

Paper IA

1. (a)
$$\begin{pmatrix} 1 & c-3 & 5 & 3 \\ -3 & 9 & -15 & s \\ 2 & c & 10 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & c-3 & 5 & 3 \\ 0 & 3c-18 & 0 & s-9 \\ 0 & c-6 & 0 & 0 \end{pmatrix} \quad (R_2 = R_2 + 3R_1, R_3 = R_3 + 2R_1)$$

$$\rightarrow \begin{pmatrix} 1 & c-3 & 5 & 3 \\ 0 & 0 & 0 & s-9 \\ 0 & c-6 & 0 & 0 \end{pmatrix} \quad (R_2 = R_2 - 3R_1)$$

$$\rightarrow \begin{pmatrix} 1 & -3 & 5 & 3 \\ 0 & 0 & 0 & s-9 \\ 0 & c-6 & 0 & 0 \end{pmatrix} \quad (R_2 = R_2 - R_3) \quad 0 = s+9$$

(+6) $c = 0$

∴ For consistency, $s = -9$, $c = \text{any real number}$.

(b) Case 1: $s = -9$, $c = -6$

$$\rightarrow \begin{pmatrix} 1 & -3 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

∴ Solution set = $\{(\alpha, \beta, \frac{3-\alpha+3\beta}{5}) \in \mathbb{R}^3 : \alpha, \beta \in \mathbb{R}\}$

Case 2: $s = -9$, $c = -6$

Solution set = $\{(\alpha, 0, \frac{3-\alpha}{5}) \in \mathbb{R}^3 : \alpha \in \mathbb{R}\}$

2. (a) Reflexive:

$$\forall (x, y) \in \mathbb{R}^2,$$

$$x - x = 0$$

$$= (x, y) - (x, y)$$

Symmetric:

$$(x_1, y_1) - (x_2, y_2) = x_1 - x_2 = n \text{ for some integer } n$$

$$= x_2 - x_1 = -n$$

$$= (x_2, y_2) - (x_1, y_1)$$

Transitive:

$$(x_1, y_1) - (x_2, y_2) \text{ and } (x_2, y_2) - (x_3, y_3)$$

$$\Rightarrow x_1 - x_2 = n \text{ and } x_2 - x_3 = m \text{ for some integers } n, m$$

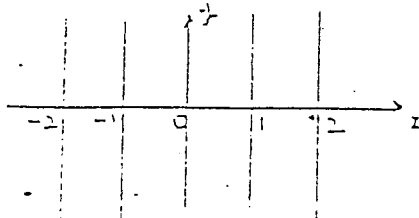
$$\Rightarrow x_1 - x_3 = n - m$$

$$\Rightarrow (x_1, y_1) - (x_3, y_3)$$

(b) $(2, 1)/- = \{(x, y) \in \mathbb{R}^2 : x - 2 = n \text{ for some } n \in \mathbb{Z}\}$

= the set of all vertical lines in \mathbb{R}^2 with

integral x-intercepts



$$3. (a) B^{-1} = \frac{1}{-2\lambda} \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$$

B^{-1} exists $\rightarrow \lambda \neq 0$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = B^{-1}AB$$

$$= -\frac{1}{2\lambda} \begin{pmatrix} \lambda & 0 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= -\frac{1}{2\lambda} \begin{pmatrix} \lambda & 0 \\ -3 & -6 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= -\frac{1}{2\lambda} \begin{pmatrix} -2\lambda & 0 \\ 0 & -6\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\therefore a = 1$$

$$b = 3$$

λ can be any non-zero number.

$$(b) A^{100} = \left(B \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} B^{-1} \right)^{100}$$

$$= B \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{100} B^{-1}$$

$$= B \begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix} B^{-1}$$

$$= -\frac{1}{2} \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix} \quad (\text{choosing } \lambda = 1)$$

$$= -\frac{1}{2} \begin{pmatrix} -2 & 0 \\ 1 & 3^{100} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$$

$$= -\frac{1}{2} \begin{pmatrix} -2 & 0 \\ 1 - 3^{100} & -3^{100} \cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{3^{100} - 1}{2} & 3^{100} \end{pmatrix}$$

$$\begin{aligned}
 4. \quad (1-i)^{2n} &= \sum_{r=0}^{2n} \binom{2n}{r} i^r \\
 &= \sum_{r=0}^n \binom{2n}{2r} i^{2r} + \sum_{r=0}^{n-1} \binom{2n}{2r+1} i^{2r+1} \\
 &= \sum_{r=0}^n \binom{2n}{2r} (-1)^r + i \sum_{r=0}^{n-1} \binom{2n}{2r+1} (-1)^r
 \end{aligned}$$

On the other hand, $(1-i)^{2n} = \left(\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right)^{2n}$

$$= 2^n \left(\cos \frac{2n\pi}{2} + i \sin \frac{2n\pi}{2}\right)$$

Alternatively:

$$\begin{aligned}
 (1-i)^{2n} &= (2i)^n \\
 &= 2^n i^n
 \end{aligned}$$

Therefore, $\sum_{r=0}^{2n} (-1)^r \binom{2n}{2r} = 2^n \cos \frac{2n\pi}{2}$

$$= \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2^n (-1)^{\frac{n}{2}} & \text{when } n \text{ is even} \end{cases}$$

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{2r+1} = 2^n \sin \frac{2n\pi}{2}$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ 2^n (-1)^{\frac{n-1}{2}} & \text{when } n \text{ is odd} \end{cases}$$

5. To show $2u_n = 2n - 1 - (-1)^n \quad \forall n = 1, 2, \dots$

For $n = 1$, L.H.S. = $2u_1 = 0$

R.H.S. = $2 - 1 - (-1) = 0$

Assume $2u_k = 2k - 1 - (-1)^k$ for some k .

For $n = k + 1$, L.H.S. = $2u_{k+1}$

$$= 2(2k - u_k)$$

$$= 2\left(2k - \frac{2k - 1 - (-1)^k}{2}\right)$$

$$= 4k - 2k - 1 - (-1)^k$$

$$= 2k - 1 - (-1)^{k+1}$$

$$= 2(k - 1) - 1 - (-1)^{k-1}$$

$$= \text{R.H.S.}$$

To find $\lim_{n \rightarrow \infty} \frac{u_n}{n}$:-

$$\lim_{n \rightarrow \infty} \frac{u_n}{n} = \lim_{n \rightarrow \infty} \frac{2n - 1 - (-1)^n}{2n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n} - \frac{(-1)^n}{2n}\right) = 1$$

5. (a) To show f^{-1} is strictly increasing:-

$$\forall x, y \in \mathbb{R},$$

$$\text{if } x < y,$$

then $f(x_0) < f(y_0)$ where $x = f(x_0)$, $y = f(y_0)$

$$- \quad x_0 < y_0 \quad (\because x_0 \geq y_0 \Rightarrow f(x_0) \geq f(y_0) = x \geq y \text{ !!})$$

$$- \quad f^{-1}(x) < f^{-1}(y)$$

To show $a_1 < f^{-1}\left(\frac{1}{n} \sum_{k=1}^n f(a_k)\right) < a_2$:-

$$\because \quad a_1 < a_k \quad (k = 2, \dots, n)$$

$$\therefore \quad f(a_1) < f(a_k) \quad (k = 2, \dots, n)$$

$$- \quad n f(a_1) < \sum_{k=1}^n f(a_k)$$

$$- \quad f(a_1) < \frac{1}{n} \sum_{k=1}^n f(a_k)$$

$$- \quad f^{-1}(f(a_1)) < f^{-1}\left(\frac{1}{n} \sum_{k=1}^n f(a_k)\right)$$

$$- \quad a_1 < f^{-1}\left(\frac{1}{n} \sum_{k=1}^n f(a_k)\right)$$

Similarly, we can show that

$$f^{-1}\left(\frac{1}{n} \sum_{k=1}^n f(a_k)\right) < a_2.$$

(b) To show $h^{-1}(x) = f^{-1}\left(\frac{x-q}{p}\right)$:-

$$h(x) = pf(x) - q$$

$$- \quad \frac{h(x) - q}{p} = f(x)$$

$$- \quad x = f^{-1}\left(\frac{h(x) - q}{p}\right)$$

$$\therefore \quad h^{-1}(x) = f^{-1}\left(\frac{x-q}{p}\right)$$

Alternatively:

$$h\left(f^{-1}\left(\frac{x-q}{p}\right)\right)$$

$$= pf\left(f^{-1}\left(\frac{x-q}{p}\right)\right) - q$$

$$= p\left(\frac{x-q}{p}\right) - q$$

$$= x$$

(b) To deduce that $h^{-1}\left(\frac{1}{n} \sum_{k=1}^n h(a_k)\right) = f^{-1}\left(\frac{1}{n} \sum_{k=1}^n f(a_k)\right)$:-

$$\begin{aligned} h^{-1}\left(\frac{1}{n} \sum_{k=1}^n h(a_k)\right) &= f^{-1}\left(\frac{\frac{1}{n} \sum_{k=1}^n h(a_k) - q}{p}\right) \\ &= f^{-1}\left(\frac{\frac{1}{n} \sum_{k=1}^n (pf(a_k) - q) - q}{p}\right) \\ &= f^{-1}\left(\frac{\frac{1}{n} \sum_{k=1}^n pf(a_k) - q - q}{p}\right) \\ &= f^{-1}\left(\frac{1}{n} \sum_{k=1}^n f(a_k)\right) \end{aligned}$$

7. (a) $\frac{C_r^n}{n^r} = \frac{n!}{(n-r)! r! n^r}$
 $= \frac{n(n-1) \cdots (n-r+1)}{r! n^r}$

$= \frac{1}{r!} \cdot 1 \cdot (1 - \frac{1}{n}) (1 - \frac{2}{n}) \cdots (1 - \frac{r-1}{n})$

$\leq \frac{1}{r!}$ (since $1 - \frac{k}{n} \in (0, 1)$ for $k = 1, \dots, r-1$)

(b) $[(1-a_1)(1-a_2) \cdots (1-a_n)]^{\frac{1}{n}} \leq \frac{(1-a_1) + (1-a_2) + \cdots + (1-a_n)}{n}$

$= \frac{(1-a_1)(1+a_2) \cdots (1-a_n)}{n} \leq \left(\frac{n-s}{n}\right)^n$

$= \left(1 - \frac{s}{n}\right)^n$

$= 1 + s - C_2^n \left(\frac{s}{n}\right)^2 - C_3^n \left(\frac{s}{n}\right)^3 - \cdots - C_n^n \left(\frac{s}{n}\right)^n$

$\leq 1 + s - \frac{1}{2!} s^2 - \frac{1}{3!} s^3 - \cdots - \frac{1}{n!} s^n$ (by (a))

(c) Since $1 - \frac{1}{2^k} > 1$, $\{c_k\}$ is increasing

To show c_n is bounded above:

Put $a_k = \frac{1}{2^k}$ for $k = 1, \dots, n$, then $s = a_1 + \cdots + a_n$

$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n$

$\leq (-1) - \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots$

$= (-1) - \frac{1}{1 - \frac{1}{2}} = 1$

$\therefore c_n = \prod_{k=1}^n (1 - a_k)$

$\leq 1 + s - \frac{s^2}{2!} - \cdots - \frac{s^n}{n!}$ (by (b))

$\leq 1 + 1 - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{n!}$

$\leq 1 + 1 - \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} - \cdots - \frac{1}{(n-1)n}$

$= 1 + 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n-1} - \frac{1}{n})$

$= 1 - 1 + 1 - \frac{1}{n}$

$= 3 - \frac{1}{n}$

≤ 3

paper I B

$$\begin{aligned}
 8. (a) \quad |u - v|^2 &= (u - v)(\overline{u - v}) \\
 &= (u - v)(\overline{u} - \overline{v}) \\
 &= u\overline{u} - u\overline{v} + \overline{u}v - v\overline{v} \\
 &= |u|^2 - u\overline{v} + \overline{u}v - |v|^2 \\
 &= |u|^2 - 2\operatorname{Re}(u\overline{v}) - |v|^2 \\
 &\leq |u|^2 - 2|u||v| - |v|^2 \\
 &= |u|^2 - 2|u||v| - |v|^2 \\
 &= (|u| - |v|)^2
 \end{aligned}$$

(b) (i) If $v = 0$, then $0 \cdot u - 1 \cdot v = 0$

If $v \neq 0$, then $u\overline{v} = \lambda \in \mathbb{R}$

$\frac{u\overline{v}}{v} = \lambda \in \mathbb{R}$

$u|v|^2 = \lambda v, \lambda \in \mathbb{R}$

$|v|^2 u - \lambda v = 0$

$\alpha u + \beta v = 0$

where $\alpha = |v|^2 \neq 0, \beta = -\lambda, \alpha, \beta \in \mathbb{R}$

(ii) $(|u| - |v|)^2 = |u|^2 - 2|u||v| - |v|^2$

$$\begin{aligned}
 &= u\overline{u} - 2\sqrt{|u||v|} - v\overline{v} \\
 &= u\overline{u} - 2\sqrt{u\overline{v} \overline{u}v} + v\overline{v} \\
 &= u\overline{u} - 2\sqrt{u\overline{v}u\overline{v}} - v\overline{v} \quad (\because u\overline{v} \in \mathbb{R}) \\
 &= u\overline{u} - 2\sqrt{(u\overline{v})^2} - v\overline{v} \\
 &= \begin{cases} u\overline{u} - 2u\overline{v} - v\overline{v} & \text{if } u\overline{v} \geq 0 \\ u\overline{u} - 2u\overline{v} - v\overline{v} & \text{if } u\overline{v} < 0 \end{cases} \\
 &= \begin{cases} u\overline{u} + u\overline{v} + \overline{u}v - v\overline{v} & \text{if } u\overline{v} \geq 0 \\ u\overline{u} - u\overline{v} - \overline{u}v - v\overline{v} & \text{if } u\overline{v} < 0 \end{cases} \\
 &= \begin{cases} (u + v)(\overline{u} + \overline{v}) & \text{if } u\overline{v} \geq 0 \\ (u - v)(\overline{u} - \overline{v}) & \text{if } u\overline{v} < 0 \end{cases} \\
 &= \begin{cases} (u - v)(\overline{u - v}) & \text{if } u\overline{v} \geq 0 \\ (u - v)(\overline{u - v}) & \text{if } u\overline{v} < 0 \end{cases} \\
 &= \begin{cases} |u - v|^2 & \text{if } u\overline{v} \geq 0 \\ |u - v|^2 & \text{if } u\overline{v} < 0 \end{cases}
 \end{aligned}$$

9. (a) For $n = 1$, L.H.S. = $\lambda^1 = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \text{R.H.S.}$

Assume $\lambda^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$

For $n = k + 1$,

$$\begin{aligned} \lambda^{k+1} &= \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos k\theta \cos \theta - \sin k\theta \sin \theta & -\sin k\theta \cos \theta - \cos k\theta \sin \theta \\ \sin k\theta \cos \theta + \cos k\theta \sin \theta & \cos k\theta \cos \theta - \sin k\theta \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{pmatrix} \end{aligned}$$

(b) (i). Let $X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $Y = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$.

$$\begin{aligned} \text{(I)} \quad XY &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \\ &= \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix} \in M \end{aligned}$$

$$\text{(II)} \quad YX = \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix}$$

$\therefore XY = YX$

$$\text{(III)} \quad X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = 0 \text{ or } b = 0$$

$\Rightarrow \det X = a^2 - b^2 \neq 0 \quad \therefore X^{-1} \text{ exists.}$

$$X^{-1} = \frac{1}{a^2 - b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} \left(\frac{a}{a^2 - b^2}\right) & \left(\frac{b}{a^2 - b^2}\right) \\ \left(\frac{-b}{a^2 - b^2}\right) & \left(\frac{a}{a^2 - b^2}\right) \end{pmatrix} \in M$$

(ii) Let $X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Case 1 : $a^2 - b^2 \neq 0$

$$X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$= \sqrt{a^2 - b^2} \cdot \begin{pmatrix} \frac{a}{\sqrt{a^2 + b^2}} & \frac{-b}{\sqrt{a^2 + b^2}} \\ \frac{b}{\sqrt{a^2 + b^2}} & \frac{a}{\sqrt{a^2 + b^2}} \end{pmatrix}$$

$$= r \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \left(\frac{b}{a} \right)$

Case 2 : $a^2 - b^2 = 0$

Then $a = b = 0$

$$X = 0 \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$X^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$r^n \cdot \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ where } X = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$r^n \cdot \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} r^n \cos n\theta = 1 \\ r^n \sin n\theta = 0 \end{cases}$$

$$r = 1, n\theta = 2k\pi, k \in \mathbb{Z}$$

$$r = 1, \theta = \frac{2k\pi}{n}, k \in \mathbb{Z}$$

$$\therefore X = \begin{pmatrix} \cos \frac{2k\pi}{n} & -\sin \frac{2k\pi}{n} \\ \sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} \end{pmatrix} \text{ where } k = 0, 1, 2, \dots, n-1.$$

(iii) To show that there exist $X \in M$

such that $Y = BX$ and $X^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

Case 1 : $\det B = 0$

Then $Y^n = B^n$

$$(\det Y)^n = (\det B)^n$$

$$\det Y = 0$$

$$\text{Let } Y = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

then $a^2 - b^2 = 0$ and $c^2 - d^2 = 0$

$$a = b = c = d = 0$$

$$Y = B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence $Y = BI$ where $I \in M$ and $I^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Case 2 : $\det B = 0$

Then B^{-1} exists and $Y^A = B^{-1}Y$

$$= (B^{-1})^A Y^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= (B^{-1})^A Y^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{by (b)(i)})$$

Hence $\dot{Y} = B(B^{-1}\dot{Y})$ where $(B^{-1}\dot{Y})^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B^{-1}Y \in \mathbb{R}^2$

$$\text{Therefore } Y = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \cos \frac{2k\pi}{n} & -\sin \frac{2k\pi}{n} \\ \sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} \end{pmatrix}$$

$$= \begin{pmatrix} \cos \frac{2k\pi}{n} + 2\sin \frac{2k\pi}{n} & 2\cos \frac{2k\pi}{n} - \sin \frac{2k\pi}{n} \\ -2\cos \frac{2k\pi}{n} - \sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} + 2\sin \frac{2k\pi}{n} \end{pmatrix}$$

$$k = 0, 1, \dots, n-1$$

$$\begin{aligned}
 \text{10. (a)} \quad \sum_{i=1}^n a_i (b_i - b_{i-1}) &= \sum_{i=1}^n (a_i b_i - a_i b_{i-1}) \\
 &= \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i b_{i-1} \\
 &= a_n b_n + \sum_{i=1}^{n-1} a_i b_i - \sum_{i=1}^n a_i b_{i-1} \\
 &= a_n b_n + \sum_{i=1}^{n-1} a_i b_i - \sum_{i=2}^n a_i b_{i-1} \quad (\because b_0 = 0) \\
 &= a_n b_n + \sum_{i=1}^{n-1} a_i b_i - \sum_{i=1}^{n-1} a_{i+1} b_i \\
 &= a_n b_n + \sum_{i=1}^{n-1} (a_i b_i - a_{i+1} b_i) \\
 &= a_n b_n + \sum_{i=1}^{n-1} (a_i - a_{i+1}) b_i
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \left| \sum_{i=1}^n a_i (b_i - b_{i-1}) \right| &= \left| a_n b_n + \sum_{i=1}^{n-1} (a_i - a_{i+1}) b_i \right| \\
 &\leq |a_n| |b_n| + \left| \sum_{i=1}^{n-1} (a_i - a_{i+1}) b_i \right| \\
 &\leq |a_n| |b_n| + \sum_{i=1}^{n-1} |a_i - a_{i+1}| |b_i| \\
 &\leq |a_n| K + \sum_{i=1}^{n-1} |a_i - a_{i+1}| K \quad (\because b_i \leq K) \\
 &= K \left\{ |a_n| + \sum_{i=1}^{n-1} |a_i - a_{i+1}| \right\} \\
 &= K \left\{ |a_n| + \sum_{i=1}^{n-1} (a_i - a_{i+1}) \right\} \quad (\because a_i \geq a_{i+1}) \\
 &= K \left\{ |a_n| + \sum_{i=1}^{n-1} a_i - \sum_{i=2}^n a_i \right\} \\
 &= K (|a_n| + a_1 - a_n) \\
 &\leq K (|a_n| - |a_1| - |a_n|) \\
 &= K (|a_1| - 2|a_n|)
 \end{aligned}$$

$$(c) \left| \sum_{j=0}^{p-1} \frac{(-1)^j}{1-j} \right| = \left| \sum_{j=1}^{p-1} \frac{(-1)^{p-1-j}}{(n-1-j)} \right| \quad (j = n-1-j)$$

$$= \left| \sum_{j=1}^{p-1} (-1)^{p-1} \frac{(-1)^j}{(n-1-j)} \right|$$

$$= \left| \sum_{j=1}^{p-1} \frac{(-1)^j}{(n-1-j)} \right|$$

$$= \left| \sum_{j=1}^{p-1} a_j (b_j - b_{j-1}) \right|$$

where $a_j = \frac{1}{n-1-j}$, $b_j = \frac{1}{2} (-1)^j$

$$\leq \frac{1}{2} \cdot (|a_1| + 2|a_{p-1}|) \quad (\text{by (b)})$$

$$= \frac{1}{2} \left(\frac{1}{n} + 2 \left(\frac{1}{n-p} \right) \right)$$

$$\leq \frac{1}{2} \left(\frac{1}{n} + \frac{2}{n} \right)$$

$$= \frac{3}{2n}$$

1M

1A

1M

1M

1A

11. (a) $y^n - 2ya^n \cos n\theta - a^{2n} = 0$

$(y - a^n(\cos n\theta - i \sin n\theta))(y - a^n(\cos n\theta + i \sin n\theta)) = 0$

$\therefore y = a^n(\cos n\theta - i \sin n\theta) \text{ or } a^n(\cos n\theta + i \sin n\theta)$

$x^{2n} - 2x^2 a^n \cos n\theta - a^{2n}$

$= (x^2 - a^n(\cos n\theta - i \sin n\theta))(x^2 - a^n(\cos n\theta + i \sin n\theta))$

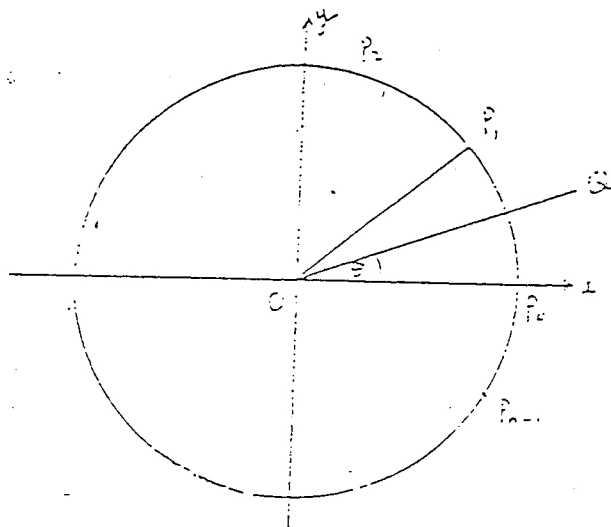
$= \left\{ \prod_{r=0}^{n-1} \left(x - a \left(\cos \left(\frac{n\theta + 2r\pi}{n} \right) - i \sin \left(\frac{n\theta + 2r\pi}{n} \right) \right) \right) \right\}$

$\left\{ \prod_{r=0}^{n-1} \left(x - a \left(\cos \left(\frac{n\theta - 2r\pi}{n} \right) - i \sin \left(\frac{n\theta - 2r\pi}{n} \right) \right) \right) \right\}$

$= \prod_{r=0}^{n-1} \left(\left(x - a \left(\cos \left(\theta + \frac{2r\pi}{n} \right) + i \sin \left(\theta + \frac{2r\pi}{n} \right) \right) \right) \left(x - a \left(\cos \left(\theta + \frac{2r\pi}{n} \right) - i \sin \left(\theta + \frac{2r\pi}{n} \right) \right) \right) \right)$

$= \prod_{r=0}^{n-1} \left(x^2 - 2x a \cos \left(\theta + \frac{2r\pi}{n} \right) + a^2 \right)$

(b) (i)



$\prod_{r=0}^{n-1} d_r^2 = \prod_{r=0}^{n-1} \left(OQ^2 + OP_r^2 - 2(OQ)(OP_r) \cos \left(\frac{2r\pi}{n} - \theta \right) \right) \text{ (cosine rule)}$

$= \prod_{r=0}^{n-1} \left(x^2 + a^2 - 2x a \cos \left(\frac{2r\pi}{n} - \theta \right) \right)$

$= x^{2n} - 2x^2 a^n \cos n(-\theta) - a^{2n} \text{ (by (a))}$

$= x^{2n} - 2x^2 a^n \cos n\theta - a^{2n}$

RESTRICTED 内部文件

PM92-14

Solutions

IB
Marks

(ii) Q lies on the positive real axis - $\theta = 0$

$$\begin{aligned} \therefore \prod_{r=0}^{n-1} d_r^2 &= x^{2n} - 2x^n a^n \cos 0 + a^{2n} \\ &= x^{2n} - 2x^n a^n + a^{2n} \\ &= (x^n - a^n)^2 \end{aligned}$$

$$\therefore \prod_{r=0}^{n-1} d_r = |x^n - a^n|$$

(iii) $\theta = \frac{2\pi}{2n} = \frac{\pi}{n}$

$$\begin{aligned} \therefore \prod_{r=0}^{n-1} d_r^2 &= x^{2n} - 2x^n a^n \cos\left(n \cdot \frac{\pi}{n}\right) + a^{2n} \\ &= x^{2n} - 2x^n a^n + a^{2n} \\ &= (x^n - a^n)^2 \end{aligned}$$

$$\therefore \prod_{r=0}^{n-1} d_r = x^n + a^n$$

1A

1M

1A

1M

12. (a) $\vec{c} \cdot \vec{a} = 0$

$(\alpha\vec{a} + \beta\vec{b}) \cdot \vec{a} = 0$

$\alpha\vec{a} \cdot \vec{a} + \beta\vec{b} \cdot \vec{a} = 0 \dots\dots\dots(1)$

$\vec{c} \cdot \vec{b} = 1$

$(\alpha\vec{a} + \beta\vec{b}) \cdot \vec{b} = 1$

$\alpha\vec{a} \cdot \vec{b} + \beta\vec{b} \cdot \vec{b} = 1 \dots\dots\dots(2)$

from (1) and (2), we have

$$\alpha = \frac{\begin{vmatrix} 0 & \vec{b} \cdot \vec{a} \\ 1 & \vec{b} \cdot \vec{b} \end{vmatrix}}{\begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}}$$

$$= \frac{-\vec{b} \cdot \vec{a}}{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$$

$$\beta = \frac{\begin{vmatrix} \vec{a} \cdot \vec{a} & 0 \\ \vec{a} \cdot \vec{b} & 1 \end{vmatrix}}{\begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}}$$

$$= \frac{\vec{a} \cdot \vec{a}}{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2}$$

(b) (i) $(\vec{x} - \vec{c}) \cdot \vec{a} = \vec{x} \cdot \vec{a} - \vec{c} \cdot \vec{a}$
 $= 0 - 0$
 $= 0$

$\therefore \vec{x} - \vec{c} \perp \vec{a}$

$(\vec{x} - \vec{c}) \cdot \vec{b} = \vec{x} \cdot \vec{b} - \vec{c} \cdot \vec{b}$
 $= 1 - 1$
 $= 0$

$\therefore \vec{x} - \vec{c} \perp \vec{b}$

(ii) $\vec{x} - \vec{c} \parallel \vec{a} \times \vec{b}$ ($\because \vec{x} - \vec{c} \perp \vec{a}, \vec{b}$)

$\vec{x} - \vec{c} = \lambda(\vec{a} \times \vec{b})$ for some $\lambda \in \mathbb{R}$

$\vec{x} = \vec{c} + \lambda(\vec{a} \times \vec{b})$ for some $\lambda \in \mathbb{R}$

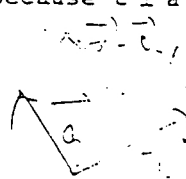
(iii) By (ii), $\vec{x} = \vec{c} + \lambda(\vec{a} \times \vec{b})$ for some $\lambda \in \mathbb{R}$.

$|\vec{x}|^2 = |\vec{c} + \lambda(\vec{a} \times \vec{b})|^2$
 $= |\vec{c}|^2 + |\lambda(\vec{a} \times \vec{b})|^2$

(by Pythagora's theorem, because $\vec{c} \perp \vec{a} \times \vec{b}$)

$\geq |\vec{c}|^2$

$\therefore |\vec{x}| \geq |\vec{c}|$



Solutions

B
Marks

(c) Let $\vec{a} = a_1\vec{i} - a_2\vec{j} - a_3\vec{k}$ and $\vec{b} = b_1\vec{i} - b_2\vec{j} - b_3\vec{k}$.

$a_1, b_1 \neq a_2, b_2 \neq \vec{a}, \vec{b}$ are linearly independent

$$\text{Consider } \vec{x} = \left(\frac{-a_2}{a_1 b_2 - a_2 b_1} \right) \vec{i} + \left(\frac{a_1}{a_1 b_2 - a_2 b_1} \right) \vec{j} + 0\vec{k}$$

$$\text{then } \vec{x} \cdot \vec{a} = \frac{-a_2 a_1}{a_1 b_2 - a_2 b_1} - \frac{a_2 a_2}{a_1 b_2 - a_2 b_1} = 0 = 0$$

$$\begin{aligned} \text{and } \vec{x} \cdot \vec{b} &= \frac{-a_2 b_1}{a_1 b_2 - a_2 b_1} - \frac{a_1 b_2}{a_1 b_2 - a_2 b_1} = 0 \\ &= \frac{-a_2 b_1 - a_1 b_2}{a_1 b_2 - a_2 b_1} = 1 \end{aligned}$$

by (b) (iii), $|\vec{x}| \geq |\vec{c}| \Rightarrow |\vec{x}|^2 \geq |\vec{c}|^2$

$$\text{Now } |\vec{x}|^2 = \left(\frac{-a_2}{a_1 b_2 - a_2 b_1} \right)^2 + \left(\frac{a_1}{a_1 b_2 - a_2 b_1} \right)^2$$

$$\text{it remains to show } |\vec{c}|^2 = \frac{\sum_{r=1}^3 a_r^2}{\left(\sum_{r=1}^3 a_r^2 \right) \left(\sum_{r=1}^3 b_r^2 \right) - \left(\sum_{r=1}^3 a_r b_r \right)^2} ;$$

$$|\vec{c}|^2 = \vec{c} \cdot \vec{c}$$

$$= \vec{c} \cdot (\alpha \vec{a} - \beta \vec{b})$$

$$= \alpha \vec{c} \cdot \vec{a} - \beta \vec{c} \cdot \vec{b}$$

$$= \alpha \cdot 0 - \beta \cdot 1$$

$$= -\beta$$

$$= \frac{\vec{a} \cdot \vec{a}}{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2} \quad (\text{by (a)})$$

$$= \frac{\sum_{r=1}^3 a_r^2}{\left(\sum_{r=1}^3 a_r^2 \right) \left(\sum_{r=1}^3 b_r^2 \right) - \left(\sum_{r=1}^3 a_r b_r \right)^2}$$

$$\begin{aligned}
 1. (a) \lim_{x \rightarrow 0} \frac{\tan x - x}{x^2 \sin x} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{2x \sin x - x^2 \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{2 \sin x + 2x \cos x - 2x \cos x - x^2 \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \sin x}{2 \sin x + 4x \cos x - x^2 \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{2 (\sec^2 x) \left(\frac{\sin x}{x} \right)}{2 \left(\frac{\sin x}{x} \right) + 4 \cos x - x \sin x} \\
 &= \frac{2(1)(1)}{2(1) + 4(1) - (0)} \\
 &= \frac{2}{6} \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 (b) \left| x \sin \frac{1}{x} \right| &= |x| \left| \sin \frac{1}{x} \right| \\
 &\leq |x| \cdot 1 \\
 &= |x| \quad \forall x \neq 0
 \end{aligned}$$

$$-|x| \leq x \sin \frac{1}{x} \leq |x|$$

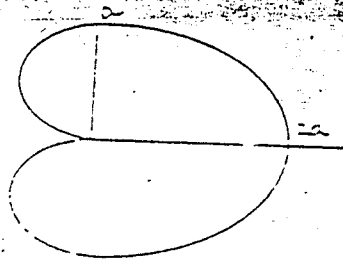
$$\therefore \lim_{x \rightarrow 0} |x| = 0$$

by Squeezing theorem,

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

$$\begin{aligned}
 \text{Hence, } \lim_{x \rightarrow 0} \frac{\frac{1}{x} + \sin \frac{1}{x}}{\frac{1}{x} - \sin \frac{1}{x}} &= \lim_{x \rightarrow 0} \frac{1 + x \sin \frac{1}{x}}{1 - x \sin \frac{1}{x}} \\
 &= \frac{1 + 0}{1 - 0} \\
 &= 1
 \end{aligned}$$

2.



$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \int_0^{2\pi} r^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} a^2 (1 - \cos\theta)^2 d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} 1 + 2\cos\theta + \frac{\cos 2\theta + 1}{2} d\theta \\
 &= \frac{a^2}{4} \int_0^{2\pi} 3 + 4\cos\theta + \cos 2\theta d\theta \\
 &= \frac{a^2}{4} \left\{ [3\theta + 4\sin\theta]_0^{2\pi} + \frac{1}{2} \int_0^{2\pi} \cos 2\theta d(2\theta) \right\} \\
 &= \frac{a^2}{4} \left\{ [3\theta + 4\sin\theta]_0^{2\pi} - \frac{1}{2} [\sin 2\theta]_0^{2\pi} \right\} \\
 &= \frac{a^2}{4} (6\pi - 0) \\
 &= \frac{3}{2} \pi a^2
 \end{aligned}$$

2

1

14

12

3. Let the lines be (L_1) and (L_2) .

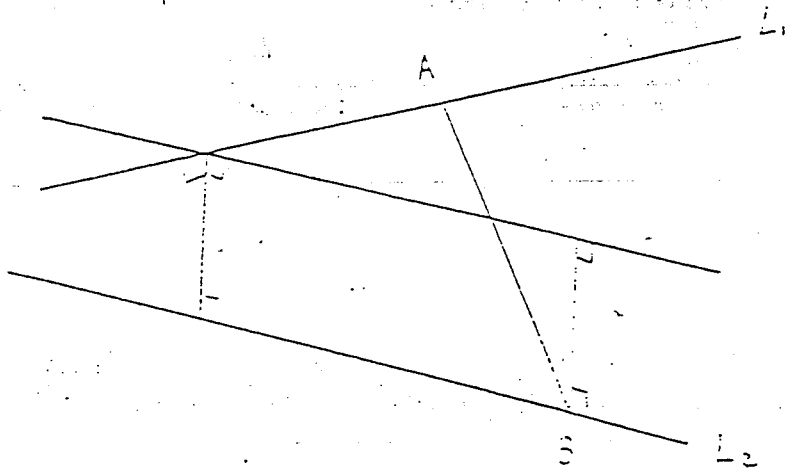
$(L_1) \perp (L_2)$

$(i - pj - k) \cdot (i - j - qk) = 0$

$1 - p + q = 0 \dots\dots\dots(1)$

1M

1A



$A = (2, 4, 4) \in (L_1)$

$B = (0, 3, 2) \in (L_2)$

Shortest distance between (L_1) and (L_2)

$= \vec{AB} \cdot (i + pj + k) \times (i - j - qk) / \sqrt{(2 - p^2)(2 - q^2)}$

1M

$= \begin{vmatrix} 2 & 1 & 2 \\ 1 & p & 1 \\ 1 & -1 & q \end{vmatrix} / \sqrt{(2 - p^2)(2 - q^2)}$

1

$= (2pq - 2 - 1 - 2p - q - 2) / \sqrt{(2 - p^2)(2 - q^2)}$

$= (2pq - 2p - q - 1) / \sqrt{(2 - p^2)(2 - q^2)}$

$\therefore (L_1)$ and (L_2) are coplanar,

$\therefore 2pq - 2p - q - 1 = 0 \dots\dots\dots(2)$

1A

$$\begin{aligned}
 4. \int_0^1 xe^{x-1} dx &= \int_0^1 xe^{-x} dx = \int_1^e xe^{-x} dx \\
 &= e \int_0^1 xe^{-x} dx = e^{-1} \int_1^e xe^{-x} dx \\
 &= e \int_0^1 (-x) e^{-x} d(-x) = \frac{1}{e} \int_1^e xe^{-x} dx \\
 &= e \int_0^{-1} ye^y dy = \frac{1}{e} \int_1^e xe^{-x} dx
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \int xe^x dx &= \int x d(e^x) \\
 &= xe^x - \int e^x dx \\
 &= xe^x - e^x + c
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_0^{-1} ye^y dy &= [ye^y - e^y]_0^{-1} \\
 &= \left(-\frac{1}{e} - \frac{1}{e}\right) - (0 - 1) \\
 &= 1 - \frac{2}{e}
 \end{aligned}$$

$$\begin{aligned}
 \int_1^e xe^{-x} dx &= [xe^{-x} - e^{-x}]_1^e \\
 &= (2e^2 - e^2) - (e - e) \\
 &= e^2
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{answer} &= e\left(1 - \frac{2}{e}\right) - \frac{1}{e}(e^2) \\
 &= e - 2 + e \\
 &= 2e - 2
 \end{aligned}$$

$$5. \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2n^2 + k^2}{n^3 + k^3} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2 + \left(\frac{k}{n}\right)^2}{1 + \left(\frac{k}{n}\right)^3} \cdot \frac{1}{n}$$

$$= \int_0^1 \frac{2 + x^2}{1 + x^3} dx$$

$$= \int_0^1 \frac{1}{1+x} + \frac{1}{x^2 - x + 1} dx$$

$$= [\ln(1+x)]_0^1 + \int_0^1 \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

$$= \ln 2 - \frac{1}{\sqrt{\frac{3}{4}}} \int_0^1 \frac{1}{1 + \left(\frac{\sqrt{\frac{4}{3}}}{2} \left(x - \frac{1}{2}\right)\right)^2} d\left(\frac{\sqrt{\frac{4}{3}}}{2} \left(x - \frac{1}{2}\right)\right)$$

$$= \ln 2 + \frac{2}{\sqrt{3}} \left[\tan^{-1} \sqrt{\frac{4}{3}} \left(x - \frac{1}{2}\right) \right]_0^1$$

$$= \ln 2 + \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{1}{\sqrt{3}} - \tan^{-1} \left(-\frac{1}{\sqrt{3}}\right) \right]$$

$$= \ln 2 + \frac{2}{\sqrt{3}} \frac{\pi}{3}$$

Marks

14-14

1A

1A

1

14-1A

1A

2

1A

6. Let $P = (c, 2a)$

OR: $a^2 = cx + 2ay$

→ slope of $OR = -\frac{c}{2a}$

Since $OR \perp OM$, where $M(p, q)$ is the mid-point of OR , we have

$\left(-\frac{c}{2a}\right)\left(\frac{q}{p}\right) = -1$

→ $c = \frac{2ap}{q}$ (1)

Let $O = (x_1, y_1)$ and $R = (x_2, y_2)$

Then $a^2 = cx_1 + 2ay_1$

and $a^2 = cx_2 + 2ay_2$

→ $2a^2 = c(x_1 + x_2) + 2a(y_1 + y_2)$

$2a^2 = 2cp + 4aq$ (2)

Substituting (1) into (2),

$2a^2 = 2\left(\frac{2ap}{q}\right)p + 4aq$

→ $aq = 2p^2 + 2a^2$

∴ M lies on the circle $2x^2 - 2y^2 - ay = 0$

$x^2 - y^2 - \frac{a}{4}y = 0$

$x^2 - \left(y - \frac{a}{4}\right)^2 = \left(\frac{a}{4}\right)^2$

∴ centre = $\left(0, \frac{a}{4}\right)$

radius = $\frac{a}{4}$

1A

1M

1

1

1A

1A

7. (a) $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$

Now $f(0) = f(0 + 0)$
 $= f(0) + f(0) - 0$
 $= 2f(0)$

$f(0) - 2f(0) = 0$

$-f(0) = 0$

$f(0) = 0$

$\therefore f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - 0}{h}$

$= \lim_{h \rightarrow 0} \frac{f(h)}{h}$

Alternatively,

$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \rightarrow 0} \frac{f(h) + f(0) - 0 - f(0)}{h}$

$= \lim_{h \rightarrow 0} \frac{f(h)}{h}$

(b) $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \rightarrow 0} \frac{f(x) + f(h) - 3xh(x+h) - f(x)}{h}$

$= \lim_{h \rightarrow 0} \frac{f(h)}{h} - \lim_{h \rightarrow 0} 3x(x+h)$

$= f'(0) - 3x^2$

Integrating, $f(x) = f'(0)x + x^3 - c$

Put $x = 0$, $0 = f(0) = 0 + 0 + c - c = 0$

$\therefore f(x) = f'(0)x + x^3$

$$\begin{aligned}
 8. \quad (a) \quad f'(x) &= \frac{d}{dx}(xe^{-x^2}) \\
 &= e^{-x^2} - xe^{-x^2} \cdot \frac{d}{dx}(-x^2) \\
 &= e^{-x^2} - 2x^2 e^{-x^2} \\
 &= e^{-x^2}(1 - 2x^2)
 \end{aligned}$$

$$\begin{aligned}
 f''(x) &= \frac{d}{dx}(e^{-x^2}(1 - 2x^2)) \\
 &= -4xe^{-x^2} - (1 - 2x^2)e^{-x^2}(-2x) \\
 &= -4xe^{-x^2} - 2x(1 - 2x^2)e^{-x^2} \\
 &= -2xe^{-x^2}(2 - 1 - 2x^2) \\
 &= -2xe^{-x^2}(3 - 2x^2)
 \end{aligned}$$

(b) (i) $f'(x) = 0$

$$e^{-x^2}(1 - 2x^2) = 0$$

$$x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

(ii) $f'(x) > 0$

$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

(iii) $f'(x) < 0$

$$x < -\frac{1}{\sqrt{2}} \quad \text{or} \quad x > \frac{1}{\sqrt{2}}$$

(iv) $f''(x) = 0$

$$-2xe^{-x^2}(3 - 2x^2) = 0$$

$$x = 0 \quad \text{or} \quad \frac{\sqrt{3}}{\sqrt{2}} \quad \text{or} \quad -\frac{\sqrt{3}}{\sqrt{2}}$$

(v) $f''(x) > 0$

$$-\frac{\sqrt{3}}{\sqrt{2}} < x < 0 \quad \text{or} \quad x > \frac{\sqrt{3}}{\sqrt{2}}$$

(vi) $f''(x) < 0$

$$x < -\frac{\sqrt{3}}{\sqrt{2}} \quad \text{or} \quad 0 < x < \frac{\sqrt{3}}{\sqrt{2}}$$

1A

1A

$\frac{1}{2}$ A

$\frac{1}{2}$ A

$\frac{1}{2}$ A

$\frac{1}{2}$ A

$\frac{1}{2}$ A

$\frac{1}{2}$ A

(c)

x	$(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$	$-\frac{3}{\sqrt{2}}$	$(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$(\frac{1}{\sqrt{2}}, 0)$	0	$(0, \frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}})$	$\frac{3}{\sqrt{2}}$	$(\frac{3}{\sqrt{2}}, -)$
f'	-	-	-	0	+	-	-	0	-	-	-
f''	-	0	-	-	+	0	-	-	-	0	-
f		Pt of inflexion		min		0		max		Pt of inflexion	

From the table, we have

$$\text{minimum point} = \left(-\frac{1}{\sqrt{2}}, f\left(-\frac{1}{\sqrt{2}}\right) \right)$$

$$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} e^{-\frac{1}{2}} \right)$$

1A

$$\text{maximum point} = \left(\frac{1}{\sqrt{2}}, f\left(\frac{1}{\sqrt{2}}\right) \right)$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} e^{-\frac{1}{2}} \right)$$

1A

Points of inflexion are

$$\left(-\frac{3}{\sqrt{2}}, f\left(-\frac{3}{\sqrt{2}}\right) \right), \left(\frac{3}{\sqrt{2}}, f\left(\frac{3}{\sqrt{2}}\right) \right) \text{ and } (0, f(0))$$

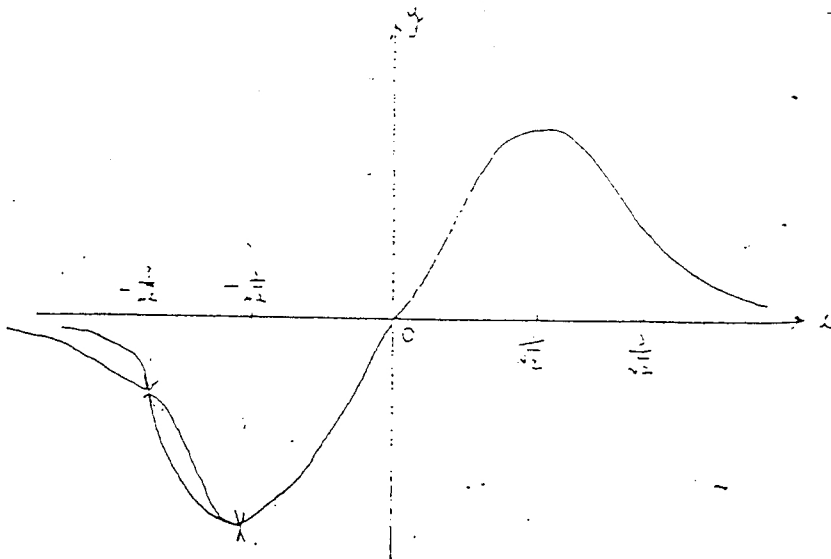
$$= \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}} e^{-\frac{9}{2}} \right), \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}} e^{-\frac{9}{2}} \right) \text{ and } (0, 0)$$

1A

1A

(d) asymptote : $y = 0$

(e)



3

(1) Consider $\begin{cases} x = (x-y)\frac{1}{\sqrt{2}} \\ y = (x+y)\frac{1}{\sqrt{2}} \end{cases}$

$\begin{cases} x = \frac{1}{\sqrt{2}}(x-y) \\ y = \frac{1}{\sqrt{2}}(x+y) \end{cases}$

then $x - y = (x - y) e^{-\frac{1}{2}(x-y)}$

$y = x e^{-x}$ which has the same graph as in (e)

On the other hand,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. Each (x, y) is transformed to (x', y') by a rotation of 45° anticlockwise.

the graph $x - y = (x - y) e^{-\frac{1}{2}(x-y)}$ is obtained by rotating the graph in (e) 45° clockwise.

$u = \frac{1}{\sqrt{2}}(x-y)$
 $x - y = \sqrt{2}u$
 put $\sqrt{2}v = x + y$

\therefore becomes
 $\sqrt{2}v = \sqrt{2}u e^{-u}$
 $\therefore v = u e^{-u}$

$u = \frac{1}{\sqrt{2}}(x-y) = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y$

$v = \frac{1}{\sqrt{2}}(x+y) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$

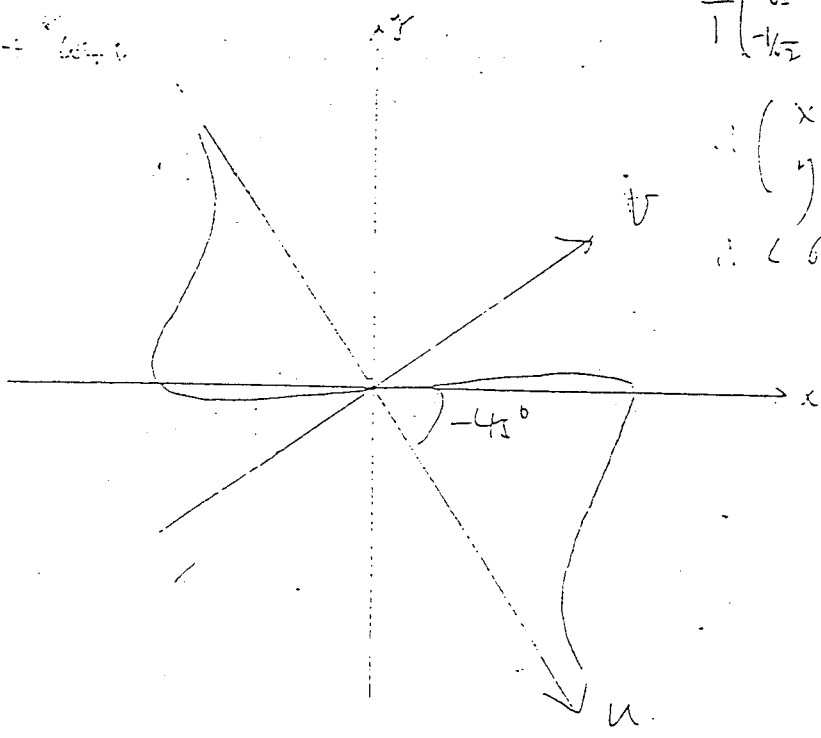
$\therefore \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\therefore \angle$ of rotation = -45°

$\begin{cases} x = X \cos \theta - Y \sin \theta \\ y = X \sin \theta + Y \cos \theta \end{cases}$



9. (a) $\int_0^x (x-t)^p g'(t) dt = \int_0^x (x-t)^p g'(t) dt$
 $= [(x-t)^p g(t)]_0^x - \int_0^x g(t) d(x-t)^p$
 $= -x^p g(0) - p \int_0^x g(t) (x-t)^{p-1} dt$

14
14

(b) $n = 1,$

L.H.S. = e^x

R.H.S. = $1 - \frac{1}{0!} \int_0^x e^{-t} dt$
 $= e^x$

Assume $e^x = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots - \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} e^{-t} dt$

1

Then $e^x = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots - \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)! n} \left\{ n \int_0^x (x-t)^{n-1} e^{-t} dt \right\}$

$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots + \frac{x^{n-1}}{(n-1)!} + \frac{1}{n!} \left\{ x^n e^0 + \int_0^x (x-t)^n e^{-t} dt \right\}$
 (by (a))

1

$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots - \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} - \frac{1}{n!} \int_0^x (x-t)^n e^{-t} dt$

$e = 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{1}{(2n)!} - \frac{1}{(2n)!} \int_0^1 (1-t)^{2n} e^{-t} dt$

1A

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \dots - \frac{1}{(2n)!} + \frac{1}{(2n)!} \int_0^{-1} (-1-t)^{2n} e^{-t} dt$

Adding,

1A

$e + \frac{1}{e}$

$= 2 \left(1 - \frac{1}{2!} + \frac{1}{4!} - \dots - \frac{1}{(2n)!} \right) - \frac{1}{(2n)!} \left\{ \int_0^1 (1-t)^{2n} e^{-t} dt - \int_0^{-1} (-1-t)^{2n} e^{-t} dt \right\}$

$\therefore \left| \left(e + \frac{1}{e} \right) - 2 \left(1 - \frac{1}{2!} + \frac{1}{4!} - \dots - \frac{1}{(2n)!} \right) \right|$

$= \frac{1}{(2n)!} \left| \int_0^1 (1-t)^{2n} e^{-t} dt - \int_0^{-1} (-1-t)^{2n} e^{-t} dt \right|$

$\leq \frac{1}{(2n)!} \left\{ \int_0^1 |1-t|^{2n} e^{-t} dt + \int_{-1}^0 |-1-t|^{2n} e^{-t} dt \right\}$

$\leq \frac{1}{(2n)!} \left\{ \int_0^1 e^{-t} dt + \int_{-1}^0 e^{-t} dt \right\}$

1A

$= \frac{1}{(2n)!} \left\{ [e^{-t}]_0^1 + [e^{-t}]_{-1}^0 \right\}$

$= \frac{1}{(2n)!} \left\{ (e^{-1} - 1) + (1 - \frac{1}{e}) \right\}$

$< \frac{e}{(2n)!}$

1A

$< \frac{1}{(2n)!}$

(c) (i) $f_0(x) = \int_0^x f_{-1}(t) dt$

$$= 1 \int_0^x (x-t)^0 f_{-1}(t) dt$$

$$= x^2 f_{-1}(0) + \int_0^x (x-t) \frac{d}{dt} f_{-1}(t) dt \quad (\text{by (a)}) \quad 1M$$

$$= \int_0^x (x-t) f_{-2}(t) dt$$

$$= \frac{1}{2} \left\{ 2 \int_0^x (x-t) f_{-2}(t) dt \right\}$$

$$= \frac{1}{2} \left\{ x^2 f_{-2}(0) + \int_0^x (x-t)^2 \frac{d}{dt} f_{-2}(t) dt \right\} \quad (\text{by (a)}) \quad 1M$$

$$= \frac{1}{2} \int_0^x (x-t)^2 f_{-3}(t) dt$$

$$= \left(\frac{1}{3}\right) \left(\frac{1}{2}\right) \int_0^x (x-t)^3 f_{-4}(t) dt \quad (\text{by similar arguments}) \quad 1M$$

$$= \dots$$

$$= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f_n(t) dt$$

(ii) Let $f_0(x) = |\sin(x^2)|$, a continuous function

Define $f_n(x) = \int_0^x f_{n-1}(t) dt$ for $n = 1, 2, \dots$ 1M

by (c) (i), $f_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} |\sin(t^2)| dt$

$f_{100}(x) = \frac{1}{99!} \int_0^x (x-t)^{99} |\sin(t^2)| dt$

$f_{99}(x) = \frac{d}{dx} f_{100}(x) = \frac{1}{99!} \frac{d}{dx} \int_0^x (x-t)^{99} |\sin(t^2)| dt$ 1M

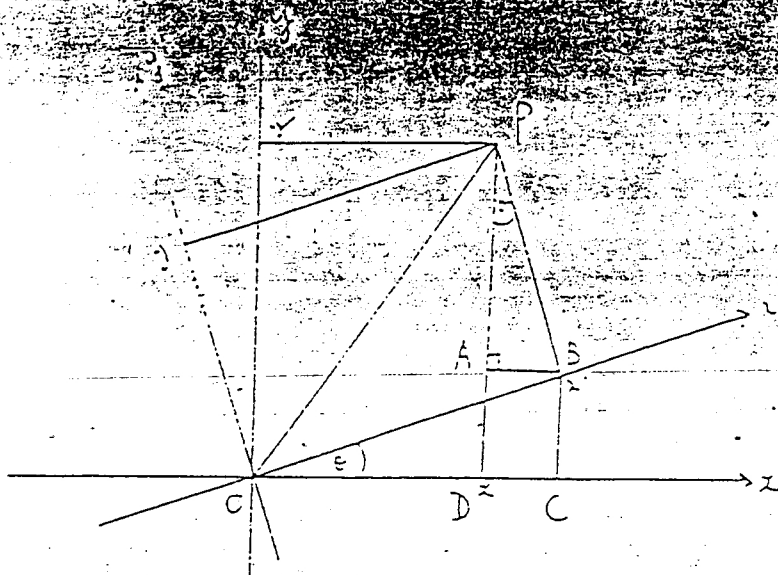
$f_{98}(x) = \frac{d}{dx} f_{99}(x) = \frac{1}{99!} \frac{d^2}{dx^2} \int_0^x (x-t)^{99} |\sin(t^2)| dt$

$f_0(x) = \frac{d}{dx} f_1(x) = \frac{1}{99!} \frac{d^{100}}{dx^{100}} \int_0^x (x-t)^{99} |\sin(t^2)| dt$

$\therefore \frac{d^{100}}{dx^{100}} \int_0^x (x-t)^{99} |\sin(t^2)| dt = 99! f_0(x)$

$= 99! |\sin(x^2)|$ 1M

10. (a) (1)



$$\begin{aligned}
 y &= PA + BC \\
 &= PB \cos \theta + BO \sin \theta \\
 &= y' \cos \theta - x' \sin \theta \\
 x &= CO - DC \\
 &= OB \cos \theta - PB \sin \theta \\
 &= x' \cos \theta - y' \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \therefore \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x' \cos \theta - y' \sin \theta \\ x' \sin \theta - y' \cos \theta \end{pmatrix} \\
 &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}
 \end{aligned}$$

(ii) Now $V = MV'$

$$\begin{aligned}
 \therefore (MV')^T A (MV) &= C \\
 \Rightarrow V'^T (M^T A M) V &= C \\
 \Rightarrow V'^T A' V &= C \quad \text{where } A' = M^T A M
 \end{aligned}$$

$$\begin{aligned}
 \text{also, } \det A' &= \det (M^T A M) \\
 &= \det M^T \det A \det M \\
 &= 1 \cdot \det A \cdot 1 \\
 &= \det A
 \end{aligned}$$

$$A' = M^{-1}AM$$

$$= \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = (c \cos \theta, s = \sin \theta)$$

$$= \begin{pmatrix} ac + sh & ch + bs \\ -sa + ch & -sh - cb \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$= \begin{pmatrix} ac^2 + sch - sch - bs^2 & -acs - s^2h - c^2h - cbs \\ -sac + c^2h - s^2h - scb & s^2a - sch - csh + c^2b \end{pmatrix}$$

$\therefore A'$ is diagonal when $-sac - c^2h - s^2h - scb = 0$

i.e. when $(b - a)cs = -(c^2 - s^2)h$

$$\Rightarrow \frac{b - a}{2} \sin 2\theta = -h \cos 2\theta \dots \dots (*)$$

if $b - a = 0$, the equation (*) is satisfied if $\cos 2\theta = 0$

if $b - a \neq 0$, the equation (*) is satisfied if $\tan 2\theta = -\frac{2h}{b - a}$

$\therefore \exists \theta \in \mathbb{R}$ such that A' is diagonal

(b) The conic section is represented by

$$(x \ y) \begin{pmatrix} 7 & h \\ h & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \text{ in } \Gamma$$

$$\text{and } (x' \ y') \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 16 \text{ in } \Gamma'$$

$$\text{where } \det \begin{pmatrix} 7 & h \\ h & 13 \end{pmatrix} = \det \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\text{i.e. } 91 - h^2 = \lambda\mu$$

(i) If the conic section is an ellipse, then

$$91 - h^2 = \lambda\mu > 0$$

$$\Rightarrow 91 > h^2$$

$$\Rightarrow -\sqrt{91} < h < \sqrt{91}$$

(ii) If the conic section is a hyperbola, then

$$91 - h^2 = \lambda\mu < 0$$

$$\Rightarrow 91 < h^2$$

$$\Rightarrow h > \sqrt{91} \text{ or } h < -\sqrt{91}$$

(iii) If the conic section is a pair of straight lines, then

$$91 - h^2 = \lambda\mu = 0$$

$$\Rightarrow h = \pm\sqrt{91}$$

$$(iv) \begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \Rightarrow \lambda = 4, \mu = 16$$

$$\therefore 91 - h^2 = 64$$

$$\Rightarrow h^2 = 27$$

$$\Rightarrow h = \pm\sqrt{27} = \pm 3\sqrt{3}$$

11. (a) Let $\deg p(x) = m > n > 1$.

Divide $p(x)$ by $(x-a)^n$. We have

$$p(x) = c_0(x-a)^n + q_1(x) \text{ for some } c_0 \in \mathbb{R} \setminus \{0\} \text{ and } \deg q_1(x) < n$$

Divide $q_1(x)$ by $(x-a)^{n-1}$. We have

$$q_1(x) = c_1(x-a)^{n-1} + q_2(x) \text{ for some } c_1 \in \mathbb{R} \text{ and } \deg q_2(x) < n-1$$

Thus

$$p(x) = c_0(x-a)^n + c_1(x-a)^{n-1} + q_2(x) \text{ for some } q_2(x)$$

such that $\deg q_2(x) < n-1$.

Repeating the same arguments, we have

$$p(x) = c_0(x-a)^n + c_1(x-a)^{n-1} + \dots + c_{n-1}(x-a) + c_n, \text{ where } c_n \in \mathbb{R}$$

Putting $x = a$, we have $0 = p(a) = c_n = c_0$.

Differentiating once and putting $x = a$, we have

$$0 = p'(a) = c_{n-1} = c_{n-1} = 0$$

Differentiating twice and putting $x = a$, we have

$$0 = p''(a) = 2c_{n-2} = c_{n-2} = 0$$

By the same arguments, since $p^{(k)}(a) = 0$ for $k = 0, 1, \dots, n-1$,

we have $c_{n-k} = 0$ for $k = 0, 1, \dots, n-1$.

Therefore,

$$\begin{aligned} p(x) &= c_0(x-a)^n + c_1(x-a)^{n-1} + \dots + c_{n-n}(x-a)^{n-(n-n)} \\ &= c_0(x-a)^n + c_1(x-a)^{n-1} + \dots + c_{n-n}(x-a)^0 \\ &= (x-a)^n [c_0(x-a)^{-n} + \dots + c_{n-n}] \end{aligned}$$

$\therefore p(x)$ is divisible by $(x-a)^n$.

<p><u>Alternatively:</u> (use M.I.)</p> <p>For $n = 1$, let $p(x) = c_1(x-a)$</p> <p>put $x = a$, we have $r = 0$</p> <p>$\therefore (x-a) p(x)$</p> <p>Assume $p^{(k)}(a) = 0$ for $k = 0, \dots, n-1 \implies (x-a)^k p(x)$</p> <p>Now if $p^{(k)}(a) = 0$ for $k = 0, \dots, n$,</p> <p>then by induction assumption, $p(x) = (x-a)^n q(x)$ for some $q(x)$</p> <p>differentiating n times and putting $x = a$,</p> <p>we have $0 = p^{(n)}(a) = n! q(a)$</p> <p>$\implies q(a) = 0$</p> <p>$\implies q(x) = (x-a) q_1(x)$ for some $q_1(x)$</p> <p>$\therefore p(x) = (x-a)^{n+1} q_1(x)$</p> <p>i.e. $(x-a)^{n+1} p(x)$</p>	<p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p>
---	--

(b) by (a), since $\deg F(x) \geq 4$, we need only to show

$$F(1) = F'(1) = F''(1) = F'''(1) = 0$$

To show $F(1) = 0$:

$$F(1) = \left(\int_1^1 p r dt \right) \left(\int_1^1 q s dt \right) - \left(\int_1^1 p q dt \right) \left(\int_1^1 r s dt \right) = 0$$

To show $F'(1) = 0$:

$$F'(x) = p r \int_1^x q s dt + q s \int_1^x p r dt - p q \int_1^x r s dt - r s \int_1^x p q dt$$

$$\therefore F'(1) = 0$$

$$F''(x) = (p r)' \int_1^x q s dt - p q r s - (q s)' \int_1^x p r dt + p q r s$$

$$- (p q)' \int_1^x r s dt - p q r s - (r s)' \int_1^x p q dt + p q r s$$

$$= (p r)' \int_1^x q s dt - (q s)' \int_1^x p r dt - (p q)' \int_1^x r s dt - (r s)' \int_1^x p q dt$$

$$\therefore F''(1) = 0$$

$$\begin{aligned}
 F''(x) &= (pr)'' \int_1^x qsdt - (pr)'(qs)' + (qs)'' \int_1^x prdt - (pr)'(qs)' \\
 &= (pq)'' \int_1^x rsdt - (ps)'(rs)' - (rs)'' \int_1^x pqdt - (pq)'(rs)' \\
 &= (pr)'' \int_1^x qsdt - (qs)'' \int_1^x prdt + (pqrs)'' \\
 &= (pq)'' \int_1^x rsdt - (rs)'' \int_1^x pqdt - (pqrs)'' \\
 &= (pr)'' \int_1^x qsdt - (qs)'' \int_1^x prdt - (pq)'' \int_1^x rsdt - (rs)'' \int_1^x pqdt
 \end{aligned}$$

$$\therefore F''(1) = 0$$

12. (a) Since $x > 0$, $\frac{1}{t^2} < \frac{1}{t} < 1 \quad \forall t \in (1, 1+x)$

$$\therefore \int_1^{1+x} \frac{1}{t^2} dt < \int_1^{1+x} \frac{1}{t} dt < \int_1^{1+x} 1 dt$$

$$- \left[-\frac{1}{t} \right]_1^{1+x} < (\ln t) \Big|_1^{1+x} < (t) \Big|_1^{1+x}$$

$$- 1 - \frac{1}{1+x} < \ln(1+x) < x$$

$$- \frac{x}{1+x} < \ln(1+x) < x$$

Replace x by $\frac{1}{x}$, we have

$$- \frac{\frac{1}{x}}{1 + \frac{1}{x}} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}$$

$$- \frac{1}{x+1} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}$$

(b) $\ln f = x \ln\left(1 + \frac{1}{x}\right)$

$$- \frac{1}{f} f' = \ln\left(1 + \frac{1}{x}\right) - \frac{x}{1 + \frac{1}{x}} \left(-\frac{1}{x^2}\right)$$

$$- f' = \left(1 + \frac{1}{x}\right)^x \left\{ \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1} \right\}$$

> 0 (by (a))

Hence f is strictly increasing.

$$\forall x > 0, \quad f(x) = \left(1 + \frac{1}{x}\right)^x$$

$$> (1+0)^x$$

$$= 1$$

$$\text{Since } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

$$= e$$

and f is strictly increasing,

we have $f(x) < e \quad \forall x > 0$.

$$(c) (i) F_0' = f' - \frac{d}{dx} \int_x^{\infty} \frac{1}{t^2 f(t)} dt$$

$$= f' - \frac{d}{dx} \int_x^{\infty} \frac{1}{t^2 f(t)} dt$$

$$= f' - \frac{1}{x^2 f(x)}$$

$$> 0 \quad (\because f' > 0, x^2 > 0 \text{ and } f > 0)$$

F_n is increasing.

$$\begin{aligned} \text{Now } F_n'(n) &= f(n) - f(n-1) - \int_n^{\infty} \frac{1}{t^2 f(t)} dt \\ &= f(n) - f(n-1) > 0 \quad (\because f \text{ is increasing.}) \end{aligned}$$

$$\begin{aligned} \text{and } F_n'(n-1) &= f(n-1) - f(n-1) - \int_{n-1}^{\infty} \frac{1}{t^2 f(t)} dt \\ &= - \int_{n-1}^{\infty} \frac{1}{t^2 f(t)} dt < 0 \end{aligned}$$

Since F_n is increasing and F_n has opposite signs at $n-1, n, \exists$ unique $\alpha_n \in \mathbb{R}$ such that $F_n(\alpha_n) = 0$.

From the above argument, $\alpha_n \in (n-1, n)$

$$\alpha_n > n-1$$

$$\therefore \lim_{n \rightarrow \infty} \alpha_n = \infty$$

$$\begin{aligned} \text{(ii) } \lim_{n \rightarrow \infty} F_n(x) &= \lim_{n \rightarrow \infty} \left\{ f(x) - f(n-1) - \int_x^n \frac{1}{t^2 f(t)} dt \right\} \\ &= f(x) - \lim_{n \rightarrow \infty} f(n-1) - \lim_{n \rightarrow \infty} \int_x^n \frac{1}{t^2 f(t)} dt \\ &= f(x) - e - \lim_{n \rightarrow \infty} \int_x^n \frac{1}{t^2 f(t)} dt \end{aligned}$$

It remains to show $\lim_{n \rightarrow \infty} \int_x^n \frac{1}{t^2 f(t)} dt$ exists :

Now $\int_x^n \frac{1}{t^2 f(t)} dt$ increases with n ($\because t^2, f > 0$)

$$\begin{aligned} \text{and } \int_x^n \frac{1}{t^2 f(t)} dt &< \int_x^{\infty} \frac{1}{t^2} dt = \left[-\frac{1}{t} \right]_x^{\infty} \\ &= \frac{1}{x} - \frac{1}{\infty} \\ &< \frac{1}{x} \end{aligned}$$

13. (a) To show $a_{k+1} \leq 2a_k$ for $k = 0, 1, 2, \dots$

For $k = 0$,

$$\begin{aligned} a_{k+1} &= a_1 \\ &= 1 \\ &\leq 2 \cdot 1 \\ &= 2 \cdot a_0 \\ &= 2a_k \end{aligned}$$

For $k > 0$,

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} \\ &\leq a_k + a_k \quad (\because a_k > 0 \text{ and } (a_k) \text{ is increasing}) \\ &= 2a_k \end{aligned}$$

To show $a_k \leq 2^k$ for $k = 0, 1, 2, \dots$:

For $k = 0$,

$$\begin{aligned} a_k &= a_0 \\ &= 1 \\ &\leq 2^0 \\ &= 2^k \end{aligned}$$

For $k > 0$,

$$\begin{aligned} a_k &= \frac{a_k}{a_{k-1}} \cdot \frac{a_{k-1}}{a_{k-2}} \cdots \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} \\ &\leq 2 \cdot 2 \cdots 2 \quad (\because a_{j+1} \leq 2a_j) \\ &= 2^k \end{aligned}$$

To show $S_2(x) < 3$ for $x = 0, 1, 2, \dots$:

$$\begin{aligned} S_2(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &< \sum_{k=0}^{\infty} 2^k \left(\frac{2}{3}\right)^k \quad (\because 0 < a_k < 2^k \text{ and } x < \frac{2}{3}) \\ &= \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \\ &< \frac{1}{1 - \frac{2}{3}} \\ &= 3 \end{aligned}$$

(b) To show $\lim_{n \rightarrow \infty} S_n(x)$ exists:

Case 1 : $x > 0$

$S_n(x)$ is increasing and bounded above

$\therefore \lim_{n \rightarrow \infty} S_n(x)$ exists.

Case 2 : $x < 0$

write $y = -x$

then $S_n(x) = \sum_{k=0}^n a_k (-y)^k$

$= U_n(x) - V_n(x)$ where $U_n(x) = \sum_{k=1,3,\dots} a_k y^k$

and $V_n(x) = \sum_{k=0,2,4,\dots} a_k y^k$

Since both $U_n(x)$ and $V_n(x)$ are monotonic increasing and bounded above, both $\lim_{n \rightarrow \infty} U_n(x)$ and $\lim_{n \rightarrow \infty} V_n(x)$ exist.

$\therefore \lim_{n \rightarrow \infty} S_n(x)$ exists.

To find $\lim_{n \rightarrow \infty} S_n(x)$:

$$\begin{aligned} S_n(x) &= \sum_{k=0}^n a_k x^k \\ &= a_0 - a_1 x + \sum_{k=2}^n a_k x^k \\ &= 1 - x + \sum_{k=2}^n (a_{k-1} + a_{k-2}) x^k \\ &= 1 - x + \sum_{k=2}^n a_{k-1} x^k + \sum_{k=2}^n a_{k-2} x^k \\ &= 1 - x + x \sum_{k=2}^n a_{k-1} x^{k-1} - x^2 \sum_{k=2}^n a_{k-2} x^{k-2} \\ &= 1 - x + x \left(\sum_{k=0}^{n-1} a_k x^k - 1 \right) - x^2 \sum_{k=0}^{n-2} a_k x^k \end{aligned}$$

Take $n \rightarrow \infty$, we have

$s = 1 - x - x(s - 1) - x^2 s$, where $s = \lim_{n \rightarrow \infty} S_n(x)$

$s = 1 - x - xs - x - x^2 s$

$s(1 - x - x^2) = 1$

$s = \frac{1}{1 - x^2 - x^2}$

$\sum a_k (-y)^k$
 $V_n(x) - V_{n-1}(x) = a_n x^n$
 $\rightarrow S_n(x) \rightarrow s(x)$

$$(c) (i) \sum_{k=0}^{\infty} a_k \left(\frac{1}{5}\right)^k = \frac{1}{1 - \frac{1}{5} - \frac{1}{25}}$$

$$= \frac{25}{19}$$

1A

$$(ii) \sum_{k=0}^{\infty} (-1)^k a_k \left(\frac{1}{5}\right)^k = \frac{1}{1 + \frac{1}{5} - \frac{1}{25}}$$

$$= \frac{25}{29}$$

1A

$$(iii) \frac{25}{19} - \frac{25}{29} = \sum_{k=0}^{\infty} a_k \left(\frac{1}{5}\right)^k - \sum_{k=0}^{\infty} a_k \left(-\frac{1}{5}\right)^k$$

$$= \sum_{k=0}^{\infty} a_{2k} \left[\left(\frac{1}{5}\right)^{2k} - \left(\frac{1}{5}\right)^{2k} \right]$$

$$= 2 \sum_{k=0}^{\infty} a_{2k} \left(\frac{1}{25}\right)^k$$

1A

$$\therefore \sum_{k=0}^{\infty} a_{2k-1} \left(\frac{1}{25}\right)^k = \frac{1}{2} \left[\frac{25}{19} + \frac{25}{29} \right]$$

1A

$$a_{k-1} = a_k \quad k > 0$$

$$= 1 \quad \sum_{k=0}^{\infty} a_k$$

$$\leq \sum_{k=0}^{\infty} a_k$$

$$\geq \sum_{k=0}^{\infty} a_k$$

$$a_k < 2^k$$

$$k=0 \quad k > 0$$

$$a_k = a_0 \quad a_k = \frac{a_0 - a_{k-1}}{2}$$

$$= 1$$

$$\leq 2^0$$

$$\frac{1}{2} < 2^k$$