 + 1)(4c^2 - 8) = 0$	
$8m^2 - c^2 + 2 = 0$ .... (1)	1A
Solving $y = mx + c$ and $y = \frac{1}{16}x^2 + 3$, we have	
$x^2 - 16mx + 48 - 16c = 0$.	
Hence $256m^2 - 4(48 - 16c) = 0$	
$4m^2 + c - 3 = 0$ .... (2)	1A
Solving (1) and (2), we have	
$c^2 + 2c - 8 = 0$	
$c = 2$ or -4 .	1A
For $c = 2$, $m = \pm\frac{1}{2}$	
For $c = -4$, $m = \pm\frac{\sqrt{7}}{2}$.	
∴ The remaining three common tangents of $\mathcal{E}$ and $\mathcal{P}$ are	
$y = -\frac{1}{2}x + 2$ and $y = \pm\frac{\sqrt{7}}{2}x - 4$.	1A+1A+1A

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Solution	Marks
10. (a) (i) By the Mean Value Theorem, for any $k \geq 1$,	
$\frac{f(k+1)-f(k)}{(k+1)-k} = f'(\xi)$ for some $\xi \in (k, k+1)$.	1A
$\because$ $f'(x)$ is strictly decreasing for $x > 0$,	
$\therefore$ $f'(k+1) < f'(\xi) < f'(k)$	1A
$\Rightarrow$ $f'(k+1) < f(k+1) - f(k) < f'(k)$	1
Alternatively,	
$\because$ $f'(x)$ is strictly decreasing for $x > 0$,	
$\therefore$ $f'(k+1) < f'(t) < f'(k)$ for $t \in (k, k+1)$.	1A
$\Rightarrow$ $\int_k^{k+1} f'(k+1)dt < \int_k^{k+1} f'(t)\,dt < \int_k^{k+1} f'(k)dt$	1A
$\Rightarrow$ $f'(k+1) < f(k+1) - f(k) < f'(k)$	1
(ii) By (i), $\sum_{k=1}^{n-1} f'(k+1) < \sum_{k=1}^{n-1}(f(k+1) - f(k)) < \sum_{k=1}^{n-1} f'(k)$	1M
$\therefore$ $f'(2) + f'(3) + \cdots + f'(n) < f(n) - f(1) < f'(1) + f'(2) + \cdots + f'(n-1)$	1
(b) (i) Let $f(x) = \ln x$ for $x > 0$, then $f'(x) = \frac{1}{x}$	1A
and $f''(x) = -\frac{1}{x^2} < 0$, i.e. $f'(x)$ is strictly decreasing.	1
By (a)(ii), $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n - 0 < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$	
$H_n - 1 < \ln n < H_n - \frac{1}{n}$ for $n \geq 2$	1
$\Rightarrow$ $1 + \frac{1}{n \ln n} < \frac{H_n}{\ln n} < 1 + \frac{1}{\ln n}$	1A
Since $\lim_{n\to\infty}\left(1 + \frac{1}{n \ln n}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{\ln n}\right) = 1$	1A
$\therefore$ $\lim_{n\to\infty}\left(\frac{H_n}{\ln n}\right) = 1$	1A
(ii) By (a)(i), $\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$	1A
$\therefore$ $\gamma_{n+1} - \gamma_n = \frac{1}{n+1} + \ln n - \ln(n+1) < \frac{1}{n+1} - \frac{1}{n+1} = 0$	
$\{\gamma_n\}$ is decreasing.	1
$\{\gamma_n\}$ is bounded below because	
$\gamma_n = H_n - \ln n > \frac{1}{n} > 0$	1A
$\because$ $\{\gamma_n\}$ is decreasing and bounded below,	
$\therefore$ $\lim_{n\to\infty} \gamma_n$ exists.	1

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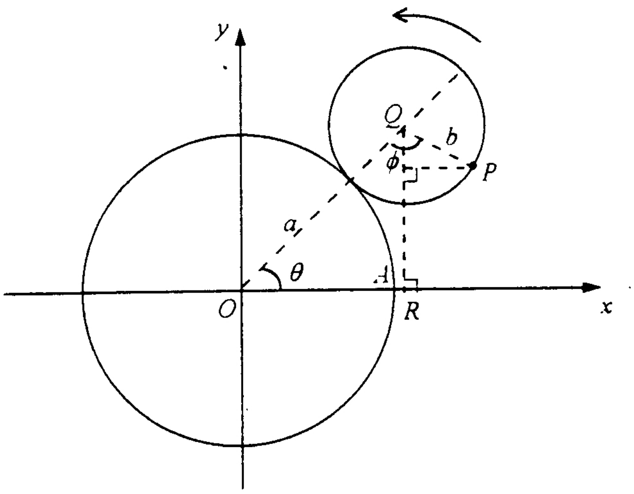
Solution	Marks

11. (a)



Let the centre of the rolling circle be Q and the projection of Q on the x-axis be R.

$$\angle PQR = \phi - \left(\frac{\pi}{2} - \theta\right)$$ 1A

$$= \frac{a}{b}\theta - \frac{\pi}{2} + \theta$$

$$= \left(\frac{a+b}{b}\right)\theta - \frac{\pi}{2}$$ 1A

$$\therefore \quad x = (a+b)\cos\theta + b\sin\left[\left(\frac{a+b}{b}\right)\theta - \frac{\pi}{2}\right]$$ 1M

$$= (a+b)\cos\theta - b\sin\left[\frac{\pi}{2} - \left(\frac{a+b}{b}\right)\theta\right]$$

$$= (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta$$ 1

$$y = (a+b)\sin\theta - b\cos\left[\left(\frac{a+b}{b}\right)\theta - \frac{\pi}{2}\right]$$ 1M

$$= (a+b)\sin\theta - b\cos\left[\frac{\pi}{2} - \left(\frac{a+b}{b}\right)\theta\right]$$

$$= (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta$$ 1

(b) (i) Locus of P when $a = 2b$:

$$\begin{cases} x = 3b\cos\theta - b\cos 3\theta \\ y = 3b\sin\theta - b\sin 3\theta \end{cases}$$ 1A

(ii)

$$\frac{dx}{d\theta} = 3b(\sin 3\theta - \sin\theta)$$

$$\frac{dy}{d\theta} = 3b(\cos\theta - \cos 3\theta)$$ 1A

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 3b\sqrt{2 - 2\sin\theta\sin 3\theta - 2\cos\theta\cos 3\theta}$$ 1A

$$= 3\sqrt{2}b\sqrt{1 - \cos 2\theta}$$ 1A

$$= 3\sqrt{2}b\sqrt{2\sin^2\theta}$$

$$= 6b\left|\sin\theta\right|$$ 1A

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Solution	Marks
(iii) P meets the fixed circle when $2b\theta = 2\pi b$	
i.e. $\theta = \pi$	1A
Distance travelled by P	
$= \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$	
$= \int_0^{\pi} 6b\lvert\sin\theta\rvert\, d\theta$	1A
$= 6b\int_0^{\pi} \sin\theta\, d\theta$	1A
$= 6b\left[-\cos\theta\right]_0^{\pi}$	
$= 12b$	1A

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Solution	Marks
12. (a) (i) $\mathrm{F}(k, 0) = \int_0^1 u^k \, \mathrm{d}u$	
$= \left[\frac{u^{k+1}}{k+1}\right]_0^1$	
$= \frac{1}{k+1}$	1
(ii) $\mathrm{F}(k, m) = \int_0^1 u^k (1-u^2)^m \, \mathrm{d}u$	
$= \int_0^1 (1-u^2)^m \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u^{k+1}}{k+1}\right) \mathrm{d}u$	1A
$= \frac{1}{k+1}\left[u^{k+1}(1-u^2)^m\right]_0^1 - \frac{1}{k+1}\int_0^1 u^{k+1} \cdot m(1-u^2)^{m-1}(-2u) \, \mathrm{d}u$	1A
$= \frac{2m}{k+1}\int_0^1 u^{k+2}(1-u^2)^{m-1} \, \mathrm{d}u$	
$= \frac{2m}{k+1}\mathrm{F}(k+2, m-1)$ for $m \geq 1$.	1
(b) By (a) (i), the statement holds for all k when $m = 0$.	1A
Assume it is also true for all k when $m = n$, then	1A
$\mathrm{F}(k, n+1) = \frac{2(n+1)}{k+1}\mathrm{F}(k+2, n)$ by (a) (ii)	1A
$= \frac{2(n+1)}{k+1} \cdot \frac{2^n (n!)}{(k+3)(k+5)\cdots(k+2n+3)}$	
$= \frac{2^{n+1} \cdot (n+1)!}{(k+1)(k+3)\cdots(k+2(n+1)+1)}$	1A
By the principle of M.I., the result follows.	
(c) Let $u = \sin\theta$, then	1A
$\int_0^{\frac{\pi}{2}} \cos^{2m+1}\theta \, \mathrm{d}\theta = \int_0^1 (1-u^2)^m \, \mathrm{d}u$	1A
$= \mathrm{F}(0, m)$	
$= \frac{2^m (m!)}{1 \cdot 3 \cdot 5 \cdots (2m+1)}$	1A
$= \frac{[2^m (m!)]^2}{(2m+1)!}$	1

Solution	Marks
(d) $F(k, m) = \int_0^1 u^k \left(\sum_{r=0}^{m} C_r^m (-u^2)^r \right) du$	1A
$= \sum_{r=0}^{m} (-1)^r C_r^m \int_0^1 u^{2r+k} \, du$	1A
$= \sum_{r=0}^{m} \frac{(-1)^r C_r^m}{2r+k+1} \left[u^{2r+k+1} \right]_0^1$	
$= \sum_{r=0}^{m} \frac{(-1)^r C_r^m}{2r+k+1}$	1

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Solution	Marks
13. (a) (i) $a_1 = x > 1$	
Assume $a_k > 1$ for some integer $k \geq 1$, then	
$(a_k - 1)^2 > 0$	1
$\Rightarrow \quad {a_k}^2 + 1 > 2a_k$	
$\Rightarrow \quad a_{k+1} = \dfrac{{a_k}^2 + 1}{2a_k} > 1$	1
By the principle of M.I., $a_n > 1$ for $n = 1, 2, 3, \ldots$	1
Now, $a_n - a_{n+1} = a_n - \dfrac{{a_n}^2 + 1}{2a_n}$	
$= \dfrac{{a_n}^2 - 1}{2a_n}$	
$= \dfrac{(a_n - 1)(a_n + 1)}{2a_n}$	1A
> 0 as $a_n > 1$.	
$\therefore\ a_n > a_{n+1}$ for $n = 1, 2, 3, \ldots$	1
(ii) By (i), $\{a_n\}$ is monotonic decreasing and bounded below.	
$\therefore \quad \lim\limits_{n \to \infty} a_n$ exists.	1
Let $\lim\limits_{n \to \infty} a_n = \ell$, then $\ell \geq 1$ and	
$\ell = \dfrac{\ell^2 + 1}{2\ell}$	1A
$\Rightarrow \quad \ell = 1$	1A
(b) For any $x > 1$, define a sequence $\{a_n\}$ as in (a).	1
Then $a_n > 1$ and $\lim\limits_{n \to \infty} a_n = 1$.	
$\because \quad f(x) = f\left(\dfrac{x^2 + 1}{2x}\right)$	
$\therefore \quad f(a_n) = f(a_{n+1})$	1A
$f(a_n)$ is a constant for all n.	1A
In particular, $f(x) = f(a_1) = f(a_n)$ for all n	1A
$\therefore \quad f(x) = \lim\limits_{n \to \infty} f(a_n)$	1
$= f(1)$ because f is continuous at 1	1
and $\lim\limits_{n \to \infty} a_n = 1$	1

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