SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C1) answer book.

1. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

(a) Evaluate $A^3 - 5A^2 + 8A - 4I$.

(b) Hence, or otherwise, find A^{-1}.

(6 marks)

2. (a) Let k and n be positive integers.
If $k > 1$, show that when $(1+k)^n$ is divided by k, the remainder is 1.

(b) If today is Tuesday, what day of the week is 8^{96} days after?

(4 marks)

3. Consider the equation

$$z^3 - az^2 + az - 1 = 0 \qquad \text{............(*)}$$

where a is real.

(a) Find a real root of (*).

(b) Find the range of values of a such that (*) has non-real roots.

(c) Show that all the non-real roots of (*) lie on the unit circle in the complex plane.

(5 marks)

4. (a) Show that $f(x) = \dfrac{x}{1+x}$ is an increasing function on $(-1, \infty)$.

(b) Using (a), or otherwise, show that $\dfrac{|r+s|}{1+|r+s|} \le \dfrac{|r|}{1+|r|} + \dfrac{|s|}{1+|s|}$

for any real numbers r and s.

(5 marks)

5. (a) Solve $\begin{cases} Z + Y \quad = a \\ Z \quad + X = b \\ \quad Y + X = c \end{cases}$ for X, Y and Z.

(b) If $a+b-c>0$, $b+c-a>0$ and $c+a-b>0$,

solve $\begin{cases} xy + xz \quad = a \\ xy \quad + yz = b \\ \quad xz + yz = c \end{cases}$ for x, y and z.

(6 marks)

6. A sequence $\{x_n\}$ is defined by $x_0 = 1$, $x_1 = 2$ and

$x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$ for $n \ge 2$.

(a) Write down the values of $x_2 - x_1$, $x_3 - x_2$ and $x_4 - x_3$.

(b) For $n = 1, 2, 3, \ldots$, guess an expression for $x_n - x_{n-1}$ in terms of n and prove it.

Hence find $\lim_{n\to\infty} x_n$.

(7 marks)

7. Let the roots of $x^3 + px^2 + qx + r = 0$ be α, β and γ.

(a) Show that $\alpha p + q = \beta\gamma - \alpha^2$.

(b) If the roots of $x^3 + Px^2 + Qx + R = 0$ are $\beta\gamma - \alpha^2$, $\gamma\alpha - \beta^2$ and $\alpha\beta - \gamma^2$, express P, Q and R in terms of p, q and r.

(7 marks)

SECTION B (60 marks)

Answer any **FOUR** questions in this section. Each question carries 15 marks.

Write your answers in the AL(C2) answer book.

8. (a) Solve the equation

$$\det\left[\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0 \qquad \text{..........(*)}.$$

(3 marks)

(b) Let λ_1, λ_2 ($\lambda_1 < \lambda_2$) be the roots of (*).

Find two non-zero vectors (x_1, y_1) and (x_2, y_2) such that

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \lambda_i\begin{pmatrix} x_i \\ y_i \end{pmatrix}, \qquad i = 1, 2.$$

Show that (x_1, y_1), (x_2, y_2) are linearly independent.

Let $P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$. Find P^{-1} and evaluate $P^{-1}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}P$.

(9 marks)

(c) Evaluate $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}^{1996}$.

(3 marks)

9. Consider the system of linear equations

$$(*): \begin{cases} x+2y-z=3 \\ x+y+2z=4 \end{cases} .$$

(a) Solve (*).

(3 marks)

(b) Find the solutions of (*) that satisfy $xy+yz+zx=2$.

(4 marks)

(c) Find all possible values of a and λ ($a, \lambda \in \mathbf{R}$) so that

$$\begin{cases} x+2y-z=3 \\ x+y+2z=4 \\ ax+y+z=\lambda \end{cases}$$

is solvable.

(4 marks)

(d) Using (b), or otherwise, find all possible values of a and λ ($a, \lambda \in \mathbf{R}$) so that

$$\begin{cases} x+2y-z=3 \\ x+y+2z=4 \\ xy+yz+zx=2 \\ ax+y+z=\lambda \end{cases}$$

has at least one solution.

(4 marks)

10. Let $\overrightarrow{OA}=\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\overrightarrow{OB}=3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\overrightarrow{OC}=5\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

(a) (i) Find $\overrightarrow{AB}\times\overrightarrow{AC}$.

(ii) Find the area of ΔABC .
Hence, or otherwise, find the distance from C to AB .

(iii) Find the equation of the plane passing through A , B and C .

(9 marks)

(b) The three altitudes of ΔABC are concurrent at D . Find $\overrightarrow{OD}$.

(6 marks)

11. Suppose the equation

$$x^4 - 3x^2 + k = 0 \qquad \text{..........}(*)$$

has two roots α, β such that $\alpha + \beta = 2$.

(a) Show that $\alpha \neq \beta$.

(5 marks)

(b) Show that α^2, β^2 are two distinct roots of the equation

$$y^2 - 3y + k = 0 \ .$$

Hence find the value of k.

(5 marks)

(c) Solve (*) and express the roots in the form of $\sqrt{a} \pm \sqrt{b}$ where a, b are rationals.

Hence find the values of α and β.

(5 marks)

12. (a) Let $a \in \mathbf{C}$ and $b \geq 0$. Show that the equation

$$z\bar{z} = a\bar{z} + \bar{a}z + b \qquad (z \in \mathbf{C})$$

can be written in the form of $|z - a| = r$ where $r \geq 0$.

(4 marks)

(b) Let A and B be two points on the complex plane representing $2 + 3i$ and $1 + 2i$ respectively. P, representing the complex number z, is a moving point so that $PA = \sqrt{2}PB$. Show that the equation of the locus of P is a circle with equation

$$\mathscr{C}: \quad z\bar{z} = i\bar{z} - iz + 3 \ .$$

Find its radius and centre.

(5 marks)

(c) Let Q, representing ω, be a point on the circle $\mathscr{C}$ in (b).

(i) Show that the circle $\left| z - \left(\omega + \dfrac{\omega - i}{|\omega - i|} \right) \right| = 1$ touches $\mathscr{C}$ at Q externally.

(ii) For any given $r > 0$, write down the equations of the two circles with radius r which touch $\mathscr{C}$ at Q.

(6 marks)

13. (a) Prove that $\left(\frac{a_1+a_2}{2}\right)^m \le \frac{a_1{}^m+a_2{}^m}{2}$

where a_1 , a_2 are positive and m is a positive integer.

(4 marks)

(b) Prove that $\left(\frac{a_1+a_2+a_3+a_4}{4}\right)^m \le \frac{a_1{}^m+a_2{}^m+a_3{}^m+a_4{}^m}{4}$

where a_1 , a_2 , a_3 , a_4 are positive and m is a positive integer.

(3 marks)

(c) For $n = 1, 2, 3, \ldots$, let $\mathrm{P}(n)$ be the statement

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m \le \frac{a_1{}^m+a_2{}^m+\cdots+a_n{}^m}{n} ,$$

where a_1 , a_2 , $\ldots$, a_n are positive and m is a positive integer.

(i) Prove that for $h = 0, 1, 2, \ldots$, if $\mathrm{P}(2^h)$ is true, then $\mathrm{P}(2^{h+1})$ is true.

(ii) Prove that for $k = 1, 2, 3, \ldots$, if $\mathrm{P}(k+1)$ is true, then $\mathrm{P}(k)$ is true.

(iii) Hence prove that $\mathrm{P}(n)$ is true for all positive integers n .

(8 marks)

END OF PAPER

96-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 1996

PURE MATHEMATICS A-LEVEL PAPER II

2.00 pm–5.00 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.

2. Answer ALL questions in Section A, using the AL(C1) answer book.

3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C1) answer book.

1. Let $f(x) = x^n e^{ax}$ where a is real and n is a positive integer.
   Evaluate $f^{(2n)}(0)$.

(4 marks)

2. Let $a_n = \int_0^1 \frac{nx^n}{1+x^2} dx$ for $n = 1, 2, 3, \ldots$.

   (a) Show that $\int_0^1 \frac{nx^n}{2} dx \le a_n \le \int_0^1 \frac{nx^{n-1}}{2} dx$ for $n = 1, 2, 3, \ldots$.

   (b) Evaluate $\lim_{n\to\infty} a_n$.

(5 marks)

3. (a) Suppose $f(x)$ is continuous on $[0, a]$.
   Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

   Further, if $f(x) + f(a-x) = K$ for all $x \in [0, a]$, where K is a constant, prove that

   (i) $K = 2f(\frac{a}{2})$;

   (ii) $\int_0^a f(x)dx = af(\frac{a}{2})$.

   (b) Hence, or otherwise, evaluate $\int_0^{2\pi} \frac{1}{e^{\sin x}+1} dx$.

(6 marks)

4. Let $f(x) = \begin{cases} x^3 \sin\frac{1}{x} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$.

   (a) Evaluate $f'(x)$ for $x \ne 0$.

   (b) Prove that $f'(0)$ exists .

   (c) Is $f'(x)$ continuous at $x = 0$? Explain.

(6 marks)

5. (a) Evaluate

   (i) $\int x \ln x \, dx$;

   (ii) $\int \left(\frac{\ln x}{x}\right) dx$.

   (b) Consider the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$ where $1 \le x \le e$.
   Find the area of the surface generated by rotating the curve about the x-axis.

(6 marks)

6. Let $f(x)$ be a function with continuous second order derivative.

(a) Prove that $\lim_{x \to a} \dfrac{\int_a^x f(t)dt - \frac{1}{2}(x-a)(f(x)+f(a))}{(x-a)^3} = -\dfrac{f''(a)}{12}$.

(b) Suppose there exists a constant K such that, for all x and a,

$$\left| \int_a^x f(t)dt - \frac{1}{2}(x-a)(f(x)+f(a)) \right| \le K(x-a)^4 .$$

Show that f is a polynomial of degree not greater than 1 .

(6 marks)

7. (a) Show that $1+ex > e^x > 1+x$ for $0 < x \le 1$.

(b) Using mathematical induction, or otherwise, prove that

$$\sum_{r=0}^{n} \frac{x^r}{r!} + \frac{ex^{n+1}}{(n+1)!} > e^x > \sum_{r=0}^{n+1} \frac{x^r}{r!}$$

for $n = 0, 1, 2, \ldots$ and $x \in (0, 1]$.

Hence show that $\lim_{n \to \infty} \left(1+1+\frac{1}{2}+\cdots+\frac{1}{n!} \right) = e$.

(7 marks)

SECTION B (60 marks)

Answer any **FOUR** questions in this section. Each question carries 15 marks.

Write your answers in the AL(C2) answer book.

8. Let $f(x) = \dfrac{(x-1)^3}{(x+1)^2}$.

(a) Find $f'(x)$ and $f''(x)$ for $x \neq -1$.

(2 marks)

(b) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$, (ii) $f'(x) < 0$,

(iii) $f''(x) > 0$, (iv) $f''(x) < 0$.

(3 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of $f(x)$.

(2 marks)

(d) Find the asymptote(s) of $f(x)$.

(2 marks)

(e) Sketch the graph of $f(x)$.

(2 marks)

(f) Let $g(x) = |f(x)|$. Does $g'(1)$ exist?

Find the asymptote(s) and sketch the graph of $g(x)$.

(4 marks)

9. Consider the curves

$$\mathcal{E} : \frac{x^2}{8}+\frac{y^2}{2}=1 \quad \text{and}$$

$$\mathcal{P} : y = kx^2+3 \quad (k>0) .$$

A common tangent L of $\mathcal{E}$ and $\mathcal{P}$ touches $\mathcal{E}$ at $(-2, 1)$.

(a) (i) Find the equation of L and the value of k .

(ii) Determine the coordinates of the point at which L touches $\mathcal{P}$.

(4 marks)

(b) Find the area enclosed by L , $\mathcal{P}$ and the y-axis.

(4 marks)

(c) Find the equations of the remaining three common tangents of $\mathcal{E}$ and $\mathcal{P}$.

(7 marks)

10. (a) Let $f(x)$ be a function such that $f'(x)$ is strictly decreasing for $x>0$.

(i) Using the Mean Value Theorem, or otherwise, show that

$$f'(k+1) < f(k+1)-f(k) < f'(k) \quad \text{for } k \geq 1 .$$

(ii) Using (i), show that for any integer $n \geq 2$,

$$\begin{aligned} f'(2)+f'(3)+\cdots+f'(n) &< f(n)-f(1) \\ &< f'(1)+f'(2)+\cdots+f'(n-1) . \end{aligned}$$

(5 marks)

(b) Define $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ for any positive integer n .

(i) Using (a), or otherwise, show that

$$H_n - 1 < \ln n < H_n - \frac{1}{n} \quad \text{for } n \geq 2 .$$

Hence, evaluate $\lim_{n\to\infty}\left(\frac{H_n}{\ln n}\right)$.

(ii) Define $\gamma_n = H_n - \ln n$.

Show that $\{\gamma_n\}$ is a decreasing sequence and $\lim_{n\to\infty}\gamma_n$ exists.

(10 marks)

11. Figure 1 shows a circle with radius b rolling externally without slipping on a fixed circle with radius a and centred at the origin. Let P be a point fixed on the rolling circle with initial position at $A(a, 0)$.

(a) Referring to Figure 2, show that the parametric equations of the locus of P are given by

$$\begin{cases} x = (a+b)\cos\theta - b\cos\left(\dfrac{a+b}{b}\right)\theta \\ y = (a+b)\sin\theta - b\sin\left(\dfrac{a+b}{b}\right)\theta \end{cases}.$$

(6 marks)

(b) Suppose $a = 2b$.

(i) Write down the parametric equations of the locus of P.

(ii) Express $\sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2}$ in terms of θ.

(iii) Find the distance travelled by P before it meets the fixed circle again.

(9 marks)

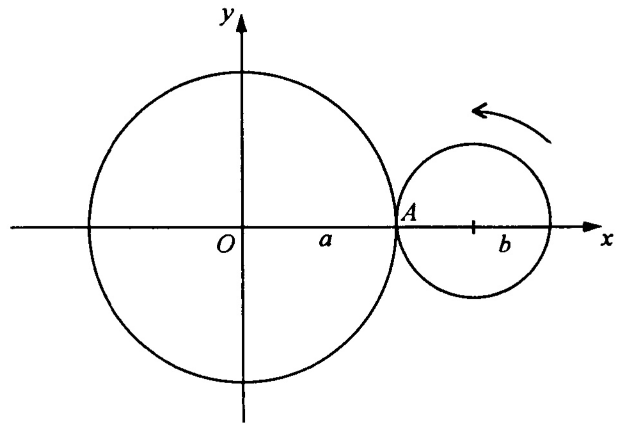

Figure 1

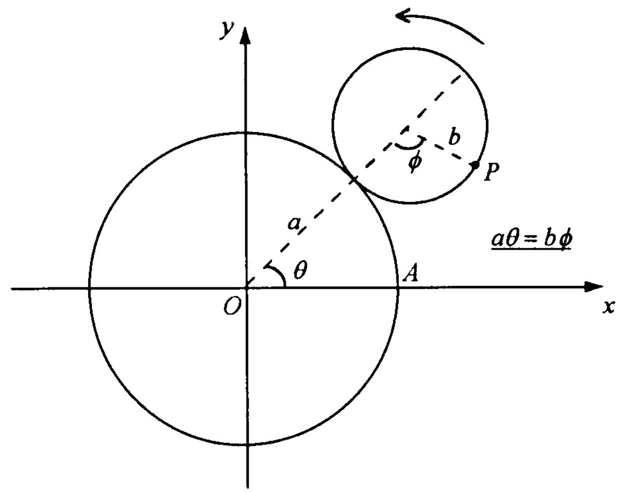

Figure 2

12. For non-negative integers k and m, define

$$F(k,m)=\int_0^1 u^k(1-u^2)^m\,du .$$

(a) Show that

(i) $F(k,0)=\dfrac{1}{k+1}$;

(ii) $F(k,m)=\dfrac{2m}{k+1}F(k+2,m-1)$ for $m\geq 1$.

(4 marks)

(b) Show that $F(k,m)=\dfrac{2^m(m!)}{(k+1)(k+3)\cdots(k+2m+1)}$.

(4 marks)

(c) Using (b), prove that $\displaystyle\int_0^{\frac{\pi}{2}}\cos^{2m+1}\theta\,d\theta=\frac{[2^m(m!)]^2}{(2m+1)!}$.

(4 marks)

(d) Show that $\displaystyle F(k,m)=\sum_{r=0}^{m}\frac{(-1)^r C_r^m}{2r+k+1}$.

(3 marks)

13. (a) Let $x>1$ and define a sequence $\{a_n\}$ by

$a_1=x$ and $a_n=\dfrac{a_{n-1}{}^2+1}{2a_{n-1}}$ for $n\geq 2$.

(i) Show that $a_n>1$ and $a_n>a_{n+1}$ for all n .

(ii) Show that $\lim\limits_{n\to\infty}a_n=1$.

(8 marks)

(b) Let $f:[1,\infty)\to\mathbf{R}$ be a continuous function satisfying

$$f(x)=f\left(\frac{x^2+1}{2x}\right)\quad\text{for all } x\geq 1 .$$

Using (a), show that $f(x)=f(1)$ for all $x\geq 1$.

(7 marks)

END OF PAPER