

**SECTION A (40 marks)**

Answer ALL questions in this section.

Write your answers in the AL(C1) answer book.

1. (a) Let  $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$  where  $a, b \in \mathbb{R}$  and  $a \neq b$ .

Prove that  $A^n = \begin{pmatrix} a^n & \frac{a^n - b^n}{a - b} \\ 0 & b^n \end{pmatrix}$  for all positive integers  $n$ .

(b) Hence, or otherwise, evaluate  $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95}$ .

(6 marks)

2. Let  $n$  be an integer and  $n > 1$ . By considering the binomial expansion of  $(1+x)^n$ , or otherwise,

(a) show that  $C_1^n + 2C_2^n + 3C_3^n + \dots + nC_n^n = 2^{n-1}n$ ;

(b) evaluate  $\frac{1}{(n-1)!} + \frac{-2}{2!(n-2)!} + \frac{3}{3!(n-3)!} + \dots + \frac{(-1)^{n-1}n}{n!}$ .

(5 marks)

3. (a) If  $a_1, a_2, a_3, a_4, p, q, \alpha, \beta$  are real numbers such that  $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = (x^2 + px + q)^2 - (\alpha x + \beta)^2$  for all  $x$ ,

show that 
$$\begin{cases} \alpha^2 = \frac{a_1^2}{4} + 2q - a_2 \\ \alpha\beta = \frac{1}{2}(a_1q - a_3) \\ \beta^2 = q^2 - a_4 \end{cases}$$

(b) Find the possible real values of  $p, q, \alpha, \beta$  such that  $x^4 + 4x^3 - 12x^2 + 24x - 9 = (x^2 + px + q)^2 - (\alpha x + \beta)^2$  for all  $x$ .

(c) Solve  $x^4 + 4x^3 - 12x^2 + 24x - 9 = 0$ .

(7 marks)

4. Let  $f: [-1, 1] \rightarrow [0, \pi]$ ,  $f(x) = \arccos x$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(x) = f(\cos x)$ .

(a) Show that  $g(x)$  is even and periodic.

(b) Find  $g(x)$  for  $x \in [0, \pi]$ . Hence sketch the graph of  $g(x)$  for  $x \in [-2\pi, 2\pi]$ .

(5 marks)

5. (a) For  $x > 0$ , prove that  $\ln x \leq x - 1$  where the equality holds if and only if  $x = 1$ .

(b) Prove that  $\ln \frac{r}{r-1} < \frac{1}{r-1}$  for  $r > 1$ .

Hence deduce that  $\ln n < \sum_{k=1}^{n-1} \frac{1}{k}$  for  $n = 2, 3, 4, \dots$ .

(7 marks)

6. Let  $\{a_n\}$  be a sequence of non-negative integers such that

$$n \leq \sum_{k=1}^n a_k^2 \leq n+1 + (-1)^n \quad \text{for } n = 1, 2, 3, \dots$$

Prove that  $a_n = 1$  for  $n \geq 1$ .

(4 marks)

7. Let  $a \in \mathbb{C}$  and  $a \neq 0$ .

(a) Show that if  $|z| = |z - a|$ , then  $\operatorname{Re}\left(\frac{z}{a}\right) = \frac{1}{2}$ .

(b) If  $|z| = |z - a| = |a|$ , express  $z$  in terms of  $a$ .

(6 marks)

### SECTION B (60 marks)

Answer any **FOUR** questions from this section. Each question carries 15 marks.

Write your answers in the AL(C2) answer book.

8. Let  $M_{mn}$  be the set of all  $m \times n$  matrices.

(a) Let  $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in M_{22}$ .

(i) Show that if  $A = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \end{pmatrix}$ , where  $u_1, u_2, s_1, s_2 \in \mathbb{R}$ , then  $\det A = 0$ .

(ii) Conversely, show that if  $\det A = 0$ , then  $A = BC$  for some  $B \in M_{21}$  and  $C \in M_{12}$ .

(5 marks)

(b) Let  $D = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in M_{33}$ .

(i) Show that if  $D = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \begin{pmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{pmatrix}$ , where  $u_i, v_i, s_i, t_i \in \mathbb{R}$  ( $i = 1, 2, 3$ ), then  $\det D = 0$ .

(ii) Suppose there are  $\alpha, \beta \in \mathbb{R}$  such that  $c_i = \alpha a_i + \beta b_i$  for  $i = 1, 2, 3$ . Find  $S \in M_{32}$  and  $T \in M_{23}$  such that  $D = ST$ .

(iii) Show that if  $\det D = 0$ , then  $D = PQ$  for some  $P \in M_{32}$  and  $Q \in M_{23}$ .

(10 marks)

9. Consider the following systems of linear equations

$$(S) : \begin{cases} 2x + 2y - z = k \\ hx - 3y - z = 0 \\ -3x + hy + z = 0 \end{cases}$$

and

$$(T) : \begin{cases} 6x + 6y - 3z = 2 \\ hx - 3y - z = 0 \\ -3x + hy + z = 0 \\ -5x - 2y + 6z = h \end{cases}$$

(a) Show that (S) has a unique solution if and only if  $h^2 \neq 9$ . Solve (S) in this case.

(3 marks)

(b) For each of the following cases, find the value(s) of  $k$  for which (S) is consistent, and solve (S):

(i)  $h = 3$ ,

(ii)  $h = -3$ .

(7 marks)

(c) Find the values of  $h$  for which (T) is consistent. Solve (T) for each of these values of  $h$ .

(5 marks)

10. Let  $\alpha$ ,  $\beta$  and  $\gamma$  be real and distinct and

$$(x - \alpha)(x - \beta)(x - \gamma) = x^3 + px^2 + qx + r.$$

(a) Show that

(i) 
$$\frac{1}{x - \alpha} + \frac{1}{x - \beta} + \frac{1}{x - \gamma} = \frac{3x^2 + 2px + q}{x^3 + px^2 + qx + r};$$

(ii) 
$$3\alpha^2 + 2p\alpha + q = (\alpha - \beta)(\alpha - \gamma).$$

(4 marks)

(b) Let  $f(x)$  be a real polynomial. Suppose  $Ax^2 + Bx + C$  is the remainder when  $(3x^2 + 2px + q)f(x)$  is divided by  $x^3 + px^2 + qx + r$ .

(i) Prove that 
$$\frac{f(\alpha)}{x - \alpha} + \frac{f(\beta)}{x - \beta} + \frac{f(\gamma)}{x - \gamma} = \frac{Ax^2 + Bx + C}{x^3 + px^2 + qx + r}.$$

(ii) Express  $A$ ,  $B$  and  $C$  in terms of  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $f(\alpha)$ ,  $f(\beta)$  and  $f(\gamma)$ .

(11 marks)

11. Let  $P, Q$  be two points on a circle with centre  $C$  such that  $P, Q, C$  are non-collinear and taken anti-clockwise.  $\angle PCQ = \alpha$  and  $M$  is the mid-point of  $PQ$ . Let  $z_P, z_Q, z_C$  and  $z_M$  be the complex numbers represented by  $P, Q, C$  and  $M$  respectively.

(a) Show that  $z_C - z_M = i(z_M - z_P) \cot \frac{\alpha}{2}$ . (5 marks)

(b) Express  $z_C$  and the radius  $r$  of the circle in terms of  $z_P, z_Q$  and  $\alpha$ . (4 marks)

(c) (i) Show that any circle in the complex plane can be represented by an equation of the form  $z\bar{z} + az + b\bar{z} + c = 0$  where  $a, b \in \mathbb{C}$  and  $c \in \mathbb{R}$ .

(ii) Let  $\mathcal{C}: z\bar{z} + az + b\bar{z} + c = 0$  be a circle passing through the points representing  $1+i$  and  $-i$ . If the chord joining these two points subtends an angle  $\frac{\pi}{3}$  at the centre, find the values of  $a, b$  and  $c$ . (6 marks)

12. Let  $p > 0$  and  $p \neq 1$ .  $\{a_n\}$  is a sequence of positive numbers

$$\text{defined by } \begin{cases} a_0 = 2 \\ a_n = \frac{1}{p\sqrt{n}} + \frac{1}{p}a_{n-1}, \quad n = 1, 2, 3, \dots \end{cases}$$

(a) Prove that  $\lim_{n \rightarrow \infty} a_n = 0$  if the limit exists. (2 marks)

(b) (i) If  $2 = a_0 < a_1 < a_2 < \dots$ , show that  $\lim_{n \rightarrow \infty} a_n$  does not exist.

(ii) If  $a_{k-1} \geq a_k$  for some  $k \geq 1$ , show that  $a_{n-1} \geq a_n$  for  $n \geq k$  and deduce that  $\lim_{n \rightarrow \infty} a_n = 0$ . (4 marks)

(c) (i) If  $0 < p < 1$ , show that  $\lim_{n \rightarrow \infty} a_n$  does not exist.

(ii) If  $p \geq 2$ , show that  $\lim_{n \rightarrow \infty} a_n = 0$ . (4 marks)

(d) Suppose  $1 < p < 2$ .

(i) Prove by mathematical induction that  $a_n < \frac{2}{p-1}$  for  $n \geq 0$ .

(ii) Prove that  $\lim_{n \rightarrow \infty} a_n = 0$ . (5 marks)

PURE MATHEMATICS A-LEVEL PAPER II

2.00 pm-5.00 pm (3 hours)

This paper must be answered in English

13. Let  $a$  and  $b$  be positive numbers.

(a) Prove that

$$a^a b^b \geq a^b b^a$$

where, if the equality holds, then  $a = b$ .

(4 marks)

(b) Using (a), or otherwise, prove that

$$\left(\frac{a+b}{2}\right)^{a+b} \geq a^b b^a$$

where, if the equality holds, then  $a = b$ .

(3 marks)

(c) Show that  $x^x(1-x)^{1-x} \geq \frac{1}{2}$  for  $0 < x < 1$

where, if the equality holds, then  $x = \frac{1}{2}$ .

Deduce that  $a^a b^b \geq \left(\frac{a+b}{2}\right)^{a+b}$

where, if the equality holds, then  $a = b$ .

(8 marks)

END OF PAPER

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

**SECTION A (40 marks)**

Answer ALL questions in this section.

Write your answers in the AL(C1) answer book.

1. (a) Evaluate  $\lim_{x \rightarrow 0} \left( \frac{a^x + b^x + 1}{3} \right)^{\frac{1}{x}}$  where  $a, b > 0$ .

(b) By considering a suitable definite integral, evaluate

$$\lim_{n \rightarrow \infty} \left( \frac{1^2}{n^3 + 1^3} + \frac{2^2}{n^3 + 2^3} + \dots + \frac{n^2}{n^3 + n^3} \right).$$

(6 marks)

2. (a) Using the substitution  $x = \sin^2 \theta$  ( $0 < \theta < \frac{\pi}{2}$ ), prove that

$$\int \frac{f(x)}{\sqrt{x(1-x)}} dx = 2 \int f(\sin^2 \theta) d\theta.$$

(b) Hence, or otherwise, evaluate

$$\int \frac{dx}{\sqrt{x(1-x)}} \quad \text{and} \quad \int \sqrt{\frac{x}{1-x}} dx.$$

(5 marks)

3. Consider the parabola  $y^2 = 4ax$ .

(a) Prove that the equation of the normal at  $P(at^2, 2at)$  is  $y + tx = 2at + at^3$ . .....

(b)  $P_i(at_i^2, 2at_i)$ ,  $i = 1, 2, 3$ , are three distinct points on the parabola. Suppose the normals at these points are concurrent. By considering (\*) as a cubic equation in  $t$ , or otherwise, show that  $t_1 + t_2 + t_3 = 0$ .

(5 marks)

4. For  $x \geq 0$ , define  $F(x) = \int_0^x \frac{\sin t}{t+1} dt$ .

(a) Find the value of  $x_0$  for which  $F(x) \leq F(x_0)$  for all  $x \in [0, 2\pi]$ .

(b) By considering  $F(0)$  and  $F(2\pi)$ , show that  $F(x) > 0$  for all  $x \in (0, 2\pi)$ .

(7 marks)

5. Figure 1 shows the graphs of the circle  $\Gamma_1 : r = -2\cos \theta$  and the cardioid  $\Gamma_2 : r = 2 + 2\cos \theta$ .

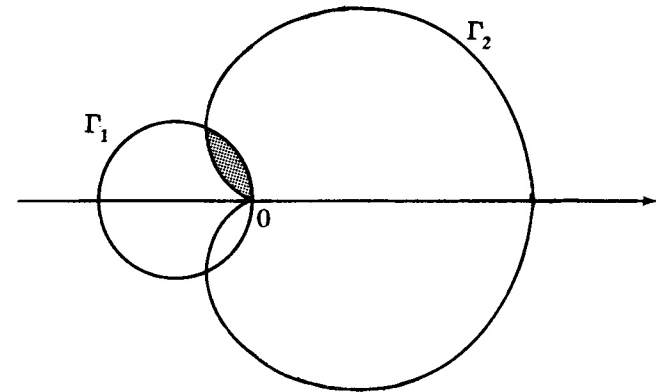


Figure 1

(a) Find the polar coordinates of all the intersecting points of  $\Gamma_1$  and  $\Gamma_2$  other than the pole.

(b) Find the area of the shaded region.

(6 marks)

6. Let  $r$  be a real number. Define  $y = \left(\frac{x+1}{x-1}\right)^r$  for  $x > 1$ .

(a) Show that  $\frac{dy}{dx} = \frac{-2ry}{x^2 - 1}$ .

(b) For  $n = 1, 2, 3, \dots$ , show that

$$(x^2 - 1)y^{(n+1)} + 2(nx + r)y^{(n)} + (n^2 - n)y^{(n-1)} = 0,$$

where  $y^{(0)} = y$  and  $y^{(k)} = \frac{d^k y}{dx^k}$  for  $k \geq 1$ .

(5 marks)

7. Let  $f: \mathbf{R} \rightarrow (-1, \infty)$  be a differentiable function.

(a) Differentiate  $\ln[1 + f(x)]$ .

(b) If  $f(x) = x^3 + \int_0^x 3t^2 f(t) dt$  for all  $x \in \mathbf{R}$ , by considering  $f'(x)$ , find  $f(x)$ .

(6 marks)

**SECTION B (60 marks)**

Answer any **FOUR** questions from this section. Each question carries 15 marks.

Write your answers in the **AL(C2) answer book**.

8. Let  $I_k = \int_0^1 \frac{(-1)^k (1-x)x^{3k}}{1+x^3} dx$ ,  $k = 0, 1, 2, \dots$ .

(a) Evaluate  $I_0$ . (4 marks)

(b) Prove that  $-\frac{1}{3k+1} \leq I_k \leq \frac{1}{3k+1}$ . (3 marks)

(c) Express  $I_{k+1} - I_k$  in terms of  $k$ . (3 marks)

(d) For  $n = 0, 1, 2, \dots$ , let  $b_n = \sum_{k=0}^n \frac{(-1)^k}{(3k+1)(3k+2)}$ .

Using (a) and (c), express  $I_{n+1}$  in terms of  $b_n$ .

Hence use (b) to evaluate  $\lim_{n \rightarrow \infty} b_n$ . (5 marks)

9. Let  $f(x) = \frac{|x|}{(x+1)^2}$ , where  $x \neq -1$ .

- (a) (i) Find  $f'(x)$  and  $f''(x)$  for  $x > 0$ .  
 (ii) Find  $f'(x)$  and  $f''(x)$  for  $x < 0$ .  
 (iii) Show that  $f'(0)$  does not exist.

(4 marks)

(b) Determine the values of  $x$  for each of the following cases:

- (i)  $f'(x) < 0$ ,           (ii)  $f'(x) > 0$ ,  
 (iii)  $f''(x) < 0$ ,       (iv)  $f''(x) > 0$ .

(4 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of  $f(x)$ .  
 (3 marks)

(d) Find the asymptote(s) and sketch the graph of  $f(x)$ .  
 (4 marks)

10. For any  $\beta > 0$ , define a sequence of real numbers as follows:

$$a_1 = \beta + 1, \quad a_n = a_{n-1} + \frac{\beta}{a_{n-1}} \quad \text{for } n > 1.$$

(a) Prove that

(i)  $a_n^2 \geq a_{n-1}^2 + 2\beta$  for  $n \geq 2$ ;

(ii)  $a_n^2 \geq \beta^2 + 2n\beta + 1$  for  $n \geq 1$ .

(2 marks)

(b) Using (a), show that for  $n \geq 2$ ,

$$a_n^2 \leq \beta^2 + 2n\beta + 1 + \sum_{k=1}^{n-1} \frac{\beta^2}{\beta^2 + 2k\beta + 1}.$$

(3 marks)

(c) Prove that for  $k \geq 1$ ,

$$\frac{1}{\beta^2 + 2k\beta + 1} \leq \int_{k-1}^k \frac{1}{\beta^2 + 2\beta x + 1} dx.$$

(2 marks)

(d) Using the above results, show that  $\lim_{n \rightarrow \infty} \frac{a_n^2}{n}$  exists and find the limit.

State with reasons whether  $\lim_{n \rightarrow \infty} \frac{a_n^2}{\sqrt{n}}$  exists.

(8 marks)



11. (a) Evaluate  $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 \sin \theta + \cos \theta + 2}$ .  
(5 marks)

(b) Let  $f(\theta) = a \sin \theta + b \cos \theta + c$  and  
 $g(\theta) = A \sin \theta + B \cos \theta + C$   
where  $A, B$  are not both zero.

Show that there exist real numbers  $p, q$  and  $r$  such that

$$f(\theta) = p g(\theta) + q g'(\theta) + r$$

for all real numbers  $\theta$ .

(5 marks)

(c) Hence, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{7 \sin \theta - 4 \cos \theta + 3}{2 \sin \theta + \cos \theta + 2} d\theta.$$

(5 marks)

12. Consider the lines

$$L_1 : \frac{x-5}{2} = \frac{y-1}{2} = \frac{z}{-1}$$

and  $L_2 : \frac{x-4}{2} = \frac{y+8}{5} = \frac{z-1}{2}$ .

(a) Show that  $L_1$  and  $L_2$  do not intersect. (2 marks)

(b) Let  $L$  be the line perpendicular to  $L_1$  and  $L_2$  intersecting  $L_1$  at  $A$  and  $L_2$  at  $B$ .

(i) Find the coordinates of  $A$  and  $B$ .

(ii) Find the equations of  $L$ .

(7 marks)

(c) Let  $\pi$  be the plane containing the point  $A$  and perpendicular to  $L_1$ .

(i) Find the equation of  $\pi$ .

(ii) Show that  $B$  lies on  $\pi$ .

(iii) Find the equations of the projection of  $L_2$  on  $\pi$ .

(6 marks)

PURE MATHEMATICS A-LEVEL PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

13. (a) Suppose  $f(x)$ ,  $g(x)$  are continuously differentiable functions such that  $f'(x) \geq 0$  for  $a \leq x \leq b$ .

(i) Let  $w(x) = \int_a^x g(t) dt$ . Show that

$$\int_a^b f(x)g(x)dx = f(b) \int_a^b g(x)dx - \int_a^b f'(x)w(x)dx.$$

(ii) Using the Theorem (\*) below, show that

$$\int_a^b f(x)g(x)dx = f(b) \int_c^b g(x)dx + f(a) \int_a^c g(x)dx$$

for some  $c \in [a, b]$ .

[ Theorem (\*) : If  $w(x)$ ,  $u(x)$  are continuous functions and  $u(x) \geq 0$  for  $a \leq x \leq b$ , then

$$\int_a^b w(x)u(x)dx = w(c) \int_a^b u(x)dx \text{ for some } c \in [a, b]. ]$$

(5 marks)

(b) Let  $F(x)$  be a function with a continuous second derivative such that  $F''(x) \geq 0$  and  $F'(x) \geq m > 0$  for  $a \leq x \leq b$ . Using (a) with  $f(x) = -\frac{1}{F'(x)}$  and  $g(x) = -F'(x) \cos F(x)$ , show that

$$\left| \int_a^b \cos F(x) dx \right| \leq \frac{4}{m}. \quad (5 \text{ marks})$$

(c) (i) Show that  $\int_0^1 \cos(x^n) dx \leq \int_0^1 \cos(x^{n+1}) dx$ .

Hence show that  $\lim_{n \rightarrow \infty} \int_0^1 \cos(x^n) dx$  exists.

(ii) Using (b), or otherwise, show that  $\lim_{n \rightarrow \infty} \int_0^{2\pi} \cos(x^n) dx$  exists.

(5 marks)

END OF PAPER