

1.  $(a_i + cb_i)^2 \geq 0 \quad \forall c \in \mathbb{R}, \forall i = 1, \dots, n$

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-  $\sum_{i=1}^n (a_i + cb_i)^2 \geq 0 \quad \forall c \in \mathbb{R}$

-  $\sum_{i=1}^n (a_i^2 + 2ca_i b_i + c^2 b_i^2) \geq 0 \quad \forall c \in \mathbb{R}$

-  $\left( \sum_{i=1}^n a_i^2 \right) + \left( 2 \sum_{i=1}^n a_i b_i \right) c + \left( \sum_{i=1}^n b_i^2 \right) c^2 \geq 0 \quad \forall c \in \mathbb{R}$

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-  $0 \geq \Delta = \left( 2 \sum_{i=1}^n a_i b_i \right)^2 - 4 \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right)$

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-  $4 \left( \sum_{i=1}^n a_i b_i \right)^2 \leq 4 \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right)$

-  $\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right)$

Put  $b_i = 1 \quad \forall i = 1, \dots, n$

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we have

$\left( \sum_{i=1}^n a_i \cdot 1 \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n 1^2 \right)$

-  $\left( \sum_{i=1}^n a_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \cdot n$

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-  $\sum_{i=1}^n a_i \leq \sqrt{n \sum_{i=1}^n a_i^2}$

-  $\frac{1}{n} \sum_{i=1}^n a_i \leq \sqrt{\frac{1}{n} \sum_{i=1}^n a_i^2}$

2. For  $n = 1$ ,

$$\text{LHS} = u_1$$

$$= 1$$

$$\text{RHS} = \alpha^1 + \beta^1$$

$$= \alpha + \beta$$

$$= 1$$

$$\therefore \text{LHS} = \text{RHS}$$

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For  $n = 2$ ,

$$\text{LHS} = u_2$$

$$= 3$$

$$\text{RHS} = \alpha^2 + \beta^2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= 1^2 - 2(-1)$$

$$= 3$$

$$\therefore \text{LHS} = \text{RHS}$$

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Assume  $u_k = \alpha^k + \beta^k$  and  $u_{k-1} = \alpha^{k-1} + \beta^{k-1}$  for some  $k \geq 2$ .

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Then for  $n = k + 1$ ,

$$\text{LHS} = u_{k+1}$$

$$= u_k + u_{k-1}$$

$$= \alpha^k + \beta^k + \alpha^{k-1} + \beta^{k-1}$$

$$= \alpha^k + \alpha^{k-1} + \beta^k + \beta^{k-1}$$

$$= \alpha^{k-1}(\alpha + 1) + \beta^{k-1}(\beta + 1)$$

$$= \alpha^{k-1}\alpha^2 + \beta^{k-1}\beta^2$$

$$= \alpha^{k+1} + \beta^{k+1}$$

$$= \text{RHS}$$

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By the principle of mathematical induction,  $u_n = \alpha^n + \beta^n \forall n \geq 1$ .

3. (a) Let  $x = \alpha$ ,  $y = \beta$ ,  $z = \gamma$  be a solution.

$$\text{Then } \begin{cases} a\alpha + b\beta + c\gamma = 1 \\ b\alpha + c\beta + a\gamma = 1 \\ c\alpha + a\beta + b\gamma = 1 \\ \alpha + \beta + \gamma = 3 \end{cases}$$

$$- \begin{cases} (a+b+c)(\alpha+\beta+\gamma) = 3 & \text{and} \\ (\alpha+\beta+\gamma) = 3 \end{cases}$$

$$- (a+b+c) \cdot 3 = 3$$

$$- a+b+c = 1$$

(b) (\*) is equivalent to

$$(*)' \begin{cases} ax + by + cz = 1 \\ bx + cy + az = 1 \\ cx + ay + bz = 1 \end{cases}$$

$$\text{Consider } \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} \quad (\because a+b+c=1)$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$$

$$= ac + ab + bc - a^2 - b^2 - c^2$$

$$= -\frac{1}{2} [(a-b)^2 + (b-c)^2 + (c-a)^2]$$

So (\*) has a unique solution if and only if  $a, b, c$  are not all equal.

(c) We have  $\Delta = 0$ , and  $a = b = c (= \frac{1}{3})$ .

Thus (\*) becomes  $x + y + z = 3$ .

$\therefore$  general solution =  $\{(s, t, 3-s-t) : s, t \in \mathbb{R}\}$

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4. (a)  $|1 + z| = |2 - z|$

$$(1 + z)(\overline{1 + z}) = (2 - z)(\overline{2 - z})$$

$$1 + z + \overline{z} + z\overline{z} = 4 - 2z - 2\overline{z} + z\overline{z}$$

$$3(z + \overline{z}) = 3$$

$$z + \overline{z} = 1$$

$$\operatorname{Re}(z) = \frac{1}{2}$$

(b) Let  $z = \frac{1}{2} + it$

Substitute it into the 1st equation.

$$\left| \frac{1}{2} + it \right|^2 - \left( \frac{1}{2} + it \right) - \overline{\left( \frac{1}{2} + it \right)} + i \left[ \left( \frac{1}{2} + it \right) - \overline{\left( \frac{1}{2} + it \right)} \right] = \frac{1}{2}$$

$$\left( \frac{1}{2} + it \right) \left( \frac{1}{2} - it \right) - \left( \frac{1}{2} + it \right) - \left( \frac{1}{2} - it \right) + i \left[ \left( \frac{1}{2} + it \right) - \left( \frac{1}{2} - it \right) \right] = \frac{1}{2}$$

$$\left( \frac{1}{2} + it \right) \left( \frac{1}{2} - it \right) - 1 + i(2it) = \frac{1}{2}$$

$$\frac{1}{4} - (it)^2 - 1 - 2t = \frac{1}{2}$$

$$\frac{1}{4} + t^2 - 1 - 2t = \frac{1}{2}$$

$$1 + 4t^2 - 4 - 8t = 2$$

$$4t^2 - 8t - 5 = 0$$

$$(2t - 5)(2t + 1) = 0$$

$$t = \frac{5}{2} \quad \text{or} \quad -\frac{1}{2}$$

$$\therefore z = \frac{1}{2} + \frac{5}{2}i \quad \text{or} \quad \frac{1}{2} - \frac{1}{2}i$$

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5. Let  $\frac{x+4}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$ .

Then  $x+4 = A(x+2) + B(x+1)$

$x+4 = (A+B)x + (2A+B)$

$$\therefore \begin{cases} A+B=1 \\ 2A+B=4 \end{cases}$$

Solving for  $A, B$ , we have  $A=3, B=-2$ .

$$\therefore \frac{x+4}{x^2+3x+2} = \frac{3}{x+1} - \frac{2}{x+2}$$

$$\sum_{k=2}^N \left\{ \frac{1}{k-1} - \frac{k+4}{k^2+3k+2} \right\}$$

$$= \sum_{k=2}^N \left\{ \frac{1}{k-1} - \frac{3}{k+1} + \frac{2}{k+2} \right\}$$

$$= \sum_{k=2}^N \frac{1}{k-1} - 3 \sum_{k=2}^N \frac{1}{k+1} + 2 \sum_{k=2}^N \frac{1}{k+2}$$

$$= \sum_{k=1}^{N-1} \frac{1}{k} - 3 \sum_{k=3}^{N+1} \frac{1}{k} + 2 \sum_{k=4}^{N+2} \frac{1}{k}$$

$$= \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right) - 3 \left( \frac{1}{3} \right) - \frac{3}{N} - \frac{3}{N+1} + \frac{2}{N} + \frac{2}{N+1} + \frac{2}{N+2}$$

$$\rightarrow \frac{5}{6} \text{ as } N \rightarrow \infty$$

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$$6. \quad (a) \quad \therefore \det A = \det (A^3)$$

$$= \det (-A)$$

$$= (-1)^3 \det A$$

$$= -\det A$$

$$\therefore \det A = 0$$

$$(b) \quad (I - B)^2 = \begin{pmatrix} 0 & 2 & -74 \\ -2 & 0 & 67 \\ 74 & -67 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 74 \\ 2 & 0 & -67 \\ -74 & 67 & 0 \end{pmatrix}$$

$$= -(I - B)$$

$$\text{by (a), } \det (I - B) = 0$$

Alternatively,

$$\det (I - B) = \det \begin{pmatrix} 0 & 2 & -74 \\ -2 & 0 & 67 \\ 74 & -67 & 0 \end{pmatrix}$$

$$= -2 \begin{vmatrix} -2 & 67 \\ 74 & 0 \end{vmatrix} - 74 \begin{vmatrix} -2 & 0 \\ 74 & -67 \end{vmatrix}$$

$$= -2(74)(67) + 74(2)(67)$$

$$= 0$$

$$\text{Now } (I - B)(I + B + B^2 + B^3) = I + B + B^2 + B^3 - B - B^2 - B^3 - B^4$$

$$= I - B^4$$

$$\text{therefore } \det (I - B^4) = \det ((I - B)(I + B + B^2 + B^3))$$

$$= \det (I - B) \det (I + B + B^2 + B^3)$$

$$= 0$$

Alternatively,

$$I - B^4 = (I - B^2)(I + B^2) = (I - B)(I + B)(I + B^2)$$

$$\therefore \det (I - B^4) = \det ((I - B)(I + B)(I + B^2))$$

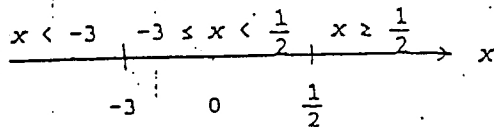
$$= \det (I - B) \det ((I + B)(I + B^2))$$

$$= 0$$

7. Substitute  $y$  into the 1st inequality.

$$|2x - 1| > |x + 3| + 1$$

Divide the real line into 3 regions as follows:



When  $x < -3$ ,

$$\begin{aligned} & |2x - 1| > |x + 3| + 1 \\ - & -2x + 1 > -x - 3 + 1 \\ - & x < 3 \end{aligned}$$

$$\therefore \begin{cases} x = t \\ y = -t - 3 \end{cases} \text{ where } t < -3$$

When  $-3 \leq x < \frac{1}{2}$

$$\begin{aligned} & |2x - 1| > |x + 3| + 1 \\ - & -2x + 1 > x + 3 + 1 \\ - & -3 > 3x \\ - & x < -1 \end{aligned}$$

$$\therefore \begin{cases} x = t \\ y = t + 3 \end{cases} \text{ where } -3 \leq t < -1$$

When  $x \geq \frac{1}{2}$

$$\begin{aligned} & |2x - 1| > |x + 3| + 1 \\ - & 2x - 1 > x + 3 + 1 \\ - & x > 5 \end{aligned}$$

$$\therefore \begin{cases} x = t \\ y = t + 3 \end{cases} \text{ where } t > 5$$

$$\therefore \text{ answer is } \begin{cases} x = t \\ y = t + 3 \end{cases} \text{ where } t > 5$$

$$\text{or } \begin{cases} x = t \\ y = t + 3 \end{cases} \text{ where } -3 \leq t < -1$$

$$\text{or } \begin{cases} x = t \\ y = -t - 3 \end{cases} \text{ where } t < -3$$

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8. (a) If  $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0$ , then the following system of linear

equations has non-zero solution:

$$(*) \begin{cases} u_1x + v_1y + w_1z = 0 \\ u_2x + v_2y + w_2z = 0 \\ u_3x + v_3y + w_3z = 0 \end{cases}$$

→  $\exists \alpha, \beta, \gamma$ , not all zero, such that

$$\begin{cases} u_1\alpha + v_1\beta + w_1\gamma = 0 \\ u_2\alpha + v_2\beta + w_2\gamma = 0 \\ u_3\alpha + v_3\beta + w_3\gamma = 0 \end{cases}$$

→  $\alpha(u_1, u_2, u_3) + \beta(v_1, v_2, v_3) + \gamma(w_1, w_2, w_3) = 0$

→  $u, v, w$  are linearly dependent

→ a contradiction

(b) Let  $x = (s_1, s_2, s_3)$ ,  $u = (u_1, u_2, u_3)$ ,  $v = (v_1, v_2, v_3)$  and  $w = (w_1, w_2, w_3)$ .

Then  $\begin{cases} u_1s_1 + u_2s_2 + u_3s_3 = 0 \\ v_1s_1 + v_2s_2 + v_3s_3 = 0 \\ w_1s_1 + w_2s_2 + w_3s_3 = 0 \end{cases}$

by (a),  $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \neq 0$

→ unique solution exists for (\*).

→  $x = 0$

(c)  $u \times (v \times w) = 0$

→  $u = \mu(v \times w)$  for some  $\mu \in \mathbb{R}$

$$\begin{cases} u \cdot v = \mu(v \times w) \cdot v = \mu 0 = 0 \\ u \cdot w = \mu(v \times w) \cdot w = \mu 0 = 0 \end{cases}$$

Similarly,  $(u \times v) \times w = 0$

→  $w = \lambda(u \times v)$  for some  $\lambda \in \mathbb{R}$

→  $w \cdot v = \lambda((u \times v) \cdot v) = \lambda 0 = 0$

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(d) Let  $r = \alpha u + \beta v + \gamma w$  for some  $\alpha, \beta, \gamma \in \mathbb{R}$

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$$\begin{aligned} \text{then } r \cdot u &= (\alpha u + \beta v + \gamma w) \cdot u \\ &= \alpha u \cdot u + \beta v \cdot u + \gamma w \cdot u \\ &= \alpha u \cdot u + 0 + 0 \\ &\Rightarrow \alpha = \frac{r \cdot u}{u \cdot u} \quad (v \cdot u = w \cdot u = 0) \end{aligned}$$

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Similarly, we can show that

$$\beta = \frac{r \cdot v}{v \cdot v}$$

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$$\gamma = \frac{r \cdot w}{w \cdot w}$$

$$\text{hence } r = \frac{r \cdot u}{u \cdot u} u + \frac{r \cdot v}{v \cdot v} v + \frac{r \cdot w}{w \cdot w} w$$

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Alternatively, consider

$$s = r - \left( \frac{r \cdot u}{u \cdot u} u + \frac{r \cdot v}{v \cdot v} v + \frac{r \cdot w}{w \cdot w} w \right)$$

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$$\text{Since } s \cdot u = r \cdot u - \frac{r \cdot u}{u \cdot u} u \cdot u = 0$$

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$$s \cdot v = r \cdot v - \frac{r \cdot v}{v \cdot v} v \cdot v = 0$$

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$$s \cdot w = r \cdot w - \frac{r \cdot w}{w \cdot w} w \cdot w = 0$$

by (b),  $s = 0$

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$$\Rightarrow r = \frac{r \cdot u}{u \cdot u} u + \frac{r \cdot v}{v \cdot v} v + \frac{r \cdot w}{w \cdot w} w$$

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9. (a) (i)  $f(0) = f(0 + 0)$

$$= f(0) + f(0)$$

$$\Rightarrow f(0) = 0$$

(ii)  $f(-x) + f(x) = f(-x + x)$

$$= f(0)$$

$$= 0$$

$$\Rightarrow f(-x) = -f(x)$$

(iii) By (i), we need only to show  $f(nx) = nf(x)$  for  $n = \pm 1, \pm 2, \dots$

Case 1:  $n > 0$

We shall use mathematical induction to show that  $f(nx) = nf(x)$ .

For  $n = 1$ ,

$$f(1 \cdot x) = f(x)$$

$$= 1 \cdot f(x)$$

Assume  $f(kx) = kf(x)$ .

Then  $f((k+1)x) = f(kx + x)$

$$= f(kx) + f(x)$$

$$= kf(x) + f(x)$$

$$= (k+1)f(x)$$

Case 2:  $n < 0$

$$f(nx) = f((-n)(-x))$$

$$= (-n)f(-x) \quad (\text{by Case 1})$$

$$= -n(-f(x)) \quad (\text{by (ii)})$$

$$= nf(x)$$

(b) If  $\exists x_0 \in \mathbb{R}$  such that  $f(x_0) \neq 0$ , then  $f(x_0) > 0$  or  $f(x_0) < 0$ .

Case 1:  $f(x_0) > 0$

Then we can choose a positive integer  $n$  such that

$$nf(x_0) > K$$

$$\Rightarrow f(nx_0) > K$$

contradicting the fact that  $f(x) < K$  for all  $x \in \mathbb{R}$ .

Case 2:  $f(x_0) < 0$

Replace  $x_0$  by  $-x_0$  and use the same arguments in Case 1.

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$$(c) (i) g(x+y) = f(x+y) - f(1)(x+y)$$

$$= f(x) + f(y) - f(1)x - f(1)y$$

$$= (f(x) - f(1)x) + (f(y) - f(1)y)$$

$$= g(x) + g(y)$$

$$(ii) g(x+1) = f(x+1) - f(1)(x+1)$$

$$= f(x) + f(1) - f(1)x - f(1)$$

$$= f(x) - f(1)x$$

$$= g(x)$$

(iii)  $\forall x \in \mathbb{R}$ , there exists  $h \in [0, 1)$  such that  $x - h$  is an integer.

$$\text{By (b)(ii), } g(x) = g(h)$$

$$= f(h) - f(1)h$$

$$< K - f(1)h$$

$$< K + |f(1)| \quad (\because 0 \leq h < 1)$$

$$\text{By (b), } g(x) = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow f(x) - f(1)x = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow f(x) = f(1)x \quad \forall x \in \mathbb{R}$$

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10. (a) (Reflexive)

$$\forall A \in M,$$

$$A = IAI^{-1}$$

$$\therefore A = A$$

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(Symmetric)

$$\forall A, B \in M,$$

$$A \sim B$$

$$\text{--- } A = PBP^{-1} \text{ for some } P$$

$$\text{--- } P^{-1}AP = B$$

$$\text{--- } B = P^{-1}A(P^{-1})^{-1}$$

$$\text{--- } B \sim A$$

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(Transitive)

$$\forall A, B, C \in M,$$

$$A \sim B \text{ and } B \sim C$$

$$\text{--- } A = PBP^{-1} \text{ and } B = QCQ^{-1} \text{ for some } P, Q$$

$$\text{--- } A = P(QCQ^{-1})P^{-1}$$

$$\text{--- } A = (PQ)C(PQ)^{-1}$$

$$\text{--- } A \sim C$$

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3(b) If  $A \sim B$ then  $A = PBP^{-1}$  for some  $P$ 

$$\text{--- } A^k = (PBP^{-1})^k$$

$$= (PBP^{-1}) \dots (PBP^{-1})$$

$$= PB(P^{-1}P)B \dots B(P^{-1}P)BP^{-1}$$

$$= PBIB \dots BIBP^{-1}$$

$$= PB^kP^{-1}$$

$$\text{--- } A^k \sim B^k$$

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2(c) (i) If  $C = 0$ then  $C = POP^{-1}$  for some  $P$ 

$$= 0$$

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(ii) Consider  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and  $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then  $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and  $BA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= 0$

$\therefore \forall$  non-singular  $P$ ,  $P(BA)P^{-1} = 0$

$\neq AB$  ( $\because AB \neq 0$ )

$\therefore AB \neq BA$

(d) (-)

If  $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

then  $\exists P$  such that  $A = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} P^{-1}$

$\therefore AP = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

Let  $P = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$

then  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$  and  $(x_3, y_3, z_3)$  are linearly independent

( $\because P$  is non-singular)

Moreover,

$A \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

$$\rightarrow \left( A \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad A \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad A \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right) = \left( \lambda_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \lambda_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad \lambda_3 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right)$$

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$$\rightarrow A \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \lambda_i \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \quad i = 1, 2, 3.$$

(-)

$$\text{Consider } P = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

Since  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$  and  $(x_3, y_3, z_3)$  are linearly independent,  $P$  is non-singular.

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Furthermore,

$$\begin{aligned} AP &= A \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \\ &= \left( A \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad A \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad A \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right) \\ &= \left( \lambda_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \lambda_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad \lambda_3 \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right) \\ &= \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \\ &= P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \end{aligned}$$

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$$\rightarrow A = P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} P^{-1} \quad (\because P \text{ is non-singular})$$

$$\rightarrow A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

11. (a)  $\Delta Z_1 Z_2 Z_3 \sim \Delta W_1 W_2 W_3$ 

$$\angle Z_3 Z_1 Z_2 = \angle W_3 W_1 W_2 \text{ and } \frac{Z_3 Z_1}{Z_2 Z_1} = \frac{W_3 W_1}{W_2 W_1}$$

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{w_3 - w_1}{w_2 - w_1}\right) \text{ and } \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{w_3 - w_1}{w_2 - w_1}\right|$$

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{w_3 - w_1}{w_2 - w_1}\right) \text{ and } \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{w_3 - w_1}{w_2 - w_1}\right|$$

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{w_3 - w_1}{w_2 - w_1}$$

1

1

1

1

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4(b) Let  $E_1, E_2, E_3$  be the points representing  $1, \epsilon, \epsilon^2$  respectively.

1

Then,  $\Delta Z_1 Z_2 Z_3$  is equilateral

$$\Delta Z_1 Z_2 Z_3 \sim \Delta E_1 E_2 E_3$$

1M

$$(z_3 - z_1)(\epsilon - 1) = (z_2 - z_1)(\epsilon^2 - 1) \text{ (by (a))}$$

1M

$$(z_3 - z_1)(\epsilon - 1) = (z_2 - z_1)(\epsilon - 1)(\epsilon + 1)$$

$$z_3 - z_1 = (z_2 - z_1)(\epsilon + 1)$$

$$z_3 - z_1 = z_2(\epsilon + 1) - z_1(\epsilon + 1)$$

$$(\epsilon + 1 - 1)z_1 - (\epsilon + 1)z_2 + z_3 = 0$$

$$\epsilon z_1 - (1 + \epsilon)z_2 + z_3 = 0$$

$$\epsilon z_1 + \epsilon^2 z_2 + z_3 = 0 \quad (\because 1 + \epsilon + \epsilon^2 = 0)$$

1

$$\epsilon^2(\epsilon z_1 + \epsilon^2 z_2 + z_3) = 0$$

$$\epsilon^3 z_1 + \epsilon^4 z_2 + \epsilon^2 z_3 = 0$$

$$z_1 + \epsilon z_2 + \epsilon^2 z_3 = 0 \quad (\because \epsilon^3 = 1)$$

1

---

5

(c) Let  $Z_1, Z_2, Z_3$  be distinct points representing  $z_j = a_j + ib_j$  with  $a_j, b_j \in \mathbb{Z}$  ( $j = 1, 2, 3$ ).

If  $\Delta Z_1 Z_2 Z_3$  is equilateral, then by (b),

$$(a_1 + b_1 i) + \epsilon(a_2 + b_2 i) + \epsilon^2(a_3 + b_3 i) = 0$$

1A

$$- (a_1 + b_1 i) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(a_2 + b_2 i) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)(a_3 + b_3 i) = 0$$

1A

$$\begin{cases} a_1 - \frac{a_2}{2} - \frac{\sqrt{3}}{2}b_2 - \frac{a_3}{2} + \frac{\sqrt{3}}{2}b_3 = 0 \\ b_1 + \frac{\sqrt{3}}{2}a_2 - \frac{b_2}{2} - \frac{\sqrt{3}}{2}a_3 - \frac{b_3}{2} = 0 \end{cases}$$

1

$$\begin{cases} (a_1 - \frac{a_2}{2} - \frac{a_3}{2}) + \frac{\sqrt{3}}{2}(b_3 - b_2) = 0 \\ (b_1 - \frac{b_2}{2} - \frac{b_3}{2}) + \frac{\sqrt{3}}{2}(a_2 - a_3) = 0 \end{cases}$$

1

$$\begin{cases} a_1 - \frac{a_2}{2} - \frac{a_3}{2} = 0 & \dots\dots\dots (1) \\ b_3 - b_2 = 0 & \dots\dots\dots (2) \\ b_1 - \frac{b_2}{2} - \frac{b_3}{2} = 0 & \dots\dots\dots (3) \\ a_2 - a_3 = 0 & \dots\dots\dots (4) \end{cases} \quad (\because a_i, b_i \in \mathbb{Z})$$

1

from (2),  $b_2 = b_3$   
from (4),  $a_2 = a_3$  }  $- z_2 = z_3$

- a contradiction ( $\because Z_2, Z_3$  are distinct)

1



12. (a)  $\forall p \in A$ ,Case-1  $p = 0$ then  $r|p$ 

1

Case 2  $p \neq 0$ 

then by Euclidean Algorithm,

 $p = qr + s$  where  $s = 0$  or  $\deg s < \deg r$ 

1

Now  $s = p - qr$ 

$$= (mf + ng) - q(m'f + n'g) \quad \text{where } p = mf + ng$$

$$\text{and } r = m'f + n'g$$

$$= (m - qm')f + (n - qn')g$$

$$\in A$$

1

 $\therefore \deg r \leq \deg s$  (by the property of  $r$ ) $\rightarrow \deg s < \deg r$  $\rightarrow s = 0$ 

1

Hence  $p = qr$  $\rightarrow r|p$ 

$$f = 1 \cdot f + 0 \cdot g \in A$$

 $\therefore r|f$ 

$$g = 0 \cdot f + 1 \cdot g \in A$$

 $\therefore r|g$ 

1

Thus  $r$  divides both  $f$  and  $g$ .

$$\text{If } \begin{cases} f = th \\ g = wh \end{cases} \text{ for some } t, w \in p$$

then  $r = m'f + n'g$ 

$$= m'th + n'wh$$

$$= (m't + n'w)h$$

1

 $\rightarrow h|r$ Hence  $r$  is a G.C.D. of  $f$  and  $g$ .

6

(b)  $\forall p \in A$ ,

$$p = mf + ng \text{ for some } m, n \in \rho$$

$$\rightarrow p = m(m'r) + n(n'r) \text{ for some } m', n' \in \rho \text{ (}\because r \text{ divides both } f \text{ and } g\text{)} \quad 1$$

$$= (mm' + nn')r$$

$$\in B$$

$$\therefore A \subset B.$$

$$\forall p \in B, p = hr$$

$$= h(mf + ng) \text{ (}\because r \in A\text{)} \quad 1$$

$$= (hm)f + (hn)g \quad 1$$

$$\in A$$

$$\therefore B \subset A$$

Therefore  $A = B$ .

---

 4

(c)  $r$  is a non-zero constant

$$\rightarrow r = m'f + n'g \text{ for some } m', n' \in \rho \text{ (}\because r \in A\text{)} \quad 1$$

$$\rightarrow 1 = m_0f + n_0g \text{ where } m_0 = \frac{m'}{r}, n_0 = \frac{n'}{r} \in \rho \quad 1$$

$$\text{By (b), } A = B$$

$$= \{hr : h \in \rho\}$$

$$= \{k : k \in \rho\} \text{ (}\because r \text{ is a non-zero constant)} \quad 1$$

$$= \rho$$

---

 5

13. (a) (i) If  $z \in G_p \cap H_p$

then  $z^p = 1$  and  $z^p = -1$ ,

a contradiction.

(ii)  $z \in G_p \cup H_p \rightarrow z \in G_p$  or  $z \in H_p$

$$\rightarrow z^p = 1 \text{ or } z^p = -1$$

$$\rightarrow z^{2p} = 1$$

$$\rightarrow z \in G_{2p}$$

(b) If  $z \in H_p \cap H_q$

then  $z \in H_p$  and  $z \in H_q$

$$\rightarrow z^p = -1 \text{ and } z^q = -1$$

we have  $z^{pq} = (z^p)^q = (-1)^q = 1$  ( $\because q$  is even)

and  $z^{pq} = (z^q)^p = (-1)^p = -1$  ( $\because p$  is odd)

$\rightarrow$  a contradiction

(c) (i)  $z \in G_q$

$$\rightarrow z^q = 1$$

$$\rightarrow z^p = z^{mq}$$

$$= (z^q)^m$$

$$= 1^m$$

$$= 1$$

$$\rightarrow z \in G_p$$

(ii)  $z \in H_q \rightarrow z^p = z^{mq}$

$$= (z^q)^m$$

$$= (-1)^m \quad (\because z \in H_q)$$

$$= -1 \quad (\because m \text{ is odd})$$

$$\rightarrow z \in H_p$$

1

1

2

1

1

2

1

1

1

(ii) To show  $G_p H_p = H_p$  :

$$z \in G_p H_p - z = st \text{ where } s \in G_p, t \in H_p$$

$$- z^p = (st)^p = s^p t^p = 1 \cdot (-1) = -1$$

$$- z \in H_p$$

1

$$z \in H_p - z = s\left(\frac{z}{s}\right) \text{ where } s \in G_p$$

$$- z \in G_p H_p \text{ (because } \left(\frac{z}{s}\right)^p = \frac{z^p}{s^p} = \frac{-1}{1} = -1)$$

1

To show  $G_p H_p = H_p G_p$  :

$$G_p H_p = \{z \in \mathbb{C} : z = st \text{ for some } s \in G_p, t \in H_p\}$$

$$= \{z \in \mathbb{C} : z = ts \text{ for some } t \in H_p, s \in G_p\}$$

$$= H_p G_p$$

1

6

$$\begin{aligned}
 (iii) \quad z \in H_q &\rightarrow z^p = z^{mq} \\
 &= (z^q)^m \\
 &= (-1)^m \quad (\because z \in H_q) \\
 &= 1 \quad (\because m \text{ is even}) \\
 &\Rightarrow z \in G_p
 \end{aligned}$$

(d) (i) To show  $G_p G_p = G_p$  :

$$\begin{aligned}
 z \in G_p G_p &\rightarrow z = st \text{ where } s, t \in G_p \\
 \rightarrow z^p &= (st)^p = s^p t^p = 1 \cdot 1 = 1 \\
 \rightarrow z &\in G_p
 \end{aligned}$$

$$\begin{aligned}
 z \in G_p &\rightarrow z^p = 1 \\
 \rightarrow z &= 1 \cdot z \text{ where } 1^p = z^p = 1 \\
 \rightarrow z &= 1 \cdot z \text{ where } 1, z \in G_p \\
 \rightarrow z &\in G_p G_p
 \end{aligned}$$

To show  $H_p H_p = G_p$  :

$$\begin{aligned}
 z \in H_p H_p &\rightarrow z = st \text{ where } s, t \in H_p \\
 \rightarrow z^p &= (st)^p = s^p t^p = (-1)(-1) = 1 \\
 \rightarrow z &\in G_p
 \end{aligned}$$

$$z \in G_p \rightarrow z = \left(\frac{z}{t}\right)(t) \text{ where } t \in H_p$$

$$\rightarrow z \in H_p H_p \text{ (because } \left(\frac{z}{t}\right)^p = \frac{z^p}{t^p} = \frac{1}{-1} = -1)$$

1. (a)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{-\sin x}$   
 $= -1$

1M

1A

(b)  $\lim_{x \rightarrow 0} \frac{(1 + mx)^n - (1 + nx)^m}{x^2} = \lim_{x \rightarrow 0} \frac{nm(1 + mx)^{n-1} - nm(1 + nx)^{m-1}}{2x}$   
 $= \lim_{x \rightarrow 0} \frac{n(n-1)m(1 + mx)^{n-2} - nm(m-1)(1 + nx)^{m-2}}{2}$  1M  
 $= \frac{n(n-1)m - nm(m-1)}{2}$   
 $= \frac{nm}{2} [(n-1) - (m-1)]$   
 $= \frac{nm(n-m)}{2}$

1M

1M

1A

Alternatively

$$\lim_{x \rightarrow 0} \frac{(1 + mx)^n - (1 + nx)^m}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{(1 + nm x + m^2 x^2 \frac{n(n-1)}{1 \cdot 2} + \dots) - (1 + nm x + n^2 x^2 \frac{m(m-1)}{1 \cdot 2} + \dots)}{x^2}$$

1M

$$= \lim_{x \rightarrow 0} (\frac{1}{2} (m^2 n(n-1) - n^2 m(m-1)) + \dots)$$

1M

$$= \frac{1}{2} mn(m(n-1) - n(m-1))$$

$$= \frac{nm(n-m)}{2}$$

1A

2. Let the equation of the plane be

$$k(x + y + z - 1) + (x + 4y + 3z) = 0$$

1M

$$(k + 1)x + (k + 4)y + (k + 3)z - k = 0$$

Since it is parallel to  $\frac{x - 1}{3} = \frac{y}{1} = \frac{z + 1}{1}$  which has direction ratio (3, 1, 1), we have

1A

$$(k + 1) \cdot 3 + (k + 4) \cdot 1 + (k + 3) \cdot 1 = 0$$

1M

$$3k + 3 + k + 4 + k + 3 = 0$$

$$5k + 10 = 0$$

$$k = -2$$

1A

the equation is

$$(-2 + 1)x + (-2 + 4)y + (-2 + 3)z - (-2) = 0$$

i.e.  $-x + 2y + z + 2 = 0$

1A

3. Since  $Q$  lies on  $OP$ , we may let  $P = (r, \theta)$  and  $Q = (s, \theta)$

1M

with  $r, s \geq 0$ .

Now  $OP \cdot OQ = a^2$

$$r \cdot s = a^2 \dots\dots\dots(1)$$

1A

But  $P$  lies on  $C$ ,

$$r = 2a \cos \theta \dots\dots\dots(2)$$

1A

Eliminating  $r$  from (1) and (2), we have

$$2s a \cos \theta = a^2$$

1A

$$s \cos \theta = \frac{a}{2}$$

$$x = \frac{a}{2} \quad (\because x = s \cos \theta)$$

1M



$$4. \frac{dx}{dt} = 3\sin^2 t \cos t$$

$$\frac{dy}{dt} = -3\cos^2 t \sin t$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \sqrt{9\sin^4 t \cos^2 t + 9\cos^4 t \sin^2 t} dt$$

$$= 3\sin t \cos t dt$$

$$\text{Surface area} = 2 \int_0^{\frac{\pi}{2}} (2\pi y) (3\sin t \cos t) dt$$

$$= 12\pi \int_0^{\frac{\pi}{2}} \cos^4 t \sin t dt$$

$$= -12\pi \int_0^{\frac{\pi}{2}} \cos^4 t d(\cos t)$$

$$= -12\pi \left[ \frac{\cos^5 t}{5} \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{12\pi}{5} (0 - 1)$$

$$= \frac{12\pi}{5}$$

1

1

1M

1M

1A

$$5. \int e^{2x} (\sin x + \cos x)^2 dx = \int e^{2x} (\sin^2 x + \cos^2 x + 2 \sin x \cos x) dx$$

$$= \int e^{2x} (1 + 2 \sin x \cos x) dx$$

$$= \int e^{2x} dx + 2 \int e^{2x} \sin x \cos x dx$$

$$= \frac{1}{2} e^{2x} + \int e^{2x} \sin 2x dx$$

$$= \frac{1}{2} e^{2x} + \frac{1}{2} \int e^y \sin y dy \quad (y = 2x)$$

$$= \frac{1}{2} e^{2x} + \frac{1}{2} I \quad \text{where } I = \int e^y \sin y dy$$

$$\text{now } I = \int e^y \sin y dy$$

$$= \int \sin y d(e^y)$$

$$= e^y \sin y - \int e^y d(\sin y)$$

$$= e^y \sin y - \int e^y \cos y dy$$

$$= e^y \sin y - \int \cos y d(e^y)$$

$$= e^y \sin y - \left\{ e^y \cos y - \int e^y d(\cos y) \right\}$$

$$= e^y \sin y - \left\{ e^y \cos y + \int e^y \sin y dy \right\}$$

$$= e^y \sin y - e^y \cos y - I$$

$$\therefore 2I = e^y \sin y - e^y \cos y$$

$$\therefore I = \frac{1}{2} e^y (\sin y - \cos y) + c'$$

$$\text{Therefore, } \int e^{2x} (\sin x + \cos x)^2 dx = \frac{1}{2} e^{2x} + \frac{1}{4} e^{2x} (\sin 2x - \cos 2x) + c$$

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1A

1M

1A

1A

1A

$$6. \quad (a) \quad \alpha > \beta \geq 0 \rightarrow \alpha\beta + \alpha > \alpha\beta + \beta \geq 0$$

$$\rightarrow \alpha(\beta + 1) > \beta(\alpha + 1) \geq 0$$

$$\rightarrow \frac{\alpha}{\alpha + 1} > \frac{\beta}{\beta + 1} \geq 0$$

$$\rightarrow \sqrt{\frac{\alpha}{\alpha + 1}} > \sqrt{\frac{\beta}{\beta + 1}}$$

$$(b) \quad u_{n+1} - u_n = \sum_{m=1}^{n+1} \frac{1}{2^m} \sqrt{\frac{n+1-m}{n+1-m+1}} - \sum_{m=1}^n \frac{1}{2^m} \sqrt{\frac{n-m}{n-m+1}}$$

$$= \sum_{m=1}^n \frac{1}{2^m} \left\{ \sqrt{\frac{n+1-m}{n+1-m+1}} - \sqrt{\frac{n-m}{n-m+1}} \right\} + \frac{1}{2^{n+1}} \sqrt{\frac{0}{1}}$$

$$> 0 \quad (\text{by (a)})$$

$$\text{Also, } u_n = \sum_{m=1}^n \frac{1}{2^m} \sqrt{\frac{n-m}{n-m+1}}$$

$$< \sum_{m=1}^n \frac{1}{2^m}$$

$$< \sum_{m=1}^{\infty} \frac{1}{2^m}$$

$$= \frac{1}{2} \cdot \frac{1}{\left(1 - \frac{1}{2}\right)}$$

$$= 1$$

Since  $(u_n)$  is increasing and bounded above,  $\lim u_n$  exists.

7. (a) By Leibniz's Theorem,  $y^{(n)} = \sum_{r=0}^n \binom{n}{r} u^{(r)}(x) (e^{\alpha x})^{(n-r)}$

$$= \sum_{r=0}^n \binom{n}{r} u^{(r)}(x) \alpha^{n-r} e^{\alpha x}$$

(b)  $u^{(r)}(x) = p^r e^{qx}$

$$\therefore y^{(n)} = \sum_{r=0}^n \binom{n}{r} p^r \alpha^{n-r} e^{(p+q)x}$$

On the other hand,

$$\begin{aligned} y^{(n)} &= (u(x) e^{\alpha x})^{(n)} = (e^{px} e^{qx})^{(n)} = (e^{(p+q)x})^{(n)} \\ &= (p+q)^n e^{(p+q)x} \end{aligned}$$

$$\text{Hence } (p+q)^n = \sum_{r=0}^n \binom{n}{r} p^r \alpha^{n-r} \quad (\because e^{(p+q)x} \neq 0)$$

1M

1A

1A

1

1

1M

8. (a)  $f(x) = (x^2 - x^3)^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}(x^2 - x^3)^{-\frac{2}{3}}(2x - 3x^2)$$

$$= \frac{2 - 3x}{3x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}}$$

$$f''(x) = -\frac{2}{9}(x^2 - x^3)^{-\frac{5}{3}}(2x - 3x^2)^2 + \frac{1}{3}(x^2 - x^3)^{-\frac{2}{3}}(2 - 6x)$$

$$= \frac{-2x^2}{9(x^2 - x^3)^{\frac{5}{3}}}$$

$$= \frac{-2}{9(1-x)(x^2 - x^3)^{\frac{2}{3}}}$$

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1A

2

(b)  $\frac{f(x) - f(0)}{x} = \frac{1}{x}(x^2 - x^3)^{\frac{1}{3}}$

$$= \left(\frac{1}{x} - 1\right)^{\frac{1}{3}}$$

$$\rightarrow \pm\infty \text{ as } x \rightarrow \pm 0$$

$\therefore f'(0)$  does not exist.

$$\frac{f(x) - f(1)}{x - 1} = \frac{1}{x - 1}(x^2 - x^3)^{\frac{1}{3}}$$

$$= \frac{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}{x - 1}$$

$$= \frac{-x^{\frac{2}{3}}}{(x - 1)^{\frac{2}{3}}}$$

$$\rightarrow -\infty \text{ as } x \rightarrow 1$$

$\therefore f'(1)$  does not exist.

1

2

(c) (i)  $f'(x) = 0 \quad - \quad x = \frac{2}{3}$

(ii)  $f'(x) > 0 \quad - \quad 0 < x < \frac{2}{3}$

(iii)  $f'(x) < 0 \quad - \quad x < 0, \frac{2}{3} < x < 1, 1 < x$

(iv)  $f''(x) = 0$  for all  $x$

(v)  $f''(x) > 0 \quad - \quad x > 1$

(vi)  $f''(x) < 0 \quad - \quad x < 0, 0 < x < 1$

$\left(\frac{1}{2}\right)$  mark each

3

3

(d)

$x$	$x < 0$	$0$	$0 < x < \frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3} < x < 1$	$1$	$x > 1$
$f'(x)$	-	undefined	+	0	-	undefined	-
$f''(x)$	-	undefined	-	-	-	undefined	+
$f(x)$		0		$\frac{1}{3} \cdot \sqrt[3]{4}$		0	

relative maximum is at  $(\frac{2}{3}, \frac{1}{3}\sqrt[3]{4})$ ,

the point of inflexion at  $(1, 0)$ ,

relative minimum is at  $(0, 0)$ .

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1A

 $\frac{1A}{3}$ (e) Let the oblique asymptote be  $y = mx + c$ 

$$m = \lim_{x \rightarrow \infty} \frac{y}{x}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x^2 - x}}{x}$$

$$= \lim_{x \rightarrow \infty} \sqrt[3]{\frac{1}{x} - 1}$$

$$= -1$$

$$c = \lim_{x \rightarrow \infty} (\sqrt[3]{x^2 - x} + x)$$

$$= \lim_{x \rightarrow \infty} \left( x \cdot \sqrt[3]{\frac{1}{x} - 1} + x \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x \cdot \left( \left( \frac{1}{x} - 1 \right) + 1 \right)}{\left( \frac{1}{x} - 1 \right)^{\frac{2}{3}} - \left( \frac{1}{x} - 1 \right)^{\frac{1}{3}} + 1}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\left( \frac{1}{x} - 1 \right)^{\frac{2}{3}} - \left( \frac{1}{x} - 1 \right)^{\frac{1}{3}} + 1}$$

$$= \frac{1}{3}$$

1A

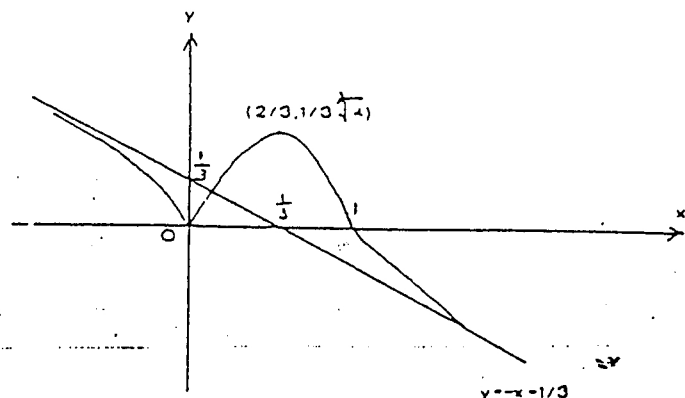
1M

1A

The oblique asymptote is  $y = -x + \frac{1}{3}$ .

3

(f)

 $\frac{2}{2}$

$$9. \quad (a) \quad (1) \quad \frac{[\frac{1}{2}a(c + \frac{1}{c})]^2}{a^2} - \frac{[\frac{1}{2}b(c - \frac{1}{c})]^2}{b^2}$$

$$= \frac{1}{4} (c + \frac{1}{c})^2 - \frac{1}{4} (c - \frac{1}{c})^2$$

$$= \frac{1}{4} [(c + \frac{1}{c}) + (c - \frac{1}{c})][(c + \frac{1}{c}) - (c - \frac{1}{c})]$$

$$= \frac{1}{4} [2c] [2(\frac{1}{c})]$$

$$= 1$$

(ii) Differentiating both sides of the equation of  $H$  with respect to  $x$ , we have

$$\frac{d}{dx} \left( \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = \frac{d}{dx} 1$$

$$= \frac{2}{a^2} x - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$= \frac{dy}{dx} = \frac{x}{y} \left( \frac{b}{a} \right)^2$$

By slope point form, equation of tangent at  $P$  is

$$\frac{y - \frac{1}{2}b(c - \frac{1}{c})}{x - \frac{1}{2}a(c + \frac{1}{c})} = \frac{\frac{1}{2}a(c + \frac{1}{c})}{\frac{1}{2}b(c - \frac{1}{c})} \left( \frac{b}{a} \right)^2$$

$$\frac{y - \frac{1}{2}b(c - \frac{1}{c})}{x - \frac{1}{2}a(c + \frac{1}{c})} = \frac{b(c + \frac{1}{c})}{a(c - \frac{1}{c})}$$

$$a(c - \frac{1}{c})y - \frac{1}{2}ab(c - \frac{1}{c})^2 = b(c + \frac{1}{c})x - \frac{1}{2}ab(c + \frac{1}{c})^2$$

$$b(c + \frac{1}{c})x - a(c - \frac{1}{c})y = \frac{ab}{2} \left[ (c + \frac{1}{c})^2 - (c - \frac{1}{c})^2 \right]$$

$$= \frac{ab}{2} \left[ (c + \frac{1}{c}) + (c - \frac{1}{c}) \right] \left[ (c + \frac{1}{c}) - (c - \frac{1}{c}) \right]$$

$$= \frac{ab}{2} (2c) \left( \frac{2}{c} \right)$$

$$= 2ab$$

$$\text{i.e.} \quad \frac{x}{2a} (c + \frac{1}{c}) - \frac{y}{2b} (c - \frac{1}{c}) = 1 \dots \dots \dots (*)$$

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(b) (i) Asymptotes :  $y = \pm \frac{b}{a}x$

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Substituting  $y = \frac{b}{a}x$  into (\*), we have

$$\frac{x}{2a} \left( t + \frac{1}{t} \right) - \frac{x}{2a} \left( t - \frac{1}{t} \right) = 1$$

$$\frac{x}{ac} = 1$$

$$x = ac$$

$$- y = bt$$

$$- S = (ac, bt)$$

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Substituting  $y = -\frac{b}{a}x$  into (\*), we have

$$\frac{x}{2a} \left( t + \frac{1}{t} \right) + \frac{x}{2a} \left( t - \frac{1}{t} \right) = 1$$

$$\frac{xt}{a} = 1$$

$$- x = \frac{a}{t}$$

$$- y = -\frac{b}{t}$$

$$- T = \left( \frac{a}{t}, -\frac{b}{t} \right)$$

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Let the equation of circle  $OST$  be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Substituting  $(0, 0)$  into the equation, we have  $c = 0$ .

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Substituting  $S, T$  into the equation, we have

$$\begin{cases} a^2t^2 + b^2t^2 + 2gat + 2fbt = 0 \\ \frac{a^2}{t^2} + \frac{b^2}{t^2} + \frac{2ga}{t} - \frac{2fb}{t} = 0 \end{cases}$$

$$- \begin{cases} a^2t^2 + b^2t^2 + 2gat + 2fbt = 0 \\ a^2 + b^2 + 2gat - 2fbt = 0 \end{cases}$$

$$- \begin{cases} 4gat = -(a^2 + b^2)(1 + t^2) \\ 4fbt = -(a^2 + b^2)(t^2 - 1) \end{cases}$$

$$- \begin{cases} g = -\frac{1}{4at} (a^2 + b^2)(1 + t^2) \\ f = -\frac{1}{4bt} (a^2 + b^2)(t^2 - 1) \end{cases}$$



Thus the centre of the circle is given by

$$\begin{cases} x = \frac{1}{4ac} (a^2 + b^2) (1 + c^2) \\ y = \frac{1}{4bc} (a^2 + b^2) (c^2 - 1) \end{cases}$$

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Eliminate  $c$  :

$$\begin{cases} x = \frac{1}{4a} (a^2 + b^2) \left( \frac{1}{c} + c \right) \\ y = \frac{1}{4b} (a^2 + b^2) \left( c - \frac{1}{c} \right) \end{cases}$$

$$\begin{cases} \frac{4ax}{a^2 + b^2} = c + \frac{1}{c} \\ \frac{4by}{a^2 + b^2} = c - \frac{1}{c} \end{cases}$$

$$\therefore \left( \frac{4ax}{a^2 + b^2} \right)^2 - \left( \frac{4by}{a^2 + b^2} \right)^2 = \left( c + \frac{1}{c} \right)^2 - \left( c - \frac{1}{c} \right)^2 = 4$$

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$$\text{or } \frac{x^2}{\left( \frac{a^2 + b^2}{2a} \right)^2} - \frac{y^2}{\left( \frac{a^2 + b^2}{2b} \right)^2} = 1$$

$$\begin{aligned} \text{(ii) } OS \cdot OT &= [(at)^2 + (bt)^2]^{\frac{1}{2}} \cdot \left[ \left( \frac{a}{c} \right)^2 + \left( \frac{b}{c} \right)^2 \right]^{\frac{1}{2}} \\ &= a^2 + b^2 \end{aligned}$$

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Let  $S' = (-at, bt)$  .

Hence  $S'OT$  are collinear and

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$$OS' \cdot OT = OS \cdot OT = a^2 + b^2$$

Let  $F, F'$  be the foci, so

$$OF \cdot OF' = a^2 + b^2 = OS' \cdot OT$$

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Therefore  $\triangle OFT = \triangle OS'F'$  and the points

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$F, F', S', T$  are concyclic. Since the centre

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of such circle lies on the  $y$ -axis,  $S, T$  and the foci are concyclic.

11

10. (a)  $x^3 + y^3 = 3axy$

$$1 + \left(\frac{y}{x}\right)^3 = 3\left(\frac{a}{x^2}\right)\left(\frac{y}{x}\right)$$

Let  $\lim_{x \rightarrow \infty} \frac{y}{x} = m$

Taking  $x \rightarrow \infty$ , we have

$$1 - m^3 = 0 \cdot m \quad (\because \lim_{x \rightarrow \infty} \frac{a}{x^2} = 0)$$

$$m = -1$$

Now  $x^3 + y^3 = 3axy$

$$x + y = \frac{3axy}{x^2 - xy + y^2}$$

$$= \frac{3a\left(\frac{y}{x}\right)}{1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2}$$

$$\rightarrow \frac{3a(-1)}{1 - (-1) + (-1)^2} = -a \text{ as } x \rightarrow \infty$$

$\therefore$  Equation of L is  $x + y + a = 0$

(b) Putting  $x = r\cos\theta$  and  $y = r\sin\theta$ ,

we have equation of  $\Gamma$  written as

$$r^3\cos^3\theta + r^3\sin^3\theta = 3ar^2\cos\theta\sin\theta$$

$$\text{i.e. } r = \frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$$

and equation of L written as

$$r\cos\theta + r\sin\theta + a = 0$$

$$\text{i.e. } r = \frac{-a}{\cos\theta + \sin\theta}$$

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$$\begin{aligned}
 \text{(c) Area of loop} &= \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 d\theta \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{9a^2 \cos^2 \theta \sin^2 \theta}{(\cos^3 \theta + \sin^3 \theta)^2} d\theta \\
 &= \frac{9}{2} a^2 \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta \sec^2 \theta}{(1 + \tan^3 \theta)^2} d\theta \\
 &= \frac{9}{2} a^2 \int_1^{\infty} \frac{dw}{3w^2} \quad \text{where } w = 1 + \tan^3 \theta \\
 &= \frac{3}{2} a^2 \left[ -\frac{1}{w} \right]_1^{\infty} \\
 &= \frac{3}{2} (0 + 1) a^2 \\
 &= \frac{3}{2} a^2
 \end{aligned}$$

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$$\begin{aligned}
 \text{(d) } A_{\phi} &= \int_{\phi}^{\pi} \frac{1}{2} r_1^2 - \frac{1}{2} r_2^2 d\theta \quad \text{where } r_1 = \frac{-a}{\cos \theta + \sin \theta} \\
 &\quad \text{and } r_2 = \frac{3a \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}
 \end{aligned}$$

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$$= \frac{a^2}{2} \left\{ \int_{\phi}^{\pi} \frac{d\theta}{(\sin \theta + \cos \theta)^2} - \int_{\phi}^{\pi} \frac{9 \sin^2 \theta \cos^2 \theta}{(\sin^3 \theta + \cos^3 \theta)^2} d\theta \right\}$$

1A

$$\begin{aligned}
 \text{Now } \int_{\phi}^{\pi} \frac{1}{(\sin \theta + \cos \theta)^2} d\theta &= \int_{\phi}^{\pi} \frac{\sec^2 \theta d\theta}{(1 + \tan \theta)^2} \\
 &= - \left[ \frac{1}{1 + \tan \theta} \right]_{\phi}^{\pi} \\
 &= \frac{1}{1 + \tan \phi} - 1
 \end{aligned}$$

1A

$$\begin{aligned}
 \text{and } \int_{\phi}^{\pi} \frac{9 \sin^2 \theta \cos^2 \theta}{(\sin^3 \theta + \cos^3 \theta)^2} d\theta &= 9 \cdot \left[ \frac{1}{-3(1 + \tan^3 \theta)} \right]_{\phi}^{\pi} \\
 &= 3 \left( \frac{1}{1 + \tan^3 \phi} - 1 \right)
 \end{aligned}$$

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$$\begin{aligned}
 \therefore A_{\phi} &= \frac{a^2}{2} \left\{ \left( \frac{1}{1 + \tan \phi} - 1 \right) - 3 \left( \frac{1}{1 + \tan^3 \phi} - 1 \right) \right\} \\
 &= \frac{a^2}{2} \left\{ \frac{1}{1 + \tan \phi} - \frac{3}{1 + \tan^3 \phi} + 2 \right\}
 \end{aligned}$$

$$\lim_{\phi \rightarrow -\frac{3\pi}{4}} A_1 = \lim_{\phi \rightarrow -\frac{3\pi}{4}} \frac{a^2}{2} \left\{ \frac{1}{1 + \tan \phi} - \frac{3}{1 + \tan^3 \phi} + 2 \right\}$$

$$= a^2 + \frac{a^2}{2} \lim_{\phi \rightarrow -\frac{3\pi}{4}} \left\{ \frac{1}{1 + \tan \phi} - \frac{3}{1 + \tan^3 \phi} \right\}$$

$$= a^2 + \frac{a^2}{2} \lim_{\phi \rightarrow -\frac{3\pi}{4}} \frac{1 - \tan \phi + \tan^2 \phi - 3}{1 + \tan^3 \phi}$$

$$= a^2 + \frac{a^2}{2} \lim_{\phi \rightarrow -\frac{3\pi}{4}} \frac{\tan^2 \phi - \tan \phi - 2}{1 + \tan^3 \phi}$$

$$= a^2 + \frac{a^2}{2} \lim_{\phi \rightarrow -\frac{3\pi}{4}} \frac{(\tan \phi + 1)(\tan \phi - 2)}{1 + \tan^3 \phi}$$

$$= a^2 + \frac{a^2}{2} \lim_{\phi \rightarrow -\frac{3\pi}{4}} \frac{\tan \phi - 2}{1 - \tan \phi + \tan^2 \phi}$$

$$= a^2 + \frac{a^2}{2} \left( \frac{(-1) - 2}{1 - (-1) + (-1)^2} \right)$$

$$= a^2 + \frac{a^2}{2} \left( \frac{-3}{3} \right)$$

$$= \frac{a^2}{2}$$

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11. (a) (i) By Mean Value Theorem,  $f(t) = f(n) + f'(\zeta)(t - n)$  for some  $\zeta \in (t, n)$  where  $t \in (n, n+1]$ .

As  $f'' \geq 0$ ,  $f'$  is increasing,  $\therefore f'(n) \leq f'(\zeta) \leq f'(n+1)$

$$\therefore f(n) + f'(n)(t - n) \leq f(t) \leq f(n) + f'(n+1)(t - n)$$

and obviously it holds for  $t = n$ .

$$(ii) \quad f(n) + f'(n)(t - n) \leq f(t) \leq f(n) + f'(n+1)(t - n) \quad \forall t \in [n, n+1],$$

$$\therefore \int_n^{n+1} f(n) + f'(n)(t - n) dt \leq \int_n^{n+1} f(t) dt \leq \int_n^{n+1} f(n) + f'(n+1)(t - n) dt$$

$$- f(n) + f'(n) \left[ \frac{(t - n)^2}{2} \right]_n^{n+1} \leq \int_n^{n+1} f(t) dt \leq f(n) + f'(n+1) \left[ \frac{(t - n)^2}{2} \right]_n^{n+1}$$

$$- f(n) + \frac{f'(n)}{2} \leq \int_n^{n+1} f(t) dt \leq f(n) + \frac{f'(n+1)}{2}$$

$$- f(n) + \frac{f'(n)}{2} - \left( \frac{f(n) - f(n+1)}{2} \right) \leq \int_n^{n+1} f(t) dt - \left( \frac{f(n) + f(n+1)}{2} \right) \\ \leq f(n) + \frac{f'(n+1)}{2} - \left( \frac{f(n) + f(n+1)}{2} \right)$$

$$- \frac{f'(n)}{2} + \frac{f(n) - f(n+1)}{2} \leq \int_n^{n+1} f(t) dt - \left( \frac{f(n) + f(n+1)}{2} \right) \\ \leq \frac{f'(n+1)}{2} + \frac{f(n) - f(n+1)}{2}$$

$$\text{Now } f(n) - f(n+1) = -(f(n+1) - f(n))$$

$$= -f'(\eta) \text{ for some } \eta \in (n, n+1)$$

since  $f'$  is increasing, we have  $-f'(n+1) \leq f(n) - f(n+1) \leq -f'(n)$

$$\text{hence } \frac{f'(n)}{2} - \frac{f'(n+1)}{2} \leq \int_n^{n+1} f(t) dt - \left( \frac{f(n) + f(n+1)}{2} \right)$$

$$\leq \frac{f'(n+1)}{2} - \frac{f'(n)}{2}$$

$$- \frac{1}{2}(f'(n+1) - f'(n)) \leq \int_n^{n+1} f(t) dt - \left( \frac{f(n) + f(n+1)}{2} \right)$$

$$\leq \frac{1}{2}(f'(n+1) - f'(n))$$

$$- \left| \int_n^{n+1} f(t) dt - \left( \frac{f(n) + f(n+1)}{2} \right) \right| \leq \frac{f'(n+1) - f'(n)}{2}$$

$$(iii) \left| \int_n^{n^2} f(t) dt - \sum_{j=n}^{n^2-1} \frac{f(j) + f(j+1)}{2} \right| g(n^2)$$

$$\leq g(n^2) \cdot \sum_{j=n}^{n^2-1} \left| \int_j^{j+1} f(t) dt - \left( \frac{f(j) + f(j+1)}{2} \right) \right|$$

$$\leq g(n^2) \cdot \sum_{j=n}^{n^2-1} \frac{f'(j+1) - f'(j)}{2}$$

$$= g(n^2) \cdot \frac{f'(n^2) - f'(n)}{2}$$

$$= \frac{1}{2} g(n^2) f''(\zeta_n) \quad \text{for some } \zeta_n \in (n, n^2).$$

$$\leq \frac{1}{2} g(\zeta_n) f''(\zeta_n) \quad \text{as } g \text{ is decreasing.}$$

$$\text{As } n \rightarrow \infty, \zeta_n \rightarrow \infty. \therefore \lim_{n \rightarrow \infty} g(\zeta_n) f''(\zeta_n) = 0 \quad (\text{by Condition C})$$

$$\therefore \lim_{n \rightarrow \infty} \left| \int_n^{n^2} f(t) dt - \sum_{j=n}^{n^2-1} \frac{f(j) + f(j+1)}{2} \right| g(n^2) = 0$$

$$\therefore \lim_{n \rightarrow \infty} \left\{ \int_n^{n^2} f(t) dt - \sum_{j=n}^{n^2-1} \frac{f(j) + f(j+1)}{2} \right\} g(n^2) = 0$$

\* Note: marks for this part have been reallocated

$$(b) \text{ Put } f(t) = \frac{1}{\ln t} \text{ and } g(t) = \frac{\ln t}{t}$$

$$\text{then } g'(t) = \frac{1 - \ln t}{t^2}$$

$$\leq 0 \quad \forall t \in (e, \infty)$$

\(\therefore\)  $g(t)$  is decreasing on  $(e, \infty)$ .

- Condition A is satisfied.

$$\text{Now } f'(t) = -\frac{1}{(\ln t)^2} \cdot \frac{1}{t}$$

$$f''(t) = \frac{2}{(\ln t)^3} \cdot \frac{1}{t^2} + \frac{1}{(\ln t)^2} \cdot \frac{1}{t^2}$$

$$\geq 0 \quad \text{on } (e, \infty)$$

$$g(t) \geq 0 \quad \text{on } (e, \infty)$$

- Condition B is satisfied.

$$\text{Also, } \lim_{t \rightarrow \infty} g(t) f''(t) = \lim_{t \rightarrow \infty} \frac{\ln t}{t} \left\{ \frac{2}{(\ln t)^3} \cdot \frac{1}{t^2} + \frac{1}{(\ln t)^2} \cdot \frac{1}{t^2} \right\}$$

$$= \lim_{t \rightarrow \infty} \left\{ \frac{2}{(\ln t)^2 t^3} + \frac{1}{(\ln t) t^3} \right\}$$

$$= 0$$

- Condition C is satisfied.

Then by (a),

$$\lim_{n \rightarrow \infty} \left\{ \int_n^{n^2} \frac{1}{\ln t} dt - \sum_{j=n}^{n^2-1} \frac{1}{2} \left( \frac{1}{\ln j} + \frac{1}{\ln(j+1)} \right) \right\} \frac{\ln n}{n^2} = 0$$

1

$$\lim_{n \rightarrow \infty} \left\{ \int_n^{n^2} \frac{1}{\ln t} dt - \sum_{j=n}^{n^2-1} \frac{1}{2} \left( \frac{1}{\ln j} + \frac{1}{\ln(j+1)} \right) \right\} \left( \frac{\ln n}{n^2} \right) = 0$$

$$\lim_{n \rightarrow \infty} \left\{ \frac{\ln n}{n^2} \int_n^{n^2} \frac{1}{\ln t} dt - \frac{\ln n}{n^2} \sum_{j=n}^{n^2-1} \frac{1}{2} \left( \frac{1}{\ln j} + \frac{1}{\ln(j+1)} \right) \right\} = 0$$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n^2} \sum_{j=n}^{n^2-1} \frac{1}{2} \left( \frac{1}{\ln j} + \frac{1}{\ln(j+1)} \right) = \frac{1}{2}$$

1

(v given that  $\lim_{n \rightarrow \infty} \frac{\ln n}{n^2} \int_n^{n^2} \frac{1}{\ln t} dt = \frac{1}{2}$ )

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n^2} \left\{ \sum_{j=n}^{n^2-1} \frac{1}{\ln(j+1)} + \frac{1}{2 \ln n} - \frac{1}{2 \ln n^2} \right\} = \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left\{ \frac{\ln n}{n^2} \sum_{j=n}^{n^2-1} \frac{1}{\ln(j+1)} + \frac{1}{2n^2} - \frac{1}{4n^2} \right\} = \frac{1}{2}$$

1

$$\lim_{n \rightarrow \infty} \left\{ \frac{\ln n}{n^2} \sum_{j=n}^{n^2-1} \frac{1}{\ln(j+1)} - \frac{1}{4n^2} \right\} = \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n^2} \sum_{j=n}^{n^2-1} \frac{1}{\ln(j+1)} = \frac{1}{2} \quad \left( \because \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0 \right)$$

6

Solutions

Marks

12. (a) Let  $S = 1 + (-t^2) + (-t^2)^2 + \dots + (-t^2)^{n-1}$

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then  $(-t^2)S = (-t^2) + (-t^2)^2 + \dots + (-t^2)^{n-1} + (-t^2)^n$

hence  $S - (-t^2)S = 1 - (-t^2)^n$

$(1 + t^2)S = 1 - (-1)^n t^{2n}$

1

$S = \frac{1}{1+t^2} - \frac{(-1)^n t^{2n}}{1+t^2}$

$\frac{1}{1+t^2} = S + \frac{(-1)^n t^{2n}}{1+t^2}$

$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \dots + (-1)^{n-1} t^{2n-2} + \frac{(-1)^n t^{2n}}{1+t^2}$

Integrating both sides from  $t = 0$  to  $t = x$ ,

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we have  $\int_0^x \frac{1}{1+t^2} dt = \int_0^x 1 - t^2 + t^4 - \dots + (-1)^{n-1} t^{2n-2} + \frac{(-1)^n t^{2n}}{1+t^2} dt$

$[\tan^{-1} t]_0^x = \int_0^x 1 - t^2 + t^4 - \dots + (-1)^{n-1} t^{2n-2} dt + \int_0^x \frac{(-1)^n t^{2n}}{1+t^2} dt$

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$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^{n-1}}{2n-1} x^{2n-1} + \int_0^x \frac{(-1)^n t^{2n}}{1+t^2} dt$

4

(b)  $\left| \tan^{-1} x - \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^{n-1}}{2n-1} x^{2n-1} \right) \right|$

$= \left| \int_0^x \frac{(-1)^n t^{2n}}{1+t^2} dt \right|$  (by (a))

1

$\leq \int_0^x \left| \frac{(-1)^n t^{2n}}{1+t^2} \right| dt$

1

$= \int_0^x \frac{t^{2n}}{1+t^2} dt$  ( $\because x \geq 0$ )

$\leq \int_0^x t^{2n} dt$

1

$= \frac{1}{2n+1} [t^{2n+1}]_0^x$

$= \frac{1}{2n+1} x^{2n+1}$

Putting  $x = 1$ , we have

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$\left| \tan^{-1} 1 - \left( 1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^{n-1}}{2n-1} \right) \right| \leq \frac{1}{2n+1}$

$\left| \frac{\pi}{4} - \left( 1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^{n-1}}{2n-1} \right) \right| \leq \frac{1}{2n+1}$

Taking  $n \rightarrow \infty$ , we have  $\lim_{n \rightarrow \infty} \left( 1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^{n-1}}{2n-1} \right) = \frac{\pi}{4}$

1M

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$

1A





13. (a) If  $f'(a) \leq 0$ ,

then  $f'(x) \leq 0 \forall x \in (a, b)$  ( $\because f'$  is decreasing) ..... (\*)

But by Mean Value Theorem,  $\exists \zeta \in (a, b)$

such that  $f'(\zeta) = \frac{f(b) - f(a)}{b - a}$

$> 0$  ( $\because f(b) > 0, f(a) < 0, b > a$ )

contradicting (\*).

Hence  $f'(a) > 0$ .

(b) By (a),  $x_1 = a - \frac{f(a)}{f'(a)} > a$

Rewrite  $f(a) = -f'(a)(x_1 - a)$  ..... (\*\*)

By MVT,  $f(b) - f(a) = f'(c)(b - a)$  for some  $c \in (a, b)$

$< f'(a)(b - a)$  ( $\because f'$  is decreasing)

$\Rightarrow 0 < f(b) < f'(a)(b - a) - f'(a)(x_1 - a)$

$= f'(a)(b - x_1)$  by (\*\*)

$\Rightarrow x_1 < b$ , since  $f'(a) > 0$  by (a).

By MVT,  $f(x_1) - f(a) = f'(c)(x_1 - a)$  for some  $c \in (a, x_1) \subset (a, b)$

$< f'(a)(x_1 - a)$  ( $\because f'$  is strictly decreasing)

$\Rightarrow f(x_1) < 0$  by (\*\*)

If  $f'(x_1) \leq 0$ , then  $f(b) - f(x_1) = f'(\eta)(b - x_1)$  for some  $\eta \in (x_1, b)$

$< 0$  ( $\because f'$  is decreasing)

$\Rightarrow f(b) < f(x_1) \leq 0$

$\Rightarrow$  a contradiction

(c) We shall use mathematical induction to show that

$x_n \in (a, b)$ ,  $f(x_n) < 0$  and  $f'(x_n) > 0 \forall n = 1, 2, \dots$

For  $n = 1$ ,

by (a), the result follows.

Assume  $x_k \in (a, b)$ ,  $f(x_k) < 0$  and  $f'(x_k) > 0$  for some  $k \geq 1$ .

Using the same arguments in (b), we can show that

$x_{k+1} \in (a, b)$ ,  $f(x_{k+1}) < 0$  and  $f'(x_{k+1}) > 0$ .

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(d) Since  $(x_n)$  is increasing and bounded above,  $\lim_{n \rightarrow \infty} x_n$  exists.

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$- 0 > f(x_n) = (x_{n+1} - x_n) f'(x_n)$$

$$> -(x_{n+1} - x_n) f'(a)$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty$$

1

$$\therefore \lim_{n \rightarrow \infty} f(x_n) = 0$$

$$- f(\lim_{n \rightarrow \infty} x_n) = 0 \quad (\because f \text{ is continuous})$$

1

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