Section A (40 marks)
Answer ALL questions in this section. Write your answers in the light yellow AL(C1) answer book.

1. Factorize the determinant

$$\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

(4 marks)

2. Let $f(x) = \dfrac{1}{(x-1)(2-x)}$.

Express $f(x)$ into partial fractions. Hence, or otherwise, determine a_k and b_k $(k = 0, 1, 2, \cdots)$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{when } |x| < 1$$

and $$f(x) = \sum_{k=0}^{\infty} \frac{b_k}{x^k} \quad \text{when } |x| > 2 .$$

(7 marks)

3. Consider the following system of linear equations:

$$\begin{cases} x + 2y + z = 1 \\ x + y + 2z = 2 \\ \quad -y + q^2 z = q \end{cases}$$

Determine all values of q for each of the following cases:

(a) The system has no solution.

(b) The system has infinitely many solutions.

(4 marks)

4. Let $a_1, a_2, \cdots, a_n$ be n distinct non-zero real numbers, where $n \geq 2$.

(a) Define
$$P(x) = a_1 \frac{(x-a_2)\cdots(x-a_n)}{(a_1-a_2)\cdots(a_1-a_n)} + \cdots + a_i \frac{(x-a_1)\cdots(x-a_{i-1})(x-a_{i+1})\cdots(x-a_n)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} + \cdots + a_n \frac{(x-a_1)\cdots(x-a_{n-1})}{(a_n-a_1)\cdots(a_n-a_{n-1})} .$$

(i) Evaluate $P(a_i)$ for $i = 1, 2, \ldots, n$.

(ii) Show that the equation $P(x) - x = 0$ has n distinct roots.

(iii) Deduce that $P(x) - x = 0$ for all $x \in \mathbf{R}$.

(b) Prove that

$$\frac{1}{(a_1-a_2)\cdots(a_1-a_n)} + \cdots + \frac{1}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} + \cdots + \frac{1}{(a_n-a_1)\cdots(a_n-a_{n-1})} = 0 .$$

(5 marks)

5. Let u, v be non-zero complex numbers.

(a) Show that

$$u\bar{v} + \bar{u}v = 0 \text{ if and only if } \frac{u}{v} = ik \text{ for some } k \in \mathbf{R} .$$

(b) If $u\bar{v} + \bar{u}v = 0$, what is the relationship between $\arg u$ and $\arg v$?

(6 marks)

6. (a) Let a, b and c be real numbers.

(i) Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(ii) Hence deduce that if $a + b + c > 0$ then

$$a^3 + b^3 + c^3 \geq 3abc .$$

(b) Let $|x| < \ln 2$.

(i) Show that

$$(e^{-x})^{\frac{1}{3}} + (2 - e^{x})^{\frac{1}{3}} + (e^{x} - e^{-x} + 1)^{\frac{1}{3}} > 0 .$$

(ii) Using (a) or otherwise, show that

$$e^{-x}(2 - e^{x})(e^{x} - e^{-x} + 1) \leq 1 .$$

(7 marks)

7. Given that $\{a_n\}$ is an increasing sequence of positive numbers and $\lim_{n \to \infty} a_n = L$. Suppose sequences $\{b_n\}$, $\{c_n\}$ are defined such that

$$b_1 = c_1 = \frac{1}{2} a_1$$

and $b_n = \frac{1}{2}(a_{n-1} + c_{n-1})$, $c_n = \sqrt{a_{n-1} b_{n-1}}$ for $n \geq 2$.

(a) Show by induction that

(i) $\{b_n\}$ and $\{c_n\}$ are strictly increasing,

(ii) $b_n < a_n$ and $c_n < a_n$ for $n \geq 1$.

(b) Show that $\{b_n\}$ and $\{c_n\}$ are convergent.

Hence evaluate $\lim_{n \to \infty} b_n$ and $\lim_{n \to \infty} c_n$.

(7 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section. Write your answers in the separate orange AL(C2) answer book.
Each question carries 15 marks.

8. (a) Let a_k, b_k $(k = 1, 2, \cdots , n)$ be non-zero real numbers.

(i) Prove the Schwarz's inequality

$$\left\{ \sum_{k=1}^{n} a_k^2 \right\} \left\{ \sum_{k=1}^{n} b_k^2 \right\} \geq \left\{ \sum_{k=1}^{n} a_k b_k \right\}^2 .$$

(ii) If $p \leq \dfrac{b_k}{a_k} \leq q$ for $k = 1, 2, \cdots , n$, prove that

$$pqa_k^2 - (p + q)a_k b_k + b_k^2 \leq 0 \quad \text{for } k = 1, 2, \cdots , n .$$

Deduce that $(p + q) \sum_{k=1}^{n} a_k b_k \geq \sum_{k=1}^{n} b_k^2 + pq \sum_{k=1}^{n} a_k^2$.

(iii) If $0 < m \leq a_k \leq M$

and $0 < m \leq b_k \leq M$ for $k = 1, 2, \cdots , n$,

prove, by using (ii) or otherwise, that

$$\left\{ \sum_{k=1}^{n} a_k^2 \right\} \left\{ \sum_{k=1}^{n} b_k^2 \right\} \leq \frac{1}{4} \left(\frac{M}{m} + \frac{m}{M}\right)^2 \left\{ \sum_{k=1}^{n} a_k b_k \right\}^2 .$$

(10 marks)

(b) Using (a) or otherwise, show that

$$\left(n + \frac{1}{9}\right)^2 < \left\{ \sum_{k=1}^{n} \left(1 + \frac{1}{3^k}\right)^2 \right\} \left\{ \sum_{k=1}^{n} \left(1 - \frac{1}{3^{k+1}}\right)^2 \right\} < \frac{169}{144} \left(n + \frac{1}{3}\right)^2 .$$

(5 marks)

9. A mapping $f: \mathbf{C} \to \mathbf{C}$ is said to be real linear if

$$f(\alpha z_1 + \beta z_2) = \alpha f(z_1) + \beta f(z_2) \text{ for all } \alpha, \beta \in \mathbf{R}, z_1, z_2 \in \mathbf{C}.$$

Let $\phi : \mathbf{C} \to \mathbf{C}$ be a real linear mapping.

(a) Prove that

(i) $\phi(\alpha z) = \alpha \phi(z)$ for $\alpha \in \mathbf{R}$ and $z \in \mathbf{C}$.

(ii) $\phi(0) = 0$.

(3 marks)

(b) Let $\psi : \mathbf{C} \to \mathbf{C}$ be a real linear mapping. Prove that

if $\phi(1) = \psi(1)$ and $\phi(i) = \psi(i)$, then $\phi = \psi$.

(2 marks)

(c) If, furthermore, ϕ is non-constant such that

$$\phi(z_1 z_2) = \phi(z_1)\phi(z_2) \text{ for all } z_1, z_2 \in \mathbf{C},$$

show that:

(i) $\phi(1) = 1$ and hence $\phi(x) = x$ for all $x \in \mathbf{R}$.

(ii) Either $\phi(z) = z$ for all $z \in \mathbf{C}$ or $\phi(z) = \bar{z}$ for all $z \in \mathbf{C}$.

(10 marks)

10. Let M be the set of all 3×1 real matrices.

For any two 3×1 matrices $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

we define $u \otimes x = \begin{pmatrix} u_2x_3 - x_2u_3 \\ u_3x_1 - x_3u_1 \\ u_1x_2 - x_1u_2 \end{pmatrix}$.

(a) Show that for any $u, x, y \in M$, and any $\alpha, \beta \in \mathbf{R}$,

(i) $u \otimes (\alpha x + \beta y) = \alpha(u \otimes x) + \beta(u \otimes y)$,

(ii) $u \otimes x = -(x \otimes u)$.

(3 marks)

(b) Show that if $u \otimes x = 0$ for all $x \in M$, then $u = 0$.

Deduce that if $u \otimes x = v \otimes x$ for all $x \in M$, then $u = v$.

(6 marks)

(c) Let A be a 3×3 real matrix and $u \in M$ such that

$$Ae_1 = u \otimes e_1, \; Ae_2 = u \otimes e_2 \text{ and } Ae_3 = u \otimes e_3$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Show that $Ax = u \otimes x$ for all $x \in M$. (4 marks)

(d) Let $u = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$. Find the matrix A such that $Ax = u \otimes x$ for all $x \in M$.

(2 marks)

11. (a) For $n = 1, 2, \cdots$, prove that there exist unique positive integers p_n and q_n such that

$$(\sqrt{3}+\sqrt{2})^{2n} = p_n + q_n\sqrt{6} \quad \cdots\cdots\cdots\cdots\cdots\cdots(*)$$

and $(\sqrt{3}-\sqrt{2})^{2n} = p_n - q_n\sqrt{6}$.

Hence deduce that $2p_n - 1 < (\sqrt{3}+\sqrt{2})^{2n} < 2p_n$.

[Hint: Use the fact that $0 < \sqrt{3} - \sqrt{2} < 1$.]

(6 marks)

(b) For $n = 1, 2, \cdots$, show that the following integers are positive multiples of 10:

(i) $2^{5n} - 2^n$,

(ii) $3^{4n} - 1$,

(iii) $2p_{2n} - (2^{3n+1})(3^n)$ where p_{2n} is given by (*).

(5 marks)

(c) By using (a) and (b) or otherwise, find the unit digit when $(\sqrt{3}+\sqrt{2})^{100}$ is expressed in the decimal form.

(4 marks)

12. A subset X of $\mathbf{R}^3$ is said to be convex if

$$u, v \in X \Rightarrow \alpha u + \beta v \in X \text{ for all } \alpha, \beta \geq 0 \text{ and } \alpha + \beta = 1 .$$

(a) Prove that the intersection of two convex sets is convex. Give an example to show that the union of two convex sets may NOT be convex.

(4 marks)

(b) For any subsets A, B of $\mathbf{R}^3$, define

$$A + B = \{w \in \mathbf{R}^3 : w = u + v \text{ for some } u \in A, v \in B\} .$$

Prove that if A, B are convex, then $A + B$ is also convex.

(2 marks)

(c) For any subset A of $\mathbf{R}^3$ and any $\gamma \in \mathbf{R}$, define

$$\gamma A = \{w \in \mathbf{R}^3 : w = \gamma u \text{ for some } u \in A\} .$$

Prove that if A is convex, then γA is also convex.

(2 marks)

(d) For any $a_1, a_2, \cdots, a_n \in \mathbf{R}^3$, define

$$\text{cov}(a_1, a_2, \cdots, a_n) = \{w \in \mathbf{R}^3 : w = \alpha_1 a_1 + \cdots + \alpha_n a_n$$
$$\text{for some } \alpha_1, \cdots, \alpha_n \geq 0 \text{ and } \alpha_1 + \cdots + \alpha_n = 1\} .$$

Prove that

(i) $\text{cov}(a_1, \cdots, a_n)$ is convex,

(ii) if S is a convex subset of $\mathbf{R}^3$ and S contains $a_1, \cdots, a_n$, then $\text{cov}(a_1, \cdots, a_n) \subset S$.

(7 marks)

13. Let $u \in \mathbf{R}^3$ be a unit vector (i.e., $u \cdot u = 1$).

A relation $\sim$ on $\mathbf{R}^3$ is defined by

$$v \sim w \text{ if and only if } v - w = ku \text{ for some } k \in \mathbf{R} .$$

(a) Show that $\sim$ is an equivalence relation.

(3 marks)

(b) Denote by $[v]$ the equivalence class containing v .

Let $f : \mathbf{R}^3/\sim \rightarrow \mathbf{R}^3$ be defined by $f([v]) = v - (v \cdot u)u$.

(i) Show that f is well-defined.

(ii) Show that f is injective.

(iii) For any non-zero w in $\mathbf{R}^3$, show that w is perpendicular to u if and only if $w \in f(\mathbf{R}^3/\sim)$.

Hence deduce that f is NOT surjective.

(10 marks)

(c) If $u = (0, 0, 1)$ and $v = (0, 1, 2)$, sketch the set $\{w \in \mathbf{R}^3 : w \sim v\}$

(2 marks)

END OF PAPER

91-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1991

PURE MATHEMATICS PAPER II

2.00 pm-5.00 pm (3 hours)
This paper must be answered in English

This paper consists of two sections BOTH of which are to be answered.

INSTRUCTIONS FOR SECTION A

1. Answer ALL questions. Write your answers in the light yellow AL(C1) answer book.
2. Write your Candidate Number, Centre Number and Seat Number in the spaces provided on the cover of the answer book.

INSTRUCTIONS FOR SECTION B

1. Answer any FOUR questions. Write your answers in the separate orange AL(C2) answer book.
2. Write your Candidate Number, Centre Number and Seat Number in the spaces provided on the cover of the answer book.

Section A (40 marks)

Answer ALL questions in this section. Write your answers in the light yellow AL(C1) answer book.

1. Chords with slope equal to 1 are drawn in the ellipse $2x^2 + 2xy + y^2 = 1$. Prove that the mid-points of these chords lie on a straight line, and find the equation of this line.

(4 marks)

2. Show that

$$(1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta)\sin\frac{\theta}{2} = \sin\frac{(n+1)\theta}{2}\cos\frac{n\theta}{2}.$$

Hence solve

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = 0, \quad 0 \le \theta < 2\pi.$$

(6 marks)

3. Consider the curve

$$\begin{cases} x = \sin^3 t \\ y = \cos^3 t \end{cases}, \quad 0 \le t \le \frac{\pi}{2}.$$

(a) Find the length of the curve.

(b) Find the area bounded by the curve, the x-axis and the y-axis.

(7 marks)

4. Evaluate the following definite integral:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + 2\sin\theta}.$$

(6 marks)

5. Find $\lim_{x\to\infty}\left(\frac{a^x - 1}{a - 1}\right)^{\frac{1}{x}}$ for each of the following cases:

(a) $0 < a < 1$,

(b) $a > 1$.

(6 marks)

6. (a) Evaluate $\frac{d}{du}\int_0^u (\sqrt{2})^{t^2} dt$.

(b) Define $F(x) = \int_{\tan x}^{\sec x} (\sqrt{2})^{t^2} dt$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solve $F'(x) = 0$.

(6 marks)

7. Let $f : \mathbf{R} \to \mathbf{R}$.

(a) If $c \in \mathbf{R}$ and $|f(x) - f(c)| \le (x - c)^2$ for all $x \in \mathbf{R}$, prove that $f'(c) = 0$.

(b) If $|f(x) - f(y)| \le (x - y)^2$ for all $x, y \in \mathbf{R}$, prove that f is a constant function.

(5 marks)

SECTION B (60 marks)
Answer any FOUR questions from this section. Write your answers in the separate orange AL(C2) answer book.
Each question carries 15 marks.

8. (a) Let $f(x) = x - \ln(1 + x)$.

Prove that $f(x) \geq 0$ for all $x > -1$.

(3 marks)

(b) Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n + 1)$

and $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$.

Show that $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing.

Hence prove that $\{a_n\}$ and $\{b_n\}$ are convergent and have the same limit.

(5 marks)

(c) (i) Using (b) or otherwise, show that

$$\lim_{n\to\infty} \left\{ \frac{1}{kn+1} + \frac{1}{kn+2} + \ldots + \frac{1}{kn+n} \right\} = \ln\left(\frac{k+1}{k}\right)$$

where k is a positive integer.

(ii) Evaluate $\lim_{n\to\infty} \left\{ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} \right\}$.

(7 marks)

9. (a) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(3 marks)

(b) Show that

$$\ln 1 + \ln 2 + \cdots + \ln(n - 1) \leq \int_1^n \ln x \, dx \leq \ln 2 + \ln 3 + \cdots + \ln n$$

where $n \geq 2$.

Deduce that

$$(n - 1)! \leq n^n e^{-n+1} \leq n! .$$

(7 marks)

(c) Using (a) and (b), or otherwise, evaluate

$$\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} .$$

(5 marks)

10. Let $f(x) = \sqrt[3]{x^3 - x^2 - x + 1}$.

(a) Find the roots of $f(x) = 0$.

(1 mark)

(b) Find $f'(x)$ for $x \neq 1, -1$.

Show that both $f'(1)$ and $f'(-1)$ do not exist.

(3 marks)

(c) Determine the values of x for each of the following cases:

(i) $f'(x) = 0$,

(ii) $f'(x) > 0$,

(iii) $f'(x) < 0$.

(3 marks)

(d) Find the relative minimum, the relative maximum and the point of inflexion of $f(x)$.

(4 marks)

(e) Find the asymptote of the graph of $f(x)$.

(2 marks)

(f) Sketch the graph of $f(x)$.

(2 marks)

11. Consider the lines

$$L_1 : \frac{x-2}{1} = \frac{y-3}{2} = \frac{z-3}{3}$$

and $$L_2 : \frac{x-4}{2} = \frac{y-6}{3} = \frac{z-11}{5} .$$

(a) Prove that L_1 and L_2 are non-coplanar.

(3 marks)

(b) (i) Find the equation of the plane π containing L_1 and parallel to L_2 .

(ii) Find the equation of the plane π' containing L_2 and perpendicular to π .

(7 marks)

(c) (i) Find the point S at which L_1 intersects π' .

(ii) Find the equations of the line passing through S and perpendicular to both L_1 and L_2 .

(5 marks)

12. For any non-negative integer n, let

$$I_n = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x \, dx .$$

(a) (i) Evaluate I_0 and express I_n in terms of I_{n-1} for $n \geq 1$.

(ii) Show by induction that

$$I_n = \frac{(n!)^2 2^{2n}}{(2n+1)!} \quad \text{for } n = 0, 1, 2, \ldots .$$

(5 marks)

(b) For any non-negative integer m, let

$$S_m = \sum_{n=0}^{m} \frac{(n!)^2 2^{n+1}}{(2n+1)!} .$$

(i) Show that

$$S_m = \int_0^{\frac{\pi}{2}} 2\cos x \frac{1 - \left(\frac{1}{2}\cos^2 x\right)^{m+1}}{1 - \frac{1}{2}\cos^2 x} dx .$$

(ii) Deduce that

$$\int_0^{\frac{\pi}{2}} \frac{2\cos x}{1 - \frac{1}{2}\cos^2 x} dx - \frac{\pi}{2^m} \leq S_m \leq \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1 - \frac{1}{2}\cos^2 x} dx .$$

(iii) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2 2^{n+1}}{(2n+1)!} = \pi$.

(10 marks)

13. For $n = 0, 1, 2, \ldots$, let $R_n(x) = (x^2 - 1)^n$ and $P_n(x) = R_n^{(n)}(x)$.

(a) (i) Show that $P_n(x)$ is a polynomial in x of degree n, $n = 0, 1, 2, \ldots$.

(ii) Prove by induction that every polynomial of degree n can be expressed as

$$\sum_{j=0}^{n} \alpha_j P_j(x) \quad \text{where } \alpha_j \in \mathbf{R} .$$

(4 marks)

(b) Prove by induction that

$$(1 - x^2) R_n^{(k+2)}(x) + 2x(n - k - 1) R_n^{(k+1)}(x) + (k+1)(2n - k) R_n^{(k)}(x) = 0$$

for $k = 0, 1, 2, \ldots$.

Hence deduce that

$$[(1 - x^2)P_n'(x)]' = -n(n+1)P_n(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

(5 marks)

(c) (i) Using (*) or otherwise, show that if $m \geq 0$, $n \geq 0$, then

$$n(n+1)\int_{-1}^{1} P_m(x)P_n(x)\,dx = \int_{-1}^{1} (1 - x^2)P_n'(x)P_m'(x)\,dx .$$

(ii) Deduce that if m, n are distinct non-negative integers, then

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 .$$

(6 marks)

END OF PAPER