| Solution | Marks |
|---|---|

1\. (a) The statement is clearly true for $n=1$. — 1

Assume it is true for $n=k$, then

$$A^{k+1} = \begin{pmatrix} a^k & \dfrac{a^k-b^k}{a-b} \\ 0 & b^k \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$$ — 1A

$$= \begin{pmatrix} a^{k+1} & a^k+\dfrac{a^k-b^k}{a-b}\cdot b \\ 0 & b^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & \dfrac{a^{k+1}-b^{k+1}}{a-b} \\ 0 & b^{k+1} \end{pmatrix}$$ — 1A

The statement is also true for $n=k+1$.

By the principle of mathematical induction,

$$A^n = \begin{pmatrix} a^n & \dfrac{a^n-b^n}{a-b} \\ 0 & b^n \end{pmatrix} \text{ for all positive integers } n.$$

(b) $$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95} = 2^{95}\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{3}{2} \end{pmatrix}^{95}$$ — 1M + 1A

$$= 2^{95}\begin{pmatrix} \left(\frac{1}{2}\right)^{95} & \dfrac{\left(\frac{1}{2}\right)^{95}-\left(\frac{3}{2}\right)^{95}}{\frac{1}{2}-\frac{3}{2}} \\ 0 & \left(\frac{3}{2}\right)^{95} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3^{95}-1 \\ 0 & 3^{95} \end{pmatrix}$$ — 1A

(6)

| Solution | Marks |
|---|---|
| (a) $(1+x)^n = 1+C_1^n x+C_2^n x^2+C_3^n x^3+\ldots+C_n^n x^n$ | 1A |
| Differentiating both sides w.r.t. $x$, we have | |
| $n(1+x)^{n-1} = C_1^n+2C_2^n x+3C_3^n x^2+\ldots+nC_n^n x^{n-1}$ ...(*) | 1A |
| Putting $x=1$ into (*), we get | |
| $C_1^n+2C_2^n+3C_3^n+\ldots+nC_n^n = 2^{n-1}n$ | 1 |
| (b) Putting $x=-1$ into (*), we get | |
| $C_1^n+2(-1)C_2^n+3C_3^n+\ldots+n(-1)^{n-1}C_n^n = 0$ | 1A |
| $\Rightarrow \frac{n!}{(n-1)!}+\frac{-2(n!)}{2!(n-2)!}+\frac{3(n!)}{3!(n-3)!}+\ldots+\frac{(-1)^{n-1}n(n!)}{n!} = 0$ | |
| $\Rightarrow \frac{1}{(n-1)!}+\frac{-2}{2!(n-2)!}+\frac{3}{3!(n-3)!}+\ldots+\frac{(-1)^{n-1}n}{n!} = 0$ | 1A |
| Alternatively, | |
| $\frac{1}{(n-1)!}+\frac{-2}{2!(n-2)!}+\frac{3}{3!(n-3)!}+\ldots+\frac{(-1)^{n-1}n}{n!}$ | |
| $= \frac{1}{n!}\left[C_1^n-2C_2^n+3C_3^n+\ldots+n(-1)^{n-1}C_n^n\right]$ | 1A |
| $= \frac{1}{n!}n(1-1)^{n-1} = 0$ | 1A |
| | (5) |

| Solution | Marks |
| --- | --- |

3. (a) Comparing coefficients on both sides, we have

$$\begin{cases} a_1 = 2p \\ a_2 = p^2+2q-\alpha^2 \\ a_3 = 2pq-2\alpha\beta \\ a_4 = q^2-\beta^2 \end{cases}$$

1A (For any 3 being co[rrect])

Eliminating $p$, we have

$$\begin{cases} \alpha^2 = \dfrac{a_1^2}{4}+2q-a_2 \\ \alpha\beta = \dfrac{1}{2}(a_1q-a_3) \\ \beta^2 = q^2-a_4 \end{cases}$$

1

(b) By (a), $\begin{cases} \alpha^2 = 2(q+8) \\ \alpha\beta = 2(q-6) \\ \beta^2 = q^2+9 \end{cases}$

1A

$\therefore 2(q^2+9)(q+8) = 4(q-6)^2$

$q(q^2+6q+33) = 0$ (or an equation useful for getting $q$, $\alpha$ or $\beta$)

1A

Since $q$ is real, therefore $q=0$.

Hence $\begin{cases} p=2 \\ q=0 \\ \alpha=4 \\ \beta=-3 \end{cases}$ or $\begin{cases} p=2 \\ q=0 \\ \alpha=-4 \\ \beta=3 \end{cases}$

1A

(c) By (b), $x^4+4x^3-12x^2+24x-9 = (x^2+2x)^2-(4x-3)^2$.

$\therefore (x^2+2x+4x-3)(x^2+2x-4x+3) = 0$

$(x^2+6x-3)(x^2-2x+3) = 0$

1A

$x = -3\pm2\sqrt{3}$ or $1\pm\sqrt{2}i$

1A

(7)

4. (a) For every $x\in\mathbb{R}$,

(i) $g(-x) = f[\cos(-x)] = f(\cos x) = g(x)$

1

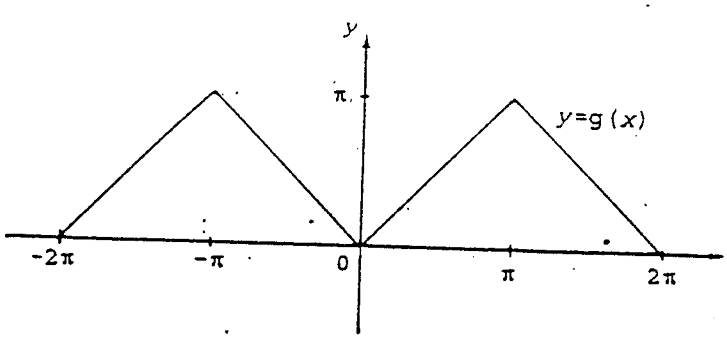
(ii) $g(x+2k\pi) = f[\cos(x+2k\pi)] = f(\cos x) = g(x)$ for any integer $k$.

1

$\therefore g(x)$ is even and periodic.

(b) For $x\in[0,\pi]$, $g(x) = \arccos(\cos x)$
$= x$

1A




1A + 1A (1A for any 2 out of the 4 segments)

(5)

| Solution | Marks |
|---|---|
| **5.** (a) Let $f(x)=\ln x-x+1$ for $x>0$, | |
| then $f'(x)=\frac{1}{x}-1 \begin{cases} >0 & \text{for } 0<x<1 \\ <0 & \text{for } x>1 \end{cases}$ | 1M + 1A (1A for differentiatio |
| Hence $f(x)$ is strictly increasing for $0<x<1$ and strictly decreasing for $x>1$. | |
| $\because f(x)$ is continuous, | |
| $\therefore f(x)\le f(1)$ for $x>0$ and the equality holds iff $x=1$. | 1 |
| i.e. $\ln x \le x-1$ for $x>0$ and the equality holds iff $x=1$. | 1 (for eithe |
| (b) For $r>1$, put $x=\frac{r}{r-1}$, then $x>1$. | |
| By (a), $\ln\frac{r}{r-1} < \frac{r}{r-1}-1 = \frac{1}{r-1}$ | 1 |
| $\sum_{k=2}^{n}[\ln r-\ln(r-1)] < \sum_{k=2}^{n}\frac{1}{k-1}$ | 1M |
| $\ln n < \sum_{k=1}^{n-1}\frac{1}{k}$ | 1 |
| | (7) |
| **6.** For $n=1$, | |
| $\because 1\le a_1^2\le 1+1+(-1)=1$ | |
| $\therefore a_1=1$ ($\because a_1$ is non-negative) | 1 |
| Assume $a_1=a_2=\dots=a_n=1$. | 1 (can be absorbed belc |
| Then $\sum_{k=1}^{n+1}a_k^2=n+a_{n+1}^2$ | 1 |
| Hence $n+1\le n+a_{n+1}^2\le n+1+1+(-1)^{n+1}$ | |
| $1\le a_{n+1}^2\le 2+(-1)^{n+1}<4$ | 1A |
| $\because a_{n+1}$ is a non-negative integer, | |
| $\therefore a_{n+1}=1$ | |
| By the principle of mathematical induction, $a_{n+1}=1$ for all $n\ge 1$. | |
| Alternatively, | |
| For any positive integer $m$, | |
| $\because 2m-1 \le \sum_{k=1}^{2m-1}a_k^2 \le (2m-1)+1+(-1)^{2m-1} = 2m-1$ | |
| $\therefore \sum_{k=1}^{2m-1}a_k^2 = 2m-1$ | 1 |
| (i) For $m=1$, $a_1^2=1 \Rightarrow a_1=1$. | 1 |
| (ii) For any positive integer $m$, | |
| $\sum_{k=1}^{2m+1}a_k^2 - \sum_{k=1}^{2m-1}a_k^2 = a_{2m+1}^2+a_{2m}^2 = 2$ | 1A |
| $\Rightarrow a_{2m+1}=a_{2m}=1$ | 1 |
| | (4) |

| Solution | Marks |
| --- | --- |
| 7. (a) $\because \lvert z\rvert = \lvert z-a\rvert$ | |
| $\therefore z\bar{z} = (z-a)\overline{(z-a)}$ | 1A |
| $z\bar{a}+a\bar{z} = a\bar{a}$ | 1A |
| $\dfrac{z}{a}+\dfrac{\bar{z}}{\bar{a}} = 1$ | |
| $\text{Re}\left(\dfrac{z}{a}\right) = \dfrac{1}{2}$ | 1 |
| (b) By (a), $\dfrac{z}{a} = \dfrac{1}{2}+yi$ for some $y\in\mathbb{R}$. | 1A |
| $\because \lvert z\rvert=\lvert a\rvert$ | |
| $\therefore \left\lvert\dfrac{z}{a}\right\rvert=1$ | |
| $\left(\dfrac{1}{2}\right)^2+y^2=1$ | |
| $y=\pm\dfrac{\sqrt{3}}{2}$ | 1A |
| $\Rightarrow z=\left(\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i\right)a$ | 1A |
| Alternatively, | |
| $\because \lvert z\rvert=\lvert z-a\rvert=\lvert a\rvert \Rightarrow$ The points representing 0, $z$, $a$ form an equilateral triangle in the Argand diagram | 1A |
| $\therefore z=ae^{\pm\frac{\pi}{3}i}$ | 1A |
| $=\left(\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i\right)a$ | 1A |
| | (6) |

<u>Note</u>

If $a$ is mistaken as a real number, no mark will be awarded for part (a). However, no mark will be deducted from part (b) in this case.

| Solution | Marks |
|---|---|
| 8. (a) (i) $A = \begin{pmatrix}u_1\\u_2\end{pmatrix}(s_1\ \ s_2)$ | |
| $= \begin{pmatrix}u_1s_1 & u_1s_2\\u_2s_1 & u_2s_2\end{pmatrix}$ | 1A |
| $\Rightarrow \det A = u_1u_2s_1s_2 - u_1u_2s_1s_2 = 0$ | 1 |
| (ii) If $\det A = 0$, then the vectors $(a_1\ a_2)$, $(b_1\ b_2)$ are linearly dependent. | 1M |
| W.l.g let $(a_1\ a_2) = k(b_1\ b_2)$ for some $k \in \mathbb{R}$, then | |
| $A = \begin{pmatrix}kb_1 & kb_2\\b_1 & b_2\end{pmatrix}$ | 1A |
| $= \begin{pmatrix}k\\1\end{pmatrix}(b_1\ \ b_2)$ | |
| Taking $B = \begin{pmatrix}k\\1\end{pmatrix} \in M_{21}$ and $C = (b_1\ b_2) \in M_{12}$, we have $A = BC$. | 1A |
| (b) (i) $D = \begin{pmatrix}u_1 & v_1\\u_2 & v_2\\u_3 & v_3\end{pmatrix}\begin{pmatrix}s_1 & s_2 & s_3\\t_1 & t_2 & t_3\end{pmatrix} = \begin{pmatrix}u_1 & v_1 & 0\\u_2 & v_2 & 0\\u_3 & v_3 & 0\end{pmatrix}\begin{pmatrix}s_1 & s_2 & s_3\\t_1 & t_2 & t_3\\0 & 0 & 0\end{pmatrix}$. | 1M+1A+1A |
| $\Rightarrow \det D = \begin{vmatrix}u_1 & v_1 & 0\\u_2 & v_2 & 0\\u_3 & v_3 & 0\end{vmatrix}\begin{vmatrix}s_1 & s_2 & s_3\\t_1 & t_2 & t_3\\0 & 0 & 0\end{vmatrix}$ | 1A |
| $= 0$ | 1 |
| <u>Alternatively</u>, | |
| $D = \begin{pmatrix}u_1s_1+v_1t_1 & u_1s_2+v_1t_2 & u_1s_3+v_1t_3\\u_2s_1+v_2t_1 & u_2s_2+v_2t_2 & u_2s_3+v_2t_3\\u_3s_1+v_3t_1 & u_3s_2+v_3t_2 & u_3s_3+v_3t_3\end{pmatrix}$ | 1A |
| $\Rightarrow \det D = \begin{vmatrix}u_1s_1 & u_1s_2+v_1t_2 & u_1s_3+v_1t_3\\u_2s_1 & u_2s_2+v_2t_2 & u_2s_3+v_2t_3\\u_3s_1 & u_3s_2+v_3t_2 & u_3s_3+v_3t_3\end{vmatrix} + \begin{vmatrix}v_1t_1 & u_1s_2+v_1t_2 & u_1s_3+v_1t_3\\v_2t_1 & u_2s_2+v_2t_2 & u_2s_3+v_2t_3\\v_3t_1 & u_3s_2+v_3t_2 & u_3s_3+v_3t_3\end{vmatrix}$ | 1M + 1A |
| $= \begin{vmatrix}u_1s_1 & u_1s_2 & u_1s_3+v_1t_3\\u_2s_1 & u_2s_2 & u_2s_3+v_2t_3\\u_3s_1 & u_3s_2 & u_3s_3+v_3t_3\end{vmatrix} + \begin{vmatrix}u_1s_1 & v_1t_2 & u_1s_3+v_1t_3\\u_2s_1 & v_2t_2 & u_2s_3+v_2t_3\\u_3s_1 & v_3t_2 & u_3s_3+v_3t_3\end{vmatrix} +$ | |
| $\begin{vmatrix}v_1t_1 & u_1s_2 & u_1s_3+v_1t_3\\v_2t_1 & u_2s_2 & u_2s_3+v_2t_3\\v_3t_1 & u_3s_2 & u_3s_3+v_3t_3\end{vmatrix} + \begin{vmatrix}v_1t_1 & v_1t_2 & u_1s_3+v_1t_3\\v_2t_1 & v_2t_2 & u_2s_3+v_2t_3\\v_3t_1 & v_3t_2 & u_3s_3+v_3t_3\end{vmatrix}$ | 1A |
| $= 0$ | 1 |

| Solution | Marks |
|---|---|
| (ii) If $c_i = \alpha a_i + \beta b_i \quad (i=1,2,3)$, then | |
| $D = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \alpha a_1 + \beta b_1 & \alpha a_2 + \beta b_2 & \alpha a_3 + \beta b_3 \end{pmatrix}$ | 1A |
| $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$ | 1A |
| Taking $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{pmatrix} \in M_{32}$ and $T = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \in M_{23}$, | |
| we have $D = ST$. | 1 |
| (iii) If $\det D=0$, then the vectors $(a_1\ a_2\ a_3)$, $(b_1\ b_2\ b_3)$ and $(c_1\ c_2\ c_3)$ are linearly dependent. | 1M |
| W.l.g. let $c_i = \alpha a_i + \beta b_i \quad (i=1,2,3)$. | |
| By (b)(ii), $D=ST$ for some $S \in M_{32}$ and $T \in M_{23}$. | |
| Taking $P=S$ and $Q=T$, the result follows. | 1 |

| Solution | Marks |
|---|---|

9. (a) For (S), $\Delta = \begin{vmatrix} 2 & 2 & -1 \\ h & -3 & -1 \\ -3 & h & 1 \end{vmatrix} = -(h^2-9)$. — 1A

(S) has a unique solution if and only if $\Delta \neq 0$, i.e. $h^2 \neq 9$. — 1

In this case, S.S. $= \left\{\left(-\frac{k}{h+3}, \frac{k}{h+3}, -k\right)\right\}$. — 1A

(b) (i) If $h=3$, the augmented matrix of (S) is

$$\left(\begin{array}{ccc|c} 2 & 2 & -1 & k \\ 3 & -3 & -1 & 0 \\ -3 & 3 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 12 & 0 & -5 & 3k \\ 1 & -5 & 0 & -k \end{array}\right)$$ — 1M

$$\text{or } \left(\begin{array}{ccc|c} 1 & -5 & 0 & -k \\ 0 & -12 & 1 & -3k \end{array}\right) \text{ or } \left(\begin{array}{ccc|c} 0 & 12 & -1 & 3k \\ 12 & 0 & -5 & 3k \end{array}\right)$$

It is consistent for all values of $k$. — 1A

S.S. $= \{(x,y,z):\ x=t,\ y=\frac{k+t}{5},\ z=\frac{12t-3k}{5},\ t\in\mathbf{R}\}$ — 2A

or $\{(x,y,z):\ x=5t-k,\ y=t,\ z=12t-3k,\ t\in\mathbf{R}\}$

or $\{(x,y,z):\ x=\frac{3k+5t}{12},\ y=\frac{3k+t}{12},\ z=t,\ t\in\mathbf{R}\}$

(ii) If $h=-3$, the augmented matrix of (S) is

$$\left(\begin{array}{ccc|c} 2 & 2 & -1 & k \\ -3 & -3 & -1 & 0 \\ -3 & -3 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \end{array}\right)$$ — 1M

It is consistent for $k = 0$ only. — 1A

S.S. $= \{(x,y,z):\ x=t,\ y=-t,\ z=0,\ t\in\mathbf{R}\}$ — 1A

or $\{(x,y,z):\ x=-t,\ y=t,\ z=0,\ t\in\mathbf{R}\}$

| Solution | Marks |
|---|---|

(c) Putting $k=\frac{2}{3}$, (S) becomes the first 3 eqtns. of (T).

(i) If $h^2\neq 9$ and $k=\frac{2}{3}$, then (S) has a unique solution. **1M**

(T) is consistent if the solution

$x=-\frac{2}{3(h+3)}$, $y=\frac{2}{3(h+3)}$, $z=-\frac{2}{3}$ satisfies the 4th eqtn.:

$$-5\left(-\frac{2}{3(h-3)}\right)-2\left(\frac{2}{3(h+3)}\right)+6\left(-\frac{2}{3}\right)=h$$ **1 for $h$**

$\Rightarrow\quad h^2+7h+10=0$

$\Rightarrow\quad h=-2$ or $-5$.

When $h=-2$, the solution of (T) is $(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3})$.

When $h=-5$, the solution of (T) is $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$. **2 for solution**

(ii) If $h=3$ and $k=\frac{2}{3}$, (S) has infinitely many solutions.

(T) is consistent if $\left(t, \frac{2+3t}{15}, \frac{36t-6}{15}\right)$, $t\in\mathbb{R}$ **1M**

satisfies the 4th eqtn.:

$$-5t-2\left(\frac{2+3t}{15}\right)+6\left(\frac{36t-6}{15}\right)=3$$

$\Rightarrow\quad 135t=85$

$\Rightarrow\quad t=\frac{17}{27}$

When $h=3$, the solution of (T) is $(\frac{17}{27}, \frac{7}{27}, \frac{10}{9})$. **1A**

| Solution | Marks |
|---|---|

10. (a) (i) $(x-\alpha)(x-\beta)(x-\gamma) = x^3+px^2+qx+r$ for all $x$.

Differentiating w.r.t. $x$ on both sides, we have

$(x-\beta)(x-\gamma)+(x-\gamma)(x-\alpha)+(x-\alpha)(x-\beta) = 3x^2+2px+q$ ......(1) — 1M + 1A

Hence

$$\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma} = \frac{(x-\beta)(x-\gamma)+(x-\gamma)(x-\alpha)+(x-\alpha)(x-\beta)}{(x-\alpha)(x-\beta)(x-\gamma)}$$

$$= \frac{3x^2+2px+q}{x^3+px^2+qx+r}$$ — 1

> Alternatively,
>
> $$\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma} = \frac{(x-\beta)(x-\gamma)+(x-\gamma)(x-\alpha)+(x-\alpha)(x-\beta)}{(x-\alpha)(x-\beta)(x-\gamma)}$$
>
> $$= \frac{3x^2-2(\alpha+\beta+\gamma)x+(\alpha\beta+\beta\gamma+\gamma\alpha)}{x^3+px^2+qx+r}$$ — 1M + 1A
>
> $$= \frac{3x^2+2px+q}{x^3+px^2+qx+r}$$ — 1

(ii) Sub. $x=\alpha$ into (1), we have $3\alpha^2+2p\alpha+q = (\alpha-\beta)(\alpha-\gamma)$. — 1

> Alternatively,
>
> Form (i), $1+\frac{x-\alpha}{x-\beta}+\frac{x-\alpha}{x-\gamma} = \frac{3x^2+2px+q}{(x-\beta)(x-\gamma)}$
>
> Sub. $x=\alpha$, we have $3\alpha^2+2p\alpha+q = (\alpha-\beta)(\alpha-\gamma)$. — 1

(b) (i) Let $(3x^2+2px+q)f(x) = (x^3+px^2+qx+r)Q(x) + Ax^2+Bx+C$ — 1M

$= (x-\alpha)(x-\beta)(x-\gamma)Q(x) + Ax^2+Bx+C$.

Then $(3\alpha^2+2p\alpha+q)f(\alpha) = A\alpha^2+B\alpha+C$ ......(2) — 1A

Let $\frac{Ax^2+Bx+C}{x^3+px^2+qx+r} = \frac{k_1}{x-\alpha}+\frac{k_2}{x-\beta}+\frac{k_3}{x-\gamma}$. — 1M

Then $Ax^2+Bx+C = k_1(x-\beta)(x-\gamma) + k_2(x-\gamma)(x-\alpha) + k_3(x-\alpha)(x-\beta)$. — 1A

Putting $x=\alpha$, we have

$A\alpha^2+B\alpha+C = k_1(\alpha-\beta)(\alpha-\gamma)$ — 1A

$(3\alpha^2+2p\alpha+q)f(\alpha) = k_1(\alpha-\beta)(\alpha-\gamma)$ by (2)

$k_1 = f(\alpha)$ by (a)(ii) — 1A

Similarly, $k_2 = f(\beta)$ and

$k_3 = f(\gamma)$. — 1M

Hence $\frac{f(\alpha)}{x-\alpha}+\frac{f(\beta)}{x-\beta}+\frac{f(\gamma)}{x-\gamma} = \frac{Ax^2+Bx+C}{x^3+px^2+qx+r}$.

(ii) From (b)(i),

$Ax^2+Bx+C = f(\alpha)(x-\beta)(x-\gamma) + f(\beta)(x-\gamma)(x-\alpha) + f(\gamma)(x-\alpha)(x-\beta)$. — 1A

Equating coefficients of $x^2$, $x$ and the constant terms, we have

$A = f(\alpha)+f(\beta)+f(\gamma)$. — 1A

$B = -[(\beta+\gamma)f(\alpha)+(\gamma+\alpha)f(\beta)+(\alpha+\beta)f(\gamma)]$ — 1A

$C = \beta\gamma f(\alpha)+\gamma\alpha f(\beta)+\alpha\beta f(\gamma)$ — 1A

| Solution | Marks |
|---|---|
| 11. (a) $\arg\left(\dfrac{z_C-z_M}{z_P-z_M}\right) = -\dfrac{\pi}{2}$ | 1M + 1A |
| $\left\lvert\dfrac{z_C-z_M}{z_P-z_M}\right\rvert = \cot\dfrac{\alpha}{2}$ | 1A |
| $\therefore \dfrac{z_C-z_M}{z_P-z_M} = -i\cot\dfrac{\alpha}{2}$ | 1A |
| $z_C-z_M = i(z_M-z_P)\cot\dfrac{\alpha}{2}$ | 1 |
| <u>Note</u>: Award 1 mark for drawing appropriate figure. | |
| (b) $\because z_M = \dfrac{1}{2}(z_P+z_Q)$ | 1A |
| $\therefore z_C = \dfrac{1}{2}(z_P+z_Q)+i\left[\dfrac{1}{2}(z_P+z_Q)-z_P\right]\cot\dfrac{\alpha}{2}$ | |
| $= \dfrac{1}{2}(z_P+z_Q)-\dfrac{1}{2}i(z_P-z_Q)\cot\dfrac{\alpha}{2}$ | 1A |
| $r = \dfrac{1}{2}\lvert z_P-z_Q\rvert\operatorname{cosec}\dfrac{\alpha}{2}$ | 1M + 1A |
| (c) (i) Any circle in the complex plane with centre $\alpha$ and radius $r$ has equation<br>$\lvert z-\alpha\rvert = r$<br>$(z-\alpha)(\overline{z}-\overline{\alpha}) = r^2$<br>$z\overline{z}-\overline{\alpha}z-\alpha\overline{z}+\alpha\overline{\alpha}-r^2 = 0$<br>which is in the form of $z\overline{z}+az+b\overline{z}+c=0$<br>where $a, b \in \mathbf{C}$ and $c \in \mathbf{R}$. | 1 |
| (ii) By (b), the radius of $\mathscr{C} = \dfrac{1}{2}\lvert 1+i-(-i)\rvert\operatorname{cosec}\dfrac{\pi}{6}$ | |
| $= \sqrt{5}$ | 1A |
| There are two possible positions for the centre of $\mathscr{C}$:<br>(1) Taking $z_P = 1+i$ and $z_Q = -i$ in (b),<br>$z_C = \dfrac{1}{2}(1+i-i)-\dfrac{1}{2}i(1+i+i)\cot\dfrac{\pi}{6}$ | |
| $= \dfrac{1}{2}+\sqrt{3}-\dfrac{\sqrt{3}}{2}i$ | 1A |
| (2) Taking $z_P = -i$ and $z_Q = 1+i$ in (b),<br>$z_C = \dfrac{1}{2}(-i+1+i)-\dfrac{1}{2}i(-i-1-i)\cot\dfrac{\pi}{6}$ | |
| $= \dfrac{1}{2}-\sqrt{3}+\dfrac{\sqrt{3}}{2}i$ | 1A |
| <u>Alternatively</u>,<br>$z_C = [1+i-(-i)]e^{\pm i\frac{\pi}{3}}+(-i)$<br>$= (1+2i)\left(\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}\right)-i$<br>$= \dfrac{1}{2}-\sqrt{3}+\dfrac{\sqrt{3}}{2}i$ or $\dfrac{1}{2}+\sqrt{3}-\dfrac{\sqrt{3}}{2}i$ | 1A + 1A |

| Solution | Marks |
|---|---|
| In case (1), | |
| $a = -\frac{1}{2}-\sqrt{3}-\frac{\sqrt{3}}{2}i$ | |
| $b = -\frac{1}{2}-\sqrt{3}+\frac{\sqrt{3}}{2}i$ | |
| $c = -1+\sqrt{3}$. | 1A |
| In case (2), | |
| $a = -\frac{1}{2}+\sqrt{3}-\frac{\sqrt{3}}{2}i$ | |
| $b = -\frac{1}{2}+\sqrt{3}-\frac{\sqrt{3}}{2}i$ | |
| $c = -1-\sqrt{3}$. | 1A |

| Solution | Marks |
|---|---|
| 12. (a) If $\lim_{n\to\infty} a_n$ exists, then | |
| $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}} + \frac{1}{p}\lim_{n\to\infty} a_{n-1}$ | |
| $= \frac{1}{p}\lim_{n\to\infty} a_n$ | 1A |
| Since $p\neq 1$, we have $\lim_{n\to\infty} a_n = 0$. | 1 |
| | ($p\neq 1$ is necessary) |
| (b) (i) Suppose on the contrary that $\lim_{n\to\infty} a_n$ exists. | |
| By (a), $\lim_{n\to\infty} a_n = 0$. | 1 |
| $2=a_0<a_1<a_2<\ldots \Rightarrow 2=a_0 \le \lim_{n\to\infty} a_n = 0$ | |
| which is a contradiction. | 1 |
| Hence $\lim_{n\to\infty} a_n$ does not exist. | |
| (ii) Given $a_{k-1} \ge a_k$ for some $k\ge 1$. | |
| Assume $a_{m-1} \ge a_m$ for some $m\ge k$, then | |
| $a_m - a_{m+1} = \left(\frac{1}{m^{\frac{1}{p}}} - \frac{1}{(m+1)^{\frac{1}{p}}}\right) + \frac{1}{p}(a_{m-1}-a_m) \ge 0$ | |
| By the principle of mathematical induction, $a_{n-1}\ge a_n$ for all $n\ge k$. | 1 |
| Thus $\{a_{n+k-1}\}$ is monotonic decreasing and bounded below by 0, hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+k-1}$ exists. | |
| By (a), $\lim_{n\to\infty} a_n = 0$. | 1 |
| (c) (i) If $0<p<1$, $a_n = \frac{1}{n^{\frac{1}{p}}} + \frac{1}{p}a_{n-1} > \frac{1}{p}a_{n-1} > a_{n-1}$ for all $n\ge 1$. | 1A |
| By (b)(i), $\lim_{n\to\infty} a_n$ does not exist. | 1 |
| (ii) If $p\ge 2$, then $a_1 = \frac{1}{1^{\frac{1}{p}}} + \frac{1}{p}a_0 = 1+\frac{2}{p} \le 2 = a_0$. | 1A |
| By (b)(ii), $\lim_{n\to\infty} a_n = 0$. | 1 |

| | Solution | Marks |
|---|---|---|
| (d) (i) | If $1<p<2$, then $0<p-1<1 \Rightarrow a_0=2<\dfrac{2}{p-1}$ | 1A |
| | Assume $a_n<\dfrac{2}{p-1}$ for some $n\geq 0$, then | |
| | $$\begin{aligned} a_{n+1} &= \frac{1}{(n+1)^{\frac{1}{p}}}+\frac{1}{p}a_n \\ &< \frac{1}{n+1}+\frac{1}{p}a_n \\ &\leq 1+\frac{1}{p}a_n \\ &< 1+\frac{1}{p}\left(\frac{2}{p-1}\right) \\ &= \frac{p^2-p+2}{p(p-1)} \\ &= \frac{(p-1)(p-2)+2p}{p(p-1)} \\ &< \frac{2p}{p(p-1)} \\ &= \frac{2}{p-1} \end{aligned}$$ | 1 |
| (ii) | By (d)(i), $\{a_n\}$ is bounded above. | |
| | Suppose $\{a_n\}$ is strictly increasing, then $\lim\limits_{n\to\infty} a_n$ exists | 1 |
| | which contradicts the result of (b)(i). | 1 |
| | Thus $\{a_n\}$ is not strictly increasing. | |
| | By (b)(ii), $\lim\limits_{n\to\infty} a_n=0$. | 1 |

| Solution | Marks |
|---|---|
| 13. (a) Without loss of generality, assume $a \ge b$. | 1M |
| Then $a-b \ge 0$, $\frac{a}{b} \ge 1$ and hence | 1 |
| $\frac{a^a b^b}{a^b b^a} = \left(\frac{a}{b}\right)^{a-b} \ge 1$ | |
| $\therefore a^a b^b \ge a^b b^a$. | 1 |
| If the equality holds, then | |
| $\left(\frac{a}{b}\right)^{a-b} = 1 \Rightarrow (a-b)\log\left(\frac{a}{b}\right) = 0$ | |
| $\Rightarrow \frac{a}{b}=1$ or $a-b=0$ | |
| $\Rightarrow a=b$. | 1 |
| (b) $\left(\frac{a+b}{2}\right)^{a+b} \ge (\sqrt{ab})^{a+b}$ | 1M |
| $= \sqrt{a^a b^b a^b b^a}$ | |
| $\ge \sqrt{a^b b^a a^b b^a}$ | |
| $= a^b b^a$ .......... (*) | 1 |
| If the equality holds, then | |
| $\frac{a+b}{2} = \sqrt{ab}$ and $a^a b^b = a^b b^a \Rightarrow a=b$. | 1 |
| (c) Let $f(x) = \ln[x^x(1-x)^{1-x}]$ | |
| $= x\ln x + (1-x)\ln(1-x)$ for $0<x<1$ | 1A |
| Then $f'(x) = \ln x - \ln(1-x)$ | 1A |
| $= \ln\frac{x}{1-x}$ | |
| $\begin{cases} <0 & \text{if } 0<x<\frac{1}{2} \\ =0 & \text{if } x=\frac{1}{2} \\ >0 & \text{if } \frac{1}{2}<x<1 \end{cases}$ | |
| $\therefore f(x) \ge f(\frac{1}{2})$ for $0<x<1$ | |
| and if the equality holds, $x=\frac{1}{2}$. | 1 |
| Hence $\ln[x^x(1-x)^{1-x}] \ge \ln[(\frac{1}{2})^{\frac{1}{2}}(1-\frac{1}{2})^{1-\frac{1}{2}}] = \ln\frac{1}{2}$ | |
| $x^x(1-x)^{1-x} \ge \frac{1}{2}$ for $0<x<1$ | |
| and if the equality holds, $x=\frac{1}{2}$. | 1 |

| Solution | Marks |
|---|---|
| Put $x=\frac{a}{a+b}$, then $0<x<1$ and $1-x=\frac{b}{a+b}$. | 1M + 1A |
| Hence $\left(\frac{a}{a+b}\right)^{\frac{a}{a+b}}\left(\frac{b}{a+b}\right)^{\frac{b}{a+b}} \geq \frac{1}{2}$ | |
| $\left(\frac{a}{a+b}\right)^{a}\left(\frac{b}{a+b}\right)^{b} \geq \left(\frac{1}{2}\right)^{a+b}$ | |
| $a^{a}b^{b} \geq \left(\frac{a+b}{2}\right)^{a+b}$ | 1 |
| If the equality holds, | |
| then $\frac{a}{a+b}=\frac{1}{2} \Rightarrow a=b$. | 1 |

| Solution | Marks |
|---|---|
| 1. (a) Let $y = \left(\frac{a^x+b^x+1}{3}\right)^{\frac{1}{x}}$, then $\ln y = \frac{1}{x}\ln\left(\frac{a^x+b^x+1}{3}\right)$ | 1A |
| $\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}\frac{3}{a^x+b^x+1}\left[\frac{a^x\ln a+b^x\ln b}{3}\right]$ | 1M |
| $= \frac{1}{3}\ln ab$ | |
| $\therefore \lim\limits_{x\to 0} y = (ab)^{\frac{1}{3}}$ | 1A |
| (b) $\lim\limits_{n\to\infty}\left(\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\dots+\frac{n^2}{n^3+n^3}\right)$ | |
| $= \lim\limits_{n\to\infty}\frac{1}{n}\left[\frac{(\frac{1}{n})^2}{1+(\frac{1}{n})^3}+\frac{(\frac{2}{n})^2}{1+(\frac{2}{n})^3}+\dots+\frac{(\frac{n}{n})^2}{1+(\frac{n}{n})^3}\right]$ | 1A |
| $= \int_0^1 \frac{x^2}{1+x^3}\,dx$ | 1M |
| $= \frac{1}{3}\left[\ln(1+x^3)\right]_0^1$ | |
| $= \frac{1}{3}\ln 2$ | 1A |
| | (6) |
| 2. (a) For $0<\theta<\frac{\pi}{2}$, let $x=\sin^2\theta$, then $dx=2\sin\theta\cos\theta\,d\theta$. | |
| $\int\frac{f(x)}{\sqrt{x(1-x)}}dx = \int\frac{f(\sin^2\theta)\cdot 2\sin\theta\cos\theta}{\sqrt{\sin^2\theta(1-\sin^2\theta)}}d\theta = 2\int f(\sin^2\theta)\,d\theta$. | 1 |
| (b) By (a), | |
| $\int\frac{dx}{\sqrt{x(1-x)}} = 2\int d\theta$ | |
| $= 2\theta + c$ | |
| $= 2\sin^{-1}\sqrt{x} + c$ | 1A |
| $\int\sqrt{\frac{x}{1-x}}\,dx = \int\frac{x}{\sqrt{x(1-x)}}dx$ | |
| $= 2\int\sin^2\theta\,d\theta$ | 1A |
| $= \int(1-\cos 2\theta)\,d\theta$ | |
| $= \theta-\frac{1}{2}\sin 2\theta + c$ | 1A |
| $= \sin^{-1}\sqrt{x}-\sqrt{x(1-x)} + c$ | 1A |
| | (5) |

| Solution | Marks |
|---|---|
| 3. (a) $y^2 = 4ax$ | |
| $2y\frac{dy}{dx} = 4a$ | |
| $\left.\frac{dy}{dx}\right\|_P = \left.\frac{2a}{y}\right\|_P = \frac{1}{t}$ | 1 (for attempting to find slope of tangent) |
| Normal at $P$ is $\frac{y-2at}{x-at^2} = -t$ | |
| $y + tx = 2at + at^3$ | 1 |
| (b) Suppose the normals at $P_i$ ($i=1,2,3$) are concurrent and intersect at $(x_0, y_0)$, then | |
| $y_0 + t_i x_0 = 2at_i + at_i^3 \quad$ for $i=1,2,3$ | 1 |
| i.e. $t_1, t_2, t_3$ are the roots of $at^3 + (2a - x_0)t - y_0 = 0$ | 1 |
| $\therefore$ Sum of the roots $= t_1 + t_2 + t_3 = 0$ | 1 |
| | (5) |
| 4. (a) For $x \geq 0$, $F'(x) = \frac{\sin x}{x+1}$ | 1A |
| $\begin{cases} > 0 & \text{for } x \in (0, \pi) \\ < 0 & \text{for } x \in (\pi, 2\pi) \end{cases}$ | 1M |
| $\therefore \quad F(x) \leq F(\pi) \quad \forall x \in [0, 2\pi]$ | |
| i.e. $x_0 = \pi$ | 1A |
| (b) $\because F(0) = 0$ and | 1A |
| $F(2\pi) = \int_0^{2\pi} \frac{\sin t}{t+1} dt$ | |
| $= \int_0^{\pi} \frac{\sin t}{t+1} dt + \int_{\pi}^{2\pi} \frac{\sin t}{t+1} dt$ | 1 |
| $= \int_0^{\pi} \frac{\sin t}{t+1} dt + \int_0^{\pi} \frac{\sin(\pi + t)}{\pi + t + 1} dt$ | |
| $= \int_0^{\pi} \frac{\sin t}{t+1} dt - \int_0^{\pi} \frac{\sin t}{\pi + t + 1} dt$ | |
| $\geq 0$ | 1 |
| $\because$ F is strictly increasing on $(0, \pi]$, $\therefore F(x) > F(0) = 0$ for $x \in (0, \pi]$. | |
| $\because$ F is strictly decreasing on $[\pi, 2\pi)$, $\therefore F(x) > F(2\pi) \geq 0$ for $x \in [\pi, 2\pi)$. | |
| Hence $F(x) > 0$ for $x \in (0, 2\pi)$. | 1 |
| | (7) |

5. (a) $\begin{cases} r = -2\cos\theta \\ r = 2+2\cos\theta \end{cases}$

$-2\cos\theta = 2+2\cos\theta$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ 1A

The intersecting points other than the pole are

$(1, \frac{2\pi}{3})$ and $(1, \frac{4\pi}{3})$. 1A

(b) Area of the shaded region

$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-2\cos\theta)^2 d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (2+2\cos\theta)^2 d\theta$ 1M + 1A

$= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\cos^2\theta\, d\theta + \int_{\frac{2\pi}{3}}^{\pi} (2+4\cos\theta+2\cos^2\theta)\, d\theta$

$= \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (1+\cos 2\theta)\, d\theta + \int_{\frac{2\pi}{3}}^{\pi} (2+4\cos\theta+1+\cos 2\theta)\, d\theta$

$= \left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} + \left[3\theta+4\sin\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{2\pi}{3}}^{\pi}$ 1 (Working out any one)

$= \left(\frac{2\pi}{3}-\frac{\pi}{2}-\frac{\sqrt{3}}{4}\right) + \left(3\pi-2\pi-2\sqrt{3}+\frac{\sqrt{3}}{4}\right)$

$= \frac{7\pi}{6} - 2\sqrt{3}$ 1A

(6)

| Solution | |
|---|---|
| 6. (a) $y=\left(\frac{x+1}{x-1}\right)^r$ | |
| $\ln y = r[\ln(x+1)-\ln(x-1)]$ | 1M |
| $\frac{1}{y}\frac{dy}{dx} = r\left[\frac{1}{x+1}-\frac{1}{x-1}\right]$ | |
| $\frac{dy}{dx} = \frac{-2ry}{x^2-1}$ | 1 |
| Alternatively, $\frac{dy}{dx} = r\left(\frac{x+1}{x-1}\right)^{r-1}\cdot\frac{(x-1)-(x+1)}{(x-1)^2}$ | |
| $= ry\cdot\left(\frac{x-1}{x+1}\right)\cdot\frac{-2}{(x-1)^2}$ | 1M |
| $= \frac{-2ry}{x^2-1}$ | 1 |
| (b) $(x^2-1)\frac{dy}{dx} = -2ry$ | |
| By Leibniz's formula, | |
| $\sum_{k=0}^{n} C_k^n (x^2-1)^{(k)}(y')^{(n-k)} = -2ry^{(n)}$ | |
| $C_0^n(x^2-1)y^{(n+1)}+C_1^n(2x)y^{(n)}+C_2^n(2)y^{(n-1)} = -2ry^{(n)}$ | 1M + 1A |
| $(x^2-1)y^{(n+1)}+2(nx+r)y^{(n)}+(n^2-n)y^{(n-1)} = 0$ | 1 |
| Alternatively, | |
| From (a), $(x^2-1)y' = -2ry$ | |
| $(x^2-1)y''+2xy' = -2ry'$ | |
| $(x^2-1)y^{(1+1)}+2(1\cdot x+r)y^{(1)}+(1^2-1)y^{(1-1)} = 0$ | 1A |
| $\therefore$ The statement holds for $n=1$. | |
| Assume $(x^2-1)y^{(n+1)}+2(nx+r)y^{(n)}+(n^2-n)y^{(n-1)} = 0$, | |
| then $(x^2-1)y^{(n+2)}+2xy^{(n+1)}+2(nx+r)y^{(n+1)}+2ny^{(n)}+(n^2-n)y^{(n)} = 0$ | 1M |
| $(x^2-1)y^{((n+1)+1)}+2[(n+1)x+r]y^{(n+1)}+[(n+1)^2-(n+1)]y^{((n+1)-1)} = 0$. | 1A |
| By the principle of M.I., the statement holds for $n\geq1$. | |
| | (5) |

| Solution | Marks |
| --- | --- |
| 7. (a) $\frac{d}{dx}\ln[1+f(x)] = \frac{f'(x)}{1+f(x)}$ | 1A |
| (b) $\because f(x) = x^3+\int_0^x 3t^2 f(t)\,dt$ | |
| $\therefore f'(x) = 3x^2+3x^2 f(x)$ | 1A |
| $= 3x^2[1+f(x)]$ | |
| $\frac{f'(x)}{1+f(x)} = 3x^2$ [ $1+f(x)>0$ as $f(x)>-1$ ] | 1M |
| $\int\frac{f'(x)}{1+f(x)}dx = \int 3x^2 dx$ | |
| $\ln[1+f(x)] = x^3+c$ | 1 |
| $\because f(0)=0$ | |
| $\therefore \ln(1+0)=0+c \Rightarrow c=0$ | 1A |
| Hence $1+f(x) = e^{x^3}$ | |
| $f(x) = e^{x^3}-1 \quad \forall x\in R$ | 1A |
| <u>Alternatively,</u> | |
| $1+f(x) = e^{x^3+c}$ | |
| $f(x) = Ae^{x^3}-1$ where $A = e^c$ | 1A |
| Putting $x=0$, we have | |
| $A-1 = 0+\int_0^0 3t^2 f(t)\,dt = 0$ | |
| $\therefore A = 1$ | |
| $\Rightarrow f(x) = e^{x^3}-1 \quad \forall x\in R$ | 1A |
| | (6) |

| Solution | Marks |
|---|---|
| 8. (a) $I_0 = \int_0^1 \frac{1-x}{1+x^3}\,dx$ | |
| $= \int_0^1 [\frac{2}{3(1+x)} - \frac{2x-1}{3(1-x+x^2)}]\,dx$ | 1M + 1A |
| $= \left[\frac{2}{3}\ln\lvert 1+x\rvert - \frac{1}{3}\ln\lvert 1-x+x^2\rvert\right]_0^1$ | 1A |
| $= \frac{2}{3}\ln 2$ | 1A |
| (b) $\lvert I_k\rvert = \left\lvert\int_0^1 \frac{(-1)^k(1-x)x^{3k}}{1+x^3}\,dx\right\rvert$ | |
| $\le \int_0^1 \left\lvert\frac{(-1)^k(1-x)x^{3k}}{1+x^3}\right\rvert dx$ | 1 |
| $= \int_0^1 \frac{(1-x)x^{3k}}{1+x^3}\,dx$ $\quad (\because \frac{(1-x)x^{3k}}{1+x^3} \ge 0\ \forall x\in[0,1])$ | |
| $\le \int_0^1 x^{3k}\,dx$ $\quad (\because \frac{1-x}{1+x^3} \le 1\ \forall x\in[0,1])$ | 1 |
| $= \frac{1}{3k+1}$ | |
| $\therefore -\frac{1}{3k+1} \le I_k \le \frac{1}{3k+1}$. | 1 |
| (c) $I_{k+1}-I_k = \int_0^1 [\frac{(-1)^{k+1}(1-x)x^{3(k+1)}}{1+x^3} - \frac{(-1)^k(1-x)x^{3k}}{1+x^3}]\,dx$ | 1M |
| $= (-1)^{k+1}\int_0^1 \frac{(1-x)x^{3k}(x^3+1)}{1+x^3}\,dx$ | |
| $= (-1)^{k+1}\int_0^1 (x^{3k}-x^{3k+1})\,dx$ | 1A |
| $= (-1)^{k+1}\left[\frac{x^{3k+1}}{3k+1} - \frac{x^{3k+2}}{3k+2}\right]_0^1$ | |
| $= \frac{(-1)^{k+1}}{(3k+1)(3k+2)}$ | 1A |
| (d) By (c), $\sum_{k=0}^{n}(I_{k+1}-I_k) = \sum_{k=0}^{n}\frac{(-1)^{k+1}}{(3k+1)(3k+2)}$ | 1M |
| $I_{n+1}-I_0 = -b_n$ | 1A |
| $\therefore I_{n+1} = \frac{2}{3}\ln 2 - b_n$ .....(*) by (a) | 1A |
| By (b), $-\frac{1}{3k+1} \le I_k \le \frac{1}{3k+1}$ and $\lim_{k\to\infty}\frac{1}{3k+1} = 0$ | |
| $\therefore \lim_{k\to\infty} I_k = 0$ by squeezing principle. | 1 |
| As $\lim_{n\to\infty} I_{n+1} = 0$, we have $\lim_{n\to\infty} b_n = \frac{2}{3}\ln 2$ by letting $n\to\infty$ in (*) | 1 |

| Solution | Marks |
|---|---|

9. (a) (i) For $x>0$, $f(x)=\dfrac{x}{(x+1)^2}$,

$f'(x) = -\dfrac{x-1}{(x+1)^3}$ and $f''(x) = \dfrac{2x-4}{(x+1)^4}$ — 1A

(ii) For $x<0$ and $x\neq-1$, $f(x) = -\dfrac{x}{(x+1)^2}$,

$f'(x) = \dfrac{x-1}{(x+1)^3}$ and $f''(x) = -\dfrac{2x-4}{(x+1)^4}$ — 1A

(iii) $\therefore \dfrac{f(x)-f(0)}{x-0} = \begin{cases} \dfrac{1}{(x+1)^2} & \text{for } x>0 \\ -\dfrac{1}{(x+1)^2} & \text{for } x<0 \text{ and } x\neq-1 \end{cases}$ — 1A

$\therefore \lim\limits_{x\to0^+} \dfrac{f(x)-f(0)}{x-0} = 1$ and $\lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0} = -1$,

$\therefore f'(0)$ does not exist. — 1

(b) (i) For $x>0$, $f'(x)<0 \Rightarrow x>1$.
For $x<0$, $f'(x)<0 \Rightarrow -1<x<0$.
$\therefore f'(x)<0$ on $(-1,0)\cup(1,\infty)$. — 1A

(ii) For $x>0$, $f'(x)>0 \Rightarrow 0<x<1$.
For $x<0$, $f'(x)>0 \Rightarrow x<-1$.
$\therefore f'(x)<0$ on $(-\infty,-1)\cup(0,1)$. — 1A

(iii) For $x>0$, $f''(x)<0 \Rightarrow 0<x<2$.
For $x<0$, $f''(x)<0 \Rightarrow$ no solution.
$\therefore f''(x)<0$ on $(0,2)$. — 1A

(iv) For $x>0$, $f''(x)>0 \Rightarrow x>2$.
For $x<0$, $f''(x)>0 \Rightarrow x<-1$ or $-1<x<0$.
$\therefore f''(x)>0$ on $(-\infty,-1)\cup(-1,0)\cup(2,\infty)$. — 1A

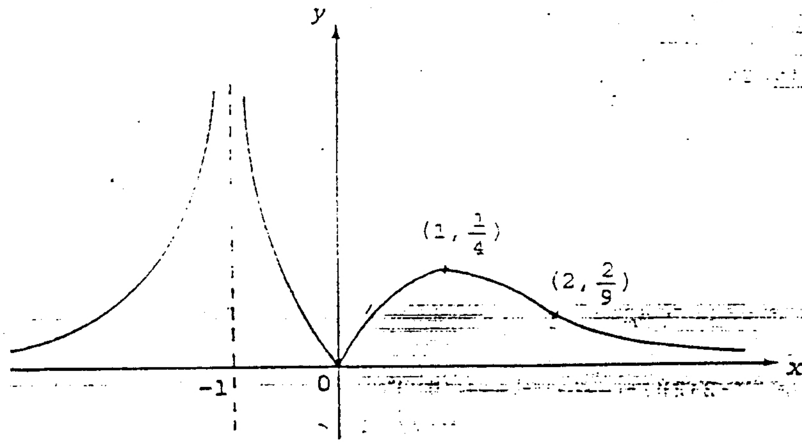
(c)

| $x$ | $(-\infty,-1)$ | $-1$ | $(-1,0)$ | 0 | $(0,1)$ | 1 | $(1,2)$ | 2 | $(2,\infty)$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | ↑ | Undefined | ↓ | 0 | ↑ | $\frac{1}{4}$ | ↓ | $\frac{2}{9}$ | ↓ |
| $f'(x)$ | + | Undefined | − | ∄ | + | 0 | − | − | − |
| $f''(x)$ | + | Undefined | + | ∄ | − | − | − | 0 | + |

$\therefore (1,\frac{1}{4})$ is a relative maximum point. — 1A

$(0,0)$ is a relative minimum point. — 1A

$(2,\frac{2}{9})$ is a point of inflexion. — 1A

| Solution | Marks |
|---|---|
| (d) The vertical symptote is $x=-1$. | 1A |
| Let the oblique/horizontal asymptote be $y=mx+c$. | |
| $m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\lvert x\rvert}{x(x+1)^3} = 0$ | |
| $c = \lim_{x\to\infty} [f(x)-0] = 0$ | |
| $\therefore$ the horizontal asymptote is $y=0$. | 1A |
|  | 1A + 1 |

10. (a) (i) For $n\geq 2$, $a_n^2 = a_{n-1}^2+2\beta+\left(\frac{\beta}{a_{n-1}}\right)^2$

$\geq a_{n-1}^2+2\beta$ 1

(ii) The statement holds for $n=1$ since $a_1^2 = (\beta+1)^2 = \beta^2+2\beta+1$.

Assume it holds for some $k\geq 1$, then

$a_{k+1}^2 \geq a_k^2+2\beta \geq \beta^2+2k\beta+1+2\beta = \beta^2+2(k+1)\beta+1$ .

By the principle of M.I., the statement holds for $n\geq 1$. 1

> Alternatively,
>
> $a_1^2 = (\beta+1)^2 = \beta^2+2\beta+1$.
>
> For $n\geq 2$, $a_n^2 \geq a_{n-1}^2+2\beta \geq a_{n-2}^2+2(2\beta) \geq \dots \geq a_1^2+(n-1)(2\beta)$
>
> $= \beta^2+2\beta+1+(n-1)(2\beta) = \beta^2+2n\beta+1$ 1

(b) For $n\geq 2$, $a_n^2 = a_{n-1}^2+2\beta+\left(\frac{\beta}{a_{n-1}}\right)^2$

$= a_{n-2}^2+2(2\beta)+\left(\frac{\beta}{a_{n-1}}\right)^2+\left(\frac{\beta}{a_{n-2}}\right)^2$ 1

$= a_1^2+(n-1)(2\beta)+\sum_{k=1}^{n-1}\frac{\beta^2}{a_k^2}$

$= \beta^2+2\beta+1+(n-1)(2\beta)+\sum_{k=1}^{n-1}\frac{\beta^2}{a_k^2}$

$= \beta^2+2n\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{a_k^2}$ 1

$\leq \beta^2+2n\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{\beta^2+2k\beta+1}$ 1

> Alternatively,
>
> $\because a_2^2 = a_1^2+2\beta+\frac{\beta^2}{a_1^2} = \beta^2+2(2)\beta+1+\frac{\beta^2}{\beta^2+2\beta+1}$
>
> $\therefore$ The statement holds for $n=2$. 1
>
> Assume it holds for $n\geq 2$, then
>
> $a_{n+1}^2 = a_n^2+2\beta+\frac{\beta^2}{a_n^2}$
>
> $\leq \beta^2+2n\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{\beta^2+2k\beta+1}+2\beta+\frac{\beta^2}{a_n^2}$
>
> $\leq \beta^2+2(n+1)\beta+1+\sum_{k=1}^{n-1}\frac{\beta^2}{\beta^2+2k\beta+1}+\frac{\beta^2}{\beta^2+2n\beta+1}$
>
> $= \beta^2+2(n+1)\beta+1+\sum_{k=1}^{n}\frac{\beta^2}{\beta^2+2k\beta+1}$. 1
>
> $\therefore$ By the principle of M.I., the statement holds for $n\geq 2$. 1

| Solution | Marks |
|---|---|
| (c) For $k \geq 1$, $\dfrac{1}{\beta^2+2k\beta+1} \leq \dfrac{1}{\beta^2+2\beta x+1} \quad \forall x \in [k-1, k]$ | 1 |
| $\therefore \displaystyle\int_{k-1}^{k} \frac{dx}{\beta^2+2k\beta+1} \leq \int_{k-1}^{k} \frac{dx}{\beta^2+2\beta x+1}$ | |
| $\Rightarrow \dfrac{1}{\beta^2+2k\beta+1} \leq \displaystyle\int_{k-1}^{k} \frac{dx}{\beta^2+2\beta x+1}$ | 1 |
| (d) By (a)(ii) and (c), | |
| $\beta^2+2n\beta+1 \leq a_n^2 \leq \beta^2+2n\beta+1+\displaystyle\sum_{k=1}^{n-1} \frac{\beta^2}{\beta^2+2k\beta+1}$ | 1 |
| $\because \displaystyle\sum_{k=1}^{n-1} \frac{\beta^2}{\beta^2+2k\beta+1} \leq \sum_{k=1}^{n-1} \int_{k-1}^{k} \frac{\beta^2}{\beta^2+2\beta x+1}\,dx$ | |
| $= \displaystyle\int_{0}^{n-1} \frac{\beta^2}{\beta^2+2\beta x+1}\,dx$ | 1A |
| $= \dfrac{\beta^2}{2\beta}\left[\ln\|\beta^2+2\beta x+1\|\right]_0^{n-1}$ | |
| $= \dfrac{\beta}{2}\ln\left[\dfrac{\beta^2+2(n-1)\beta+1}{\beta^2+1}\right]$ | 1A |
| $\therefore \dfrac{\beta^2+1}{n}+2\beta \leq \dfrac{a_n^2}{n} \leq \dfrac{\beta^2+1}{n}+2\beta+\dfrac{\beta}{2n}\ln\left[\dfrac{\beta^2+2(n-1)\beta+1}{\beta^2+1}\right]$ | |
| As $\displaystyle\lim_{n\to\infty} \frac{\beta}{2n}\ln\left[\frac{\beta^2+2(n-1)\beta+1}{\beta^2+1}\right]$ | |
| $= \displaystyle\lim_{n\to\infty} \frac{\beta}{2}\cdot\frac{\beta^2+1}{\beta^2+2(n-1)\beta+1}\cdot\frac{2\beta}{\beta^2+1}$ | 1M |
| $= 0$ | 1 |
| By squeezing principle, $\displaystyle\lim_{n\to\infty} \frac{a_n^2}{n}$ exists and | |
| $\displaystyle\lim_{n\to\infty} \frac{a_n^2}{n} = 2\beta$ | 1A |
| Suppose $\displaystyle\lim_{n\to\infty} \frac{a_n^2}{\sqrt{n}} = \ell$ exists, then | |
| $\displaystyle\lim_{n\to\infty} \frac{a_n^2}{n} = \lim_{n\to\infty} \frac{a_n^2}{\sqrt{n}} \cdot \lim_{n\to\infty} \frac{1}{\sqrt{n}}$ | 1 |
| $= \ell \cdot 0$ | |
| $= 0$ | |
| $\neq 2\beta \quad (\because \beta > 0)$ | |
| $\therefore \displaystyle\lim_{n\to\infty} \frac{a_n^2}{\sqrt{n}}$ does not exist. | 1 |

II-8 (a) Let $t=\tan\frac{\theta}{2}$, then $dt=\frac{1}{2}\sec^2\frac{\theta}{2}d\theta$ or $d\theta=\frac{2}{1+t^2}dt$. 1M + 1A

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin\theta+\cos\theta+2} = \int_0^1 \frac{2dt}{t^2+4t+3}$$ 1A

$$= \int_0^1 \left(\frac{1}{t+1}-\frac{1}{t+3}\right)dt$$ 1A

$$= \left[\ln\left|\frac{t+1}{t+3}\right|\right]_0^1$$

$$= \ln\frac{3}{2}$$ 1A

(b) $pg(\theta)+qg'(\theta)+r$

$= p(A\sin\theta-B\cos\theta+C)+q\frac{d}{d\theta}(A\sin\theta-B\cos\theta+C)+r$

$= (Ap-Bq)\sin\theta+(Bp+Aq)\cos\theta+(Cp+r)$ 1A

There exist real numbers $p$, $q$, $r$ such that

$$a\sin\theta+b\cos\theta+c = pg(\theta)+qg'(\theta)+r$$

if $\begin{cases} Ap-Bq=a \\ Bp+Aq=b \\ Cp+r=c \end{cases}$ is consistent. 1M

Since $\Delta = \begin{vmatrix} A & -B & 0 \\ B & A & 0 \\ C & 0 & 1 \end{vmatrix}$

$= A^2+B^2$ 1A

$\neq 0$ as $A$, $B$ are not both zero. 1

Therefore the system of linear equations is consistent and hence $p$, $q$, $r$ exist. 1

Alternatively,

Comparing coefficients of

$$a\sin\theta+b\cos\theta+c = pg(\theta)+qg'(\theta)+r,$$

we have (*): $\begin{cases} Ap-Bq=a \\ Bp+Aq=b \\ Cp+r=c \end{cases}$ 1M

Solving (*) and using the fact that $A^2+B^2\neq 0$,

we have $p=\frac{Aa+Bb}{A^2+B^2}$, $q=\frac{Ab-Ba}{A^2+B^2}$ and $r=c-C\left(\frac{Aa+Bb}{A^2+B^2}\right)$. 1A+1A+1A

$\therefore$ $p$, $q$, $r$ exist.

(c) Let $7\sin\theta - 4\cos\theta + 3 = p(2\sin\theta + \cos\theta + 2) + q\dfrac{d}{d\theta}(2\sin\theta + \cos\theta + 2) + r$,

then $\begin{cases} 2p - q = 7 \\ p + 2q = -4 \\ 2p + r = 3 \end{cases}$

$\Rightarrow \quad p = 2, \; q = -3, \; r = -1.$ 1A

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \frac{7\sin\theta - 4\cos\theta + 3}{2\sin\theta + \cos\theta + 2} d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{2(2\sin\theta + \cos\theta + 2) - 3\frac{d}{d\theta}(2\sin\theta + \cos\theta + 2) - 1}{2\sin\theta + \cos\theta + 2} d\theta$$ 1A

$$= \int_0^{\frac{\pi}{2}} 2d\theta - 3[\ln|2\sin\theta + \cos\theta + 2|]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin\theta + \cos\theta + 2}$$ 1A + 1A

$$= \pi - 3\ln\frac{4}{3} - \ln\frac{3}{2}$$

$$= \pi - 5\ln 2 + 2\ln 3$$ 1A

12. (a) Equations of $L_1$ and $L_2$ are

(A) $\begin{cases} x = 2t+5 \\ y = 2t+1 \\ z = -t \end{cases}$ and (B) $\begin{cases} x = 2s+4 \\ y = 5s-8 \\ x = 2s+1 \end{cases}$ respectively.

Putting (A) into (B), we have

(C) $\begin{cases} 2t-2s = -1 & \ldots(1) \\ 2t-5s = -9 & \ldots(2) \\ t+2s = -1 & \ldots(3) \end{cases}$ 1M

Solving (1) and (2), $s=\frac{8}{3}$ and solving (2) and (3), $s=\frac{7}{9}$.

[ OR $\left(\begin{array}{cc|c} 2 & -2 & -1 \\ 2 & -5 & -9 \\ 1 & 2 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 2 & -2 & -1 \\ 0 & 1 & \frac{8}{3} \\ 0 & 1 & -\frac{1}{5} \end{array}\right)$ ]

$\therefore$ (C) is inconsistent $\Rightarrow$ $L_1$ and $L_2$ do not intersect. 1

<u>Alternatively,</u>

From $L_1$, $x-y=4$ and $x+2z=5$.

From $L_2$, $5x-2y=36$ and $x-z=3$.

$\begin{cases} x-y=4 \\ 5x-2y=36 \end{cases} \Rightarrow x=\frac{28}{3}$ 1M

$\begin{cases} x+2z=5 \\ x-z=3 \end{cases} \Rightarrow x=\frac{11}{3}$

The equations are not consistent $\Rightarrow$ $L_1$ and $L_2$ do not intersect. 1

(b) Let the direction ratios of $L$ be $[a:b:c]$, then

$\begin{cases} 2a+2b-c = 0 \\ 2a+5b+2c = 0 \end{cases}$ 1M

$\Rightarrow$ $a:b:c = 3:-2:2$ 1A

Let $A = (2t+5,\ 2t+1,\ -t)$ and $B = (2s+4,\ 5s-8,\ 2s+1)$, then

$\frac{2s-2t-1}{3} = \frac{5s-2t-9}{-2} = \frac{2s+t+1}{2}$ 1M

$\Rightarrow$ $t = -1$ and $s = 1$ 1A

Therefore $A = (3,\ -1,\ 1)$

$B = (6,\ -3,\ 3)$ 1A

and eqtn. of $L$ is $\frac{x-3}{3} = \frac{y+1}{-2} = \frac{z-1}{2}$ 1A

or $\frac{x-6}{3} = \frac{y+3}{-2} = \frac{z-3}{2}$.

(c) (i) Equation of $\pi$ is

$$2(x-3)+2(y+1)-(z-1) = 0 \qquad \text{1M}$$

or $\quad 2x+2y-z-3 = 0.$ 1A

(ii) Sub. $B = (6, -3, 3)$ into the equation of $\pi$,

L.H.S. $= 2(6)+2(-3)-(3)-3 = 0.$

$\therefore B$ lies on $\pi$. 1

(ii) Let the projection of $L_2$ on $\pi$ be $\dfrac{x-6}{p} = \dfrac{y+3}{q} = \dfrac{z-3}{r}$, then

$$\begin{cases} 2p+2q-r = 0 \\ 3p-2q+2r = 0 \end{cases} \qquad \text{1M + 1A}$$

$p:q:r = -2:7:10$ 1A

Hence the equation required is

$$\frac{x-6}{-2} = \frac{y-3}{7} = \frac{z-3}{10}. \qquad \text{1A}$$

13. (a) (i) $\int_a^b f(x)\,g(x)\,dx = \int_a^b f(x)\,w'(x)\,dx$ 1 (for applying integration by pa

$= [f(x)\,w(x)]_a^b - \int_a^b w(x)\,f'(x)\,dx$

$= f(b)\int_a^b g(x)\,dx - \int_a^b w(x)\,f'(x)\,dx$ 1

(ii) By (a)(i) and Theorem (*),

$\int_a^b f(x)\,g(x)\,dx = f(b)\int_a^b g(x)\,dx - w(c)\int_a^b f'(x)\,dx$

for some $c\in[a,b]$ 1A

$= f(b)\int_a^b g(x)\,dx - \left(\int_a^c g(x)\,dx\right)(f(b)-f(a))$ 1A

$= f(b)\int_c^b g(x)\,dx + f(a)\int_a^c g(x)\,dx$ 1

(b) Putting $f(x) = -\dfrac{1}{F'(x)}$ and $g(x) = -F'(x)\cos F(x)$,

then $f(x)$ and $g(x)$ are continuously differentiable and

$f'(x) = \dfrac{F''(x)}{(F'(x))^2} \ge 0$ for $a\le x\le b$. By (a), 1

$$\left|\int_a^b \cos F(x)\,dx\right| = \left|\frac{-1}{F'(b)}\int_c^b(-F'(x)\cos F(x))\,dx + \frac{-1}{F'(a)}\int_a^c(-F'(x)\cos F(x))\,dx\right|$$ 1A

$$= \left|\frac{1}{F'(b)}[\sin F(x)]_c^b + \frac{1}{F'(a)}[\sin F(x)]_a^c\right|$$ 1M

$$\le \left|\frac{1}{F'(b)}\right||\sin F(b)| + \left|\frac{1}{F'(b)}\right||\sin F(c)| + \left|\frac{1}{F'(a)}\right||\sin F(c)| + \left|\frac{1}{F'(a)}\right||\sin F(b)|$$ 1A

$$\le \frac{4}{m}$$ 1

(c) (i) For $0\le x\le 1$ and $n\ge 1$, we have

$x^{n+1} \le x^n \Rightarrow \cos(x^n) \le \cos(x^{n+1})$

$\Rightarrow \int_0^1 \cos(x^n)\,dx \le \int_0^1 \cos(x^{n+1})\,dx$ 1

$\because \int_0^1 \cos(x^n)\,dx \le \int_0^1 dx = 1$

$\therefore \left\{\int_0^1 \cos(x^n)\,dx\right\}$ is monotonic increasing and bounded above.

Hence $\lim_{n\to\infty}\int_0^1 \cos(x^n)\,dx$ exists. 1

(ii) Let $F(x)=x^n$ for $x\in[1,2\pi]$. 1A

When $n\ge 2$, $F'(x)=nx^{n-1}\ge n>0$ and $F''(x)=n(n-1)x^{n-2}\ge 0$.

$\therefore \left|\int_1^{2\pi} \cos(x^n)\,dx\right| \le \frac{4}{n}$ by (b)

$\Rightarrow \lim_{n\to\infty}\left|\int_1^{2\pi} \cos(x^n)\,dx\right| = 0$

$\Rightarrow \lim_{n\to\infty}\int_1^{2\pi} \cos(x^n)\,dx = 0$ 1M (for either)

Hence $\lim_{n\to\infty}\int_0^{2\pi} \cos(x^n)\,dx$

$= \lim_{n\to\infty}\left(\int_0^1 \cos(x^n)\,dx + \int_1^{2\pi} \cos(x^n)\,dx\right)$

$= \lim_{n\to\infty}\int_0^1 \cos(x^n)\,dx$ exists. 1