Section A (40 marks)
Answer ALL questions in this section, using AL(C1) answer book.

1. Prove the following Schwarz's inequality:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right),$$

where a_i , $b_i \in \mathbf{R}$ and $n \in \mathbf{N}$.

Hence, or otherwise, prove that

$$\frac{1}{n}\sum_{i=1}^{n} a_i \le \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2} .$$

(5 marks)

2. Let $u_1 = 1$, $u_2 = 3$ and $u_n = u_{n-2} + u_{n-1}$ for $n \ge 3$.

Using mathematical induction, or otherwise, prove that

$$u_n = \alpha^n + \beta^n \text{ for } n \ge 1 ,$$

where α and β are the roots of $x^2 - x - 1 = 0$. (5 marks)

3. Suppose the following system of linear equations is consistent:

$$(*) \begin{cases} ax + by + cz = 1 \\ bx + cy + az = 1 \\ cx + ay + bz = 1 \\ x + y + z = 3 \end{cases} , \text{ where } a , b , c \in \mathbf{R} .$$

(a) Show that $a + b + c = 1$.

(b) Show that (*) has a unique solution if and only if a , b and c are not all equal.

(c) If $a = b = c$, solve (*). (6 marks)

4. (a) If $|1 + z| = |2 - z|$, find $\operatorname{Re} z$.

(b) Find all $z \in \mathbf{C}$ such that

$$\begin{cases} |z|^2 - z - \bar{z} + i(z - \bar{z}) = \dfrac{1}{2} \text{ and} \\ |1 + z| = |2 - z| . \end{cases}$$

(5 marks)

5. Express $\dfrac{x + 4}{x^2 + 3x + 2}$ in partial fractions.

Hence evaluate $\displaystyle\sum_{k=2}^{\infty}\left\{\frac{1}{k - 1} - \frac{k + 4}{k^2 + 3k + 2}\right\}$.

(6 marks)

6. (a) Show that if A is a 3×3 matrix such that $A^t = -A$, then $\det A = 0$.

(b) Given that

$$B = \begin{pmatrix} 1 & -2 & 74 \\ 2 & 1 & -67 \\ -74 & 67 & 1 \end{pmatrix} ,$$

use (a), or otherwise, to show $\det(I - B) = 0$.

Hence deduce that $\det(I - B^4) = 0$. (7 marks)

7. Find all (x, y) in $\mathbf{R}^2$ satisfying the following two conditions:

$$\begin{cases} |2x - 1| > y + 1 \\ y = |x + 3| . \end{cases}$$

(6 marks)

SECTION B (60 marks)
Answer any FOUR questions from this section, using AL(C2) answer book.
Each question carries 15 marks.

8. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors in $\mathbf{R}^3$.

Show that:

(a) If $\mathbf{u} = (u_1\ u_2\ u_3)$, $\mathbf{v} = (v_1\ v_2\ v_3)$ and $\mathbf{w} = (w_1\ w_2\ w_3)$,

then $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0$.

(4 marks)

(b) If $\mathbf{s} \in \mathbf{R}^3$ such that $\mathbf{s} \cdot \mathbf{u} = \mathbf{s} \cdot \mathbf{v} = \mathbf{s} \cdot \mathbf{w} = 0$,

then $\mathbf{s} = \mathbf{0}$.

(3 marks)

(c) If $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$,

then $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$.

(4 marks)

(d) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$,

then $\mathbf{r} = \dfrac{\mathbf{r} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} + \dfrac{\mathbf{r} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} + \dfrac{\mathbf{r} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}$ for all $\mathbf{r} \in \mathbf{R}^3$.

(4 marks)

9. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x , y \in \mathbf{R}$.

(a) Show that

(i) $f(0) = 0$,

(ii) $f(-x) = -f(x)$ for all $x \in \mathbf{R}$,

(iii) $f(nx) = nf(x)$ for all $n \in \mathbf{Z}$ and $x \in \mathbf{R}$.

(5 marks)

(b) Show that if there exists $K > 0$ such that $f(x) < K$ for all $x \in \mathbf{R}$, then $f(x) = 0$ for all $x \in \mathbf{R}$.

(3 marks)

(c) Suppose there exists $K > 0$ such that

$$f(x) < K \text{ for all } x \in [0, 1) .$$

Let $g(x) = f(x) - f(1)x$ for all $x \in \mathbf{R}$.

Show that, for all $x , y \in \mathbf{R}$,

(i) $g(x + y) = g(x) + g(y)$,

(ii) $g(x + 1) = g(x)$,

(iii) $g(x) < K + |f(1)|$.

Hence, or otherwise, show that

$$f(x) = f(1)x \text{ for all } x \in \mathbf{R} .$$

(7 marks)

10. Let M be the set of all 3×3 real matrices. A relation $\sim$ is defined on M as follows:

For any $A, B \in M$, $A \sim B$ if there is a non-singular matrix P such that $A = PBP^{-1}$.

(a) Show that $\sim$ is an equivalence relation on M.

(3 marks)

(b) Show that if $A \sim B$, then $A^k \sim B^k$ for any positive integer k.

(2 marks)

(c) (i) Show that if $C \sim 0$, then $C = 0$.

(ii) Find two matrices A and B in M such that $AB \sim BA$ is NOT true.

(4 marks)

(d) Let $A \in M$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{R}$.

Show that

$$A \sim \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

if and only if there exist linearly independent vectors $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$ and $(x_3\ y_3\ z_3)$ such that

$$A\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \lambda_i \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \quad \text{for } i = 1, 2 \text{ and } 3.$$

(6 marks)

11. Let Z_1, Z_2 and Z_3 be 3 distinct points representing the complex numbers z_1, z_2 and z_3 respectively.

(a) Suppose W_1, W_2 and W_3 are 3 distinct points representing the complex numbers w_1, w_2 and w_3 respectively. Prove that $\triangle Z_1Z_2Z_3$ is similar to $\triangle W_1W_2W_3$ (vertices anticlockwise) if and only if

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{w_3 - w_1}{w_2 - w_1}.$$

(4 marks)

(b) Using (a), or otherwise, show that $\triangle Z_1Z_2Z_3$ (vertices anticlockwise) is equilateral if and only if

$$z_1 + \epsilon z_2 + \epsilon^2 z_3 = 0$$

where $\epsilon = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$.

(5 marks)

(c) A point representing $a + ib$ is said to be an *integral* point if a and b are integers. Using (b), or otherwise, show that no triangle with distinct integral points as vertices can be equilateral.

(6 marks)

12. Let $\wp$ be the set of all polynomials with real coefficients.

Let $f, g \in \wp \backslash \{0\}$ and

$$A = \{mf + ng : m, n \in \wp\} .$$

Suppose $r \in A \backslash \{0\}$ has the property that

$$\deg r \le \deg p \quad \text{for all } p \in A \backslash \{0\} .$$

(a) Show that r divides every polynomial in A.

Deduce that r is a G.C.D. of f and g (i.e. r divides both f and g, and if h divides both f and g then h divides r).

(6 marks)

(b) Let $B = \{hr : h \in \wp\}$.

Show that $A = B$.

(4 marks)

(c) If $\deg r = 0$, i.e. r is a non-zero constant, show that there exist $m_0, n_0 \in \wp$ such that

$$m_0 f + n_0 g = 1 ,$$

and also $A = \wp$.

(5 marks)

13. For any $n = 1, 2, \ldots$, the sets G_n and H_n are defined by

$$G_n = \{ z \in \mathbf{C} : z^n = 1 \} , \quad H_n = \{ z \in \mathbf{C} : z^n = -1 \} .$$

Let p, q be any two positive integers.

(a) Show that (i) $G_p \cap H_p = \phi$,

(ii) $G_p \cup H_p = G_{2p}$.

(2 marks)

(b) Show that if p is odd and q is even, then $H_p \cap H_q = \phi$.

(2 marks)

(c) Suppose $p = mq$ where m is an integer.

Show that (i) $G_q \subset G_p$;

(ii) if m is odd, then $H_q \subset H_p$;

(iii) if m is even, then $H_q \subset G_p$.

(5 marks)

(d) For any $S, T \subset \mathbf{C}$, define ST by

$$ST = \{ z \in \mathbf{C} : z = st \text{ for some } s \in S \text{ and } t \in T \} .$$

Show that (i) $G_p G_p = H_p H_p = G_p$,

(ii) $G_p H_p = H_p G_p = H_p$.

(6 marks)

END OF PAPER

93-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1993

PURE MATHEMATICS PAPER II

2.00 pm-5.00 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Section A: Answer ALL questions in the AL(C1) answer book.
3. Section B: Answer any FOUR questions in the AL(C2) answer book.

Section A (40 marks)
Answer ALL questions in this section, using AL(C1) answer book.

1. Evaluate

(a) $$\lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\cos x},$$

(b) $$\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2}, \text{ where } m, n \ge 2 .$$

(5 marks)

2. Find the equation of the plane passing through the line of intersection of the planes

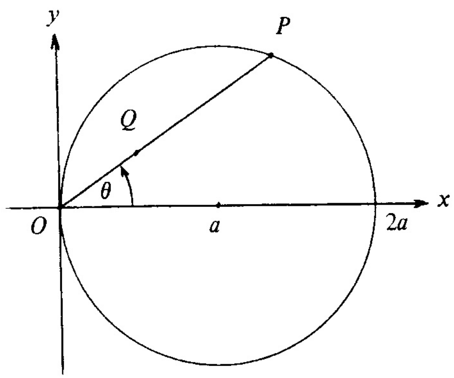
$$x+y+z-1=0 \quad\text{and}\quad x+4y+3z=0$$

and parallel to the straight line

$$x-1=3y=3(z+1) .$$

(5 marks)

3.



Let C be the circle given by the polar equation $r = 2a\cos\theta$ (where $a > 0$), P be a variable point on C and O be the origin. Let Q be a point lying on the line through O and P such that P and Q are on the same side of O and

$$OP \cdot OQ = a^2 .$$

Show that the Cartesian equation of the locus of Q is $x = \frac{a}{2}$.

(5 marks)

4. Find the area of the surface obtained by rotating the following curve about the x-axis:

$$\begin{cases} x = \sin^3 t \\ y = \cos^3 t \end{cases} , \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2} .$$

(5 marks)

5. Evaluate $\int e^{2x}(\sin x + \cos x)^2\,dx$.

(7 marks)

6. (a) Show that if $\alpha > \beta \ge 0$, then

$$\sqrt{\frac{\alpha}{\alpha + 1}} > \sqrt{\frac{\beta}{\beta + 1}} .$$

(b) Let $u_n = \sum_{m=1}^{n} \frac{1}{2^m}\sqrt{\frac{n - m}{n - m + 1}}$, $n = 1, 2, \dots$.

Use (a), or otherwise, to show that

$$u_n < u_{n+1} \quad \text{for } n = 1, 2, \dots .$$

Hence show that $\lim_{n\to\infty} u_n$ exists.

(7 marks)

7. Let n be a positive integer and $u(x)$ be a function such that $u'(x)$, $u''(x)$, $\cdots$, $u^{(n)}(x)$ exist.

(a) Given that $y(x) = u(x)e^{qx}$, where q is a real number, express $y^{(n)}(x)$ in terms of $u(x)$, $u'(x)$, $u''(x)$, $\cdots$, $u^{(n)}(x)$.

(b) By putting $u(x) = e^{px}$, where p is a real number, use (a) to prove the Binomial Theorem, i.e. $(p + q)^n = \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r}$.

(6 marks)

SECTION B (60 marks)
Answer any FOUR questions from this section, using AL(C2) answer book.
Each question carries 15 marks.

8. Let $f(x) = \sqrt[3]{x^2 - x^3}$.

(a) Find $f'(x)$ and $f''(x)$.

(2 marks)

(b) Show that both $f'(0)$ and $f'(1)$ do not exist.

(2 marks)

(c) Determine the sets of values of x such that:

(i) $f'(x) = 0$, (ii) $f'(x) > 0$, (iii) $f'(x) < 0$,

(iv) $f''(x) = 0$, (v) $f''(x) > 0$, (vi) $f''(x) < 0$.

(3 marks)

(d) Find the relative extremum point(s) and the point(s) of inflexion on the curve $y = f(x)$.

(3 marks)

(e) Find the asymptote(s) of the curve $y = f(x)$.

(3 marks)

(f) Sketch the curve $y = f(x)$.

(2 marks)

9. The equation of the hyperbola H is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ , where } a , b > 0 .$$

Let $P = \left(\frac{1}{2}a(t + \frac{1}{t}), \frac{1}{2}b(t - \frac{1}{t})\right)$, where $t \neq 0$.

(a) (i) Show that P lies on H .

(ii) Find the equation of the tangent to H at P .

(4 marks)

(b) Let the tangent to H at P meet the asymptotes of H at the points S and T. Let O be the origin.

(i) Show that as t varies the locus of the centre of the circle passing through O , S and T is a hyperbola.

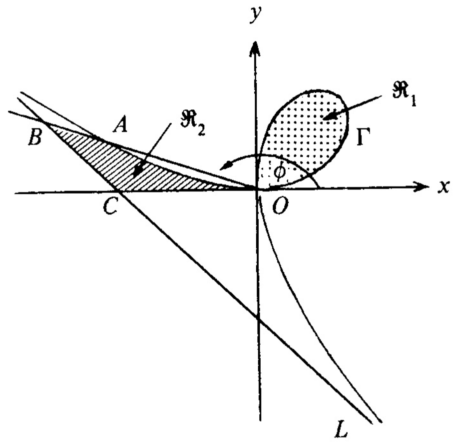
(ii) Prove that

$$OS \cdot OT = a^2 + b^2 .$$

Hence show that S , T and the two foci of H are concyclic.

(11 marks)

10.



In the figure, O is the origin and Γ is the curve whose equation is $x^3 + y^3 = 3axy$ $(a > 0)$. L is the asymptote of Γ.

(a) Evaluate $\lim_{x \to \infty} \frac{y}{x}$, where $(x, y) \in \Gamma$.

(You may assume that $\lim_{x \to \infty} \frac{y}{x}$ exists.)

Hence, or otherwise, show that the equation of L is

$x + y + a = 0$. (3 marks)

(b) Find the polar equations of Γ and L. (3 marks)

(c) Find the area of the region enclosed by Γ (i.e. $\Re_1$). (3 marks)

(d) Suppose a straight line through O cuts Γ at A and L at B in the second quadrant. Let ϕ be the angle between OB and the positive x-axis. Let A_ϕ be the area of the region bounded by the x-axis, Γ, L and AB (i.e. $\Re_2$).

Show that $A_\phi = \frac{a^2}{2}\left\{ \frac{1}{1 + \tan\phi} - \frac{3}{1 + \tan^3\phi} + 2 \right\}$.

Hence evaluate $\lim_{\phi \to \frac{3\pi}{4}} A_\phi$. (6 marks)

11. (a) Let f and g be real-valued functions defined on (a, ∞) where $a > 0$, and f be twice differentiable satisfying the following conditions:

A. g is decreasing,

B. $g(t) \geq 0$ and $f''(t) \geq 0$ for all $t \in (a, \infty)$,

C. $\lim_{t \to \infty} g(t)f'(t) = 0$.

(i) Use the Mean Value Theorem to show that

$$f(n) + f'(n)(t - n) \leq f(t) \leq f(n) + f'(n + 1)(t - n)$$

for all $t \in [n, n + 1]$, where n is a positive integer greater than a.

(ii) Hence, or otherwise, show that

$$\left| \int_n^{n+1} f(t)\,dt - \left(\frac{f(n) + f(n + 1)}{2} \right) \right| \leq \frac{f'(n + 1) - f'(n)}{2},$$

where n is a positive integer greater than a.

(iii) Show that

$$\lim_{n \to \infty} \left\{ \int_n^{n^2} f(t)\,dt - \sum_{j=n}^{n^2-1} \frac{f(j) + f(j + 1)}{2} \right\} g(n^2) = 0.$$

(9 marks)

(b) Using (a) and the fact that $\lim_{n \to \infty} \frac{\ln n}{n^2} \int_n^{n^2} \frac{1}{\ln t}\,dt = \frac{1}{2}$,

or otherwise, evaluate

$$\lim_{n \to \infty} \left\{ \frac{1}{\ln(n + 1)} + \frac{1}{\ln(n + 2)} + \cdots + \frac{1}{\ln(n^2)} \right\} \frac{\ln n}{n^2}.$$

(6 marks)

12. (a) Show that

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^{n-1}t^{2n-2} + \frac{(-1)^n t^{2n}}{1+t^2}$$

for all $t \in \mathbf{R}$ and $n = 1, 2, 3, \ldots$.

Deduce that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1} + \int_0^x \frac{(-1)^n t^{2n}}{1+t^2}\,dt$$

for all $x \in \mathbf{R}$ and $n = 1, 2, 3, \ldots$. (4 marks)

(b) Using (a), or otherwise, show that

$$\left|\tan^{-1}x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1}\right)\right| \le \frac{x^{2n+1}}{2n+1}$$

for all $x \ge 0$ and $n = 1, 2, 3, \ldots$.

Hence find $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. (6 marks)

(c) Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$.

Deduce that

$$\left|\frac{\pi}{4} - \left[\left(\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right) - \cdots + \frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{2^{2n-1}}+\frac{1}{3^{2n-1}}\right)\right]\right| \le \frac{1}{n \cdot 2^{2n+1}}$$

for $n = 1, 2, 3, \ldots$. (5 marks)

13. Let $a, b \in \mathbf{R}$ and $a < b$. Let $f(x)$ be a differentiable function on $\mathbf{R}$ such that $f(a) < 0$, $f(b) > 0$ and $f'(x)$ is strictly decreasing.

(a) Show that $f'(a) > 0$.

(2 marks)

(b) Let $x_0 = a$ and $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Show that $a < x_1 < b$, $f(x_1) < 0$ and $f'(x_1) > 0$.

(6 marks)

(c) Let $x_0 = a$ and $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$ for $n = 1, 2, 3, \ldots$.

Show that $a < x_n < b$, $f(x_n) < 0$ and $f'(x_n) > 0$ for $n = 1, 2, \ldots$

(4 marks)

(d) Show that $\lim_{n\to\infty} x_n$ exists and is a zero of $f(x)$.

(3 marks)

END OF PAPER