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SECTION A

1. (\*)  $\begin{cases} 3x - y + z = 1 & (1) \\ 2x - 4y - 5z = 1 & (2) \\ 4x + 2y + 7z = c & (3) \end{cases}$

$3 \times (2), 3 \times (3) :$   $\begin{cases} 3x - y + z = 1 & (1) \\ 6x - 12y - 15z = 3 & (4) \\ 12x + 6y + 21z = 3c & (5) \end{cases}$

$(4) - 2 \times (1), (5) - 4 \times (1) :$   $\begin{cases} 3x - y + z = 1 & (1) \\ -10y - 17z = 1 & (6) \\ 10y + 17z = 3c - 4 & (7) \end{cases}$

$(7) + (6) :$   $\begin{cases} 3x - y + z = 1 & (1) \\ -10y - 17z = 1 & (6) \\ 0 = 3c - 3 & (8) \end{cases}$

If the system (\*) is consistent,  $3c - 3 = 0$   
 $c = 1$

Put  $z = t$ , from (6) and (1)

$$y = \frac{-17t - 1}{10}$$

and  $x = \frac{-9t + 3}{10}$

i.e.  $\begin{cases} x = \frac{-9t + 3}{10} \\ y = \frac{-17t - 1}{10} \\ z = t \end{cases}$  where  $t \in \mathbb{R}$

2.  $\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

$1 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$   
 $= (A+B+C)x^2 + (3A+2B+C)x + 2A$

Hence  $\begin{cases} A+B+C=0 \\ 3A+2B+C=0 \\ 2A=1 \end{cases}$

On solving  $A = \frac{1}{2}$   
 $B = -1$   
 $C = \frac{1}{2}$

$\therefore \frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$

(b)  $\sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \sum_{k=1}^n \left[ \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} \right]$   
 $= \sum_{k=1}^n \frac{1}{2k} - \sum_{k=1}^n \frac{1}{k+1} + \sum_{k=1}^n \frac{1}{2(k+2)}$   
 $= \frac{n-1}{2} \left( \frac{1}{2(k+1)} \right) - \sum_{k=1}^n \frac{1}{k+1} + \frac{n+1}{2} \left( \frac{1}{2(k+1)} \right)$   
 $= \left( \frac{1}{2} - \frac{1}{2} + \frac{n-1}{2} \left( \frac{1}{2(k+1)} \right) \right) - \left( \frac{1}{2} - \frac{n-1}{2} \left( \frac{1}{k+1} \right) + \frac{1}{n+1} \right)$   
 $+ \left( \frac{n-1}{2} \left( \frac{1}{2(k+1)} \right) + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \right)$   
 $= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{n+1} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$   
 $+ \frac{n-1}{2} \left( \frac{1}{2(k+1)} - \frac{1}{k+1} + \frac{1}{2(k+1)} \right)$   
 $= \frac{1}{2} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$

$\therefore \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)(k+2)} = \frac{1}{4}$

3. (a)  $\begin{cases} a + b + c = -A & (1) \\ a^2 + 8b + c = 8 & (2) \\ a^2 b = -C & (3) \end{cases}$

$a^2 + 8b + c = (a+b+c)^2 - 2(ab+bc+ca)$   
 $= A^2 - 2B$  (by (1) and (2))

$a^2 b^2 + 8^2 c^2 + c^2 a^2 = (a^2 + 8b + c)^2 - 2ac(a+8b+c)$   
 $= B^2 - 2AC$  (by (1), (2) and (3))

(b) The required equation is

$$x^3 - (a^2 + b^2 + c^2)x^2 + (a^2b^2 + b^2c^2 + c^2a^2)x - a^2b^2c^2 = 0$$

$$x^3 - [0^2 - 2(-3)]x^2 + [(-3)^2 - 2(0)(1)]x - (1)^2 = 0$$

$$x^3 - 6x^2 + 9x - 1 = 0$$

$$\begin{aligned} 4. (a) C_k^n &= \frac{n!}{k!(n-k)!} \\ &= \frac{(n+1)(n!)}{(k+1)k!(n-k)!} \cdot \frac{(k+1)}{(n+1)} \\ &= \frac{(n+1)!}{(k+1)k![(n+1)-(k+1)]!} \cdot \frac{(k+1)}{(n+1)} \\ &= \frac{k+1}{n+1} C_{k+1}^{n+1} \end{aligned}$$

$$(b) \text{ Consider } (1+x)^{n+1} = \sum_{k=0}^{n+1} C_k^{n+1} x^k$$

$$\text{Set } x = -1, 0 = \sum_{k=0}^{n+1} (-1)^k C_k^{n+1}$$

$$\begin{aligned} (c) \sum_{k=0}^n \frac{(-1)^k}{k+1} C_k^n &= \sum_{k=0}^n \frac{(-1)^k}{n+1} C_{k+1}^{n+1} \quad (\text{by (a)}) \\ &= \frac{n+1}{n+1} \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} C_k^{n+1} = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} C_k^{n+1} - \frac{(-1)^{-1}}{n+1} \\ &= \frac{-1}{n+1} \left( \sum_{k=0}^{n+1} (-1)^k C_k^{n+1} - 1 \right) \\ &= \frac{-1}{n+1} (0 - 1) \quad (\text{by (b)}) \\ &= \frac{1}{n+1} \end{aligned}$$

5. (a) Let  $z = \cos \theta + i \sin \theta$

$$\begin{cases} \frac{1}{z} = \cos \theta - i \sin \theta \\ z^3 = \cos 3\theta + i \sin 3\theta \\ \frac{1}{z^3} = \cos 3\theta - i \sin 3\theta \end{cases}$$

$$\begin{aligned} \cos 3\theta &= \frac{1}{2} \left( z^3 + \frac{1}{z^3} \right) \\ &= \frac{1}{2} [(\cos \theta + i \sin \theta)^3 + (\cos \theta - i \sin \theta)^3] \\ &= \frac{1}{2} [2\cos^3 \theta - 2\cos \theta \sin^2 \theta + 10\cos \theta \sin^2 \theta] \end{aligned}$$

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$$= \cos^3 \theta - 10\cos \theta \sin^2 \theta + 5\cos \theta (1 - \cos^2 \theta)^2$$

$$= 16\cos^3 \theta - 20\cos \theta + 5\cos \theta$$

(b) Consider  $\cos 5\theta = 0 \dots \dots (*)$

$$5\theta = (2n+1)\frac{\pi}{2}, \text{ where } n = 0, \pm 1, \pm 2, \dots$$

$$\theta = (2n+1)\frac{\pi}{10}$$

Hence, the solutions of the equation

$$16\cos^3 \theta - 20\cos \theta + 5\cos \theta = 0$$

for  $\theta$  between 0 and  $2\pi$  are  $\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$ .

Since  $16\cos^3 \theta - 20\cos \theta + 5\cos \theta = \cos \theta (16\cos^2 \theta - 20\cos \theta + 5)$ ,

the solutions of the equation  $16\cos^3 \theta - 20\cos \theta + 5 = 0$  for  $\theta$  between 0 and  $2\pi$  are  $\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$ .

$\therefore \cos \frac{\pi}{10}, \cos \frac{3\pi}{10}, \cos \frac{7\pi}{10}, \cos \frac{9\pi}{10}$  are roots of the equation  $16t^3 - 20t^2 + 5 = 0$

$$\text{Hence } \cos \frac{\pi}{10} \cos \frac{3\pi}{10} \cos \frac{7\pi}{10} \cos \frac{9\pi}{10} = \frac{5}{16}$$

$$\cos \frac{\pi}{10} \cos \frac{3\pi}{10} (-\cos \frac{3\pi}{10}) (-\cos \frac{\pi}{10}) = \frac{5}{16}$$

$$\cos^2 \frac{\pi}{10} \cos^2 \frac{3\pi}{10} = \frac{5}{16}$$

$$6. |x-1| - |x+2| > 2$$

$$|x-1| > 2 + |x+2|$$

$$(x-1)^2 > [2 + |x+2|]^2$$

$$(x-1)^2 > 4 + 4|x+2| + (x+2)^2$$

$$-6x-7 > 4|x+2|$$

$$\text{case (i) } x \leq -2$$

$$-6x-7 > -4(x+2)$$

$$1 > 2x$$

$$\frac{1}{2} > x$$

In this case, the solution is  $x < -2$ .

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case (ii)  $x > -2$

$$-6x - 7 > 4(x+2)$$

$$-15 > 10x$$

$$-\frac{3}{2} > x$$

$\therefore$  In this case, the solution is  $-2 < x < -\frac{3}{2}$ .

Hence, the solution of the given inequality is

$$\{x \in \mathbb{R} : x < -\frac{3}{2}\}$$

7. Induce on  $n$

(i) when  $n = 0$ ,  $\frac{1}{\sqrt{5}}(a^0 - b^0) = 0$

$$\begin{aligned} \text{when } n = 1, \frac{1}{\sqrt{5}}(a - b) &= \frac{1}{\sqrt{5}} \sqrt{(a+b)^2 - 4ab} \\ &= \frac{1}{\sqrt{5}} \sqrt{(-1)^2 - 4(-1)} \\ &= 1 \end{aligned}$$

$\therefore$  It is true for  $n=0$  and  $n=1$ .

(ii) Assume  $a_k = \frac{1}{\sqrt{5}}(a^k - b^k)$  and  $a_{k+1} = \frac{1}{\sqrt{5}}(a^{k+1} - b^{k+1})$

$$\begin{aligned} \text{when } n=k+2, \frac{1}{\sqrt{5}}(a^{k+2} - b^{k+2}) &= \frac{1}{\sqrt{5}}[(a^{k+1} - b^{k+1})(a+b) - ab(a^k - b^k)] \\ &= \frac{1}{\sqrt{5}}[a^{k+1}(a+b) - b^{k+1}(a+b) - ab(a^k - b^k)] \\ &= a_{k+1}(-1) - a_k(-1) \\ &= -a_{k+1} + a_k \\ &= a_{k+2} \end{aligned}$$

$\therefore$  It is true for  $n=k+2$

Hence, from the Principle of Mathematical Induction, it is true for all non-negative integers  $n$ .

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As  $\alpha, \beta$  are roots of the equation  $x^2 + x - 1 = 0$  and  $\alpha > 0, \beta < 0$ ,

$$\alpha = \frac{-1 + \sqrt{5}}{2} \text{ and } \beta = \frac{-1 - \sqrt{5}}{2}$$

$$\begin{aligned} \text{Consider } \frac{\alpha}{\beta} &= \frac{-1 + \sqrt{5}}{-1 - \sqrt{5}} \\ &= \frac{(-1 + \sqrt{5})^2}{-4} \\ &= \frac{6 - 2\sqrt{5}}{-4} \end{aligned}$$

$$\therefore \left| \frac{\alpha}{\beta} \right| < 1$$

$$\begin{aligned} \text{Hence } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{5}}(a^{n+1} - b^{n+1})}{\frac{1}{\sqrt{5}}(a^n - b^n)} \\ &= \lim_{n \rightarrow \infty} \frac{(\frac{\alpha}{\beta})^{n+1} - 1}{(\frac{\alpha}{\beta})^n - 1} \\ &= \frac{-1}{-1} \quad \left( \because \lim_{n \rightarrow \infty} \left(\frac{\alpha}{\beta}\right)^n = 0 \right) \\ &= 1 \end{aligned}$$

## SECTION B

$$\begin{aligned} \text{S. (1)} \quad (i) \quad (X+Y)^2 &= X^2 + XY + YX + Y^2 \\ &= X^2 + 2XY + Y^2 \quad (\because XY = YX) \end{aligned}$$

(ii) Induce on  $n$

$$\begin{aligned} (1) \text{ when } n=3, (X+Y)^3 &= (X+Y)^2(X+Y) \\ &= (X^2 + 2XY + Y^2)(X+Y) \\ &= X^3 + X^2Y + 2XYX + 2XY^2 \\ &\quad + Y^2X + Y^3 \\ &= X^3 + X^2Y + 2XXY + 2XY^2 \\ &\quad + YXY + Y^3 \quad (\because XY = YX) \\ &= X^3 + 3X^2Y + 2XY^2 + XY^2 + Y^3 \\ &\quad (\because XY = YX) \end{aligned}$$

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$$= X^3 + 3X^2Y + 3XY^2 + Y^3$$

(2) Assume  $(X+Y)^k = \sum_{r=0}^k C_r^k X^{k-r} Y^r$  where  $k$  is an integer greater than 2

$$\begin{aligned} (X+Y)^{k+1} &= (X+Y)(X+Y)^k \\ &= (X+Y) \sum_{r=0}^k C_r^k X^{k-r} Y^r \\ &= \sum_{r=0}^k C_r^k X^{k+1-r} Y^r + \sum_{r=0}^k C_r^k X^{k-r} Y^{r+1} \\ &\quad (\because XY=YX, \therefore YX^{k-r}Y^r = X^{k-r}Y^{r+1}) \end{aligned}$$

$$\begin{aligned} &= \sum_{r=0}^k C_r^k X^{k+1-r} Y^r + \sum_{r=0}^{k+1} C_{r-1}^k X^{k+1-r} Y^r \\ &= C_0^k X^{k+1} + \sum_{r=1}^k (C_r^k + C_{r-1}^k) X^{k+1-r} Y^r + C_k^k Y^{k+1} \\ &= C_0^{k+1} X^{k+1} + \sum_{r=1}^k C_r^{k+1} X^{k+1-r} Y^r + C_{k+1}^{k+1} Y^{k+1} \end{aligned}$$

$$(\because C_0^k = C_0^{k+1} = 1,$$

$$C_k^k = C_{k+1}^{k+1} = 1,$$

$$= \sum_{r=0}^{k+1} C_r^{k+1} X^{k+1-r} Y^r \quad C_r^{k+1} = C_r^k + C_{r-1}^k)$$

$\therefore$  From (1) and (2), by the Principle of Mathematical Induction, it's true for  $n = 3, 4, 5, \dots$

$$(b) \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y^2 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \sum_{r=0}^{\infty} C_r^{\infty} X^{\infty-r} Y^r$$

$$= \sum_{r=0}^{\infty} C_r^{\infty} Y^r \quad (\because Y^r = 0 \text{ for } r=3, 4, 5, \dots)$$

$$= \begin{pmatrix} 1 & 200 & 30100 \\ 0 & 1 & 300 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(c) (i) (X+Y)^2 = X^2 + 2XY + Y^2$$

$$X^2 + XY + YX + Y^2 = X^2 + 2XY + Y^2$$

$$YX = XY$$

$$(ii) \text{ Let } X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$X+Y = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

$$(X+Y)^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{3}{2} & 0 \end{pmatrix}$$

$$(X+Y)^3 = \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{3}{2} & 0 \end{pmatrix}$$

$$X^2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$Y^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

$$Y^3 = \begin{pmatrix} -\frac{1}{8} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \therefore X^2 + 3X^2Y + 3XY^2 + Y^3 &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + 3 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{8} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \frac{1}{8} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$= (X + Y)^2$$

$$XY = \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

$$YX = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore XY \neq YX$$

Hence,  $(X + Y)^2 = X^2 + 3XY + 3XY^2 + Y^2$  does not

imply  $XY = YX$ .

(a)  $\forall \underline{u} \in S, \underline{u} = a_1 \underline{w}_1 + a_2 \underline{w}_2$ , where  $a_1, a_2 \in \mathbb{R}$

$$\begin{aligned} & (\underline{u} \cdot \underline{w}_1) \underline{w}_1 + (\underline{u} \cdot \underline{w}_2) \underline{w}_2 \\ &= [(a_1 \underline{w}_1 + a_2 \underline{w}_2) \cdot \underline{w}_1] \underline{w}_1 + [(a_1 \underline{w}_1 + a_2 \underline{w}_2) \cdot \underline{w}_2] \underline{w}_2 \\ &= (a_1 \underline{w}_1 \cdot \underline{w}_1 + a_2 \underline{w}_2 \cdot \underline{w}_1) \underline{w}_1 + (a_1 \underline{w}_1 \cdot \underline{w}_2 + a_2 \underline{w}_2 \cdot \underline{w}_2) \underline{w}_2 \\ &= a_1 \underline{w}_1 + a_2 \underline{w}_2 \quad (\because \underline{w}_1 \cdot \underline{w}_2 = 0, \underline{w}_1 \cdot \underline{w}_1 = \underline{w}_2 \cdot \underline{w}_2 = 1) \\ &= \underline{u} \end{aligned}$$

$$\begin{aligned} \text{(b) (i) } (\underline{v} - \underline{w}) \cdot \underline{u} &= \underline{v} \cdot \underline{u} - \underline{w} \cdot \underline{u} \\ &= \underline{v} \cdot [(\underline{u} \cdot \underline{w}_1) \underline{w}_1 + (\underline{u} \cdot \underline{w}_2) \underline{w}_2] - [(\underline{v} \cdot \underline{w}_1) \underline{w}_1 \\ &\quad + (\underline{v} \cdot \underline{w}_2) \underline{w}_2] \cdot \underline{u} \\ &= (\underline{u} \cdot \underline{w}_1)(\underline{v} \cdot \underline{w}_1) + (\underline{u} \cdot \underline{w}_2)(\underline{v} \cdot \underline{w}_2) \\ &\quad - (\underline{v} \cdot \underline{w}_1)(\underline{u} \cdot \underline{w}_1) - (\underline{v} \cdot \underline{w}_2)(\underline{u} \cdot \underline{w}_2) \\ &= (\underline{u} \cdot \underline{w}_1)(\underline{v} \cdot \underline{w}_1) + (\underline{u} \cdot \underline{w}_2)(\underline{v} \cdot \underline{w}_2) \\ &\quad - (\underline{u} \cdot \underline{w}_1)(\underline{v} \cdot \underline{w}_1) - (\underline{u} \cdot \underline{w}_2)(\underline{v} \cdot \underline{w}_2) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{(ii) } \underline{v} \cdot \underline{w} &= [(\underline{v} \cdot \underline{w}_1) \underline{w}_1 + (\underline{v} \cdot \underline{w}_2) \underline{w}_2] \cdot [(\underline{w} \cdot \underline{w}_1) \underline{w}_1 + (\underline{w} \cdot \underline{w}_2) \underline{w}_2] \\ &= (\underline{v} \cdot \underline{w}_1)^2 \underline{w}_1 \cdot \underline{w}_1 + 2(\underline{v} \cdot \underline{w}_1)(\underline{v} \cdot \underline{w}_2)(\underline{w}_1 \cdot \underline{w}_2) \\ &\quad + (\underline{v} \cdot \underline{w}_2)^2 \underline{w}_2 \cdot \underline{w}_2 \\ &= (\underline{v} \cdot \underline{w}_1)^2 + (\underline{v} \cdot \underline{w}_2)^2 \quad (\because \underline{w}_1 \cdot \underline{w}_1 = 0, \underline{w}_1 \cdot \underline{w}_2 = \underline{w}_2 \cdot \underline{w}_1 = 1) \\ &= \underline{v} \cdot [(\underline{v} \cdot \underline{w}_1) \underline{w}_1] + \underline{v} \cdot [(\underline{v} \cdot \underline{w}_2) \underline{w}_2] \\ &= \underline{v} \cdot [(\underline{v} \cdot \underline{w}_1) \underline{w}_1 + (\underline{v} \cdot \underline{w}_2) \underline{w}_2] \\ &= \underline{v} \cdot \underline{w} \quad \dots (*) \end{aligned}$$

$$\begin{aligned} \underline{v} \cdot \underline{v} - \underline{w} \cdot \underline{w} &= \underline{v} \cdot \underline{v} - \underline{w} \cdot \underline{v} + \underline{w} \cdot \underline{v} - \underline{w} \cdot \underline{w} \\ &= \underline{v} \cdot \underline{v} - \underline{w} \cdot \underline{v} + \underline{w} \cdot \underline{v} - \underline{w} \cdot \underline{v} \quad (\text{by } (*)) \\ &= (\underline{v} - \underline{w}) \cdot (\underline{v} - \underline{w}) \\ &= |\underline{v} - \underline{w}|^2 \\ &> 0 \end{aligned}$$

$$\begin{aligned} \therefore \underline{v} \cdot \underline{v} &> \underline{w} \cdot \underline{w} \\ \underline{v} \cdot \underline{v} &= \underline{w} \cdot \underline{w} \\ \Rightarrow |\underline{v} - \underline{w}| &= 0 \\ \Rightarrow \underline{v} - \underline{w} &= 0 \\ \Rightarrow \underline{v} &= \underline{w} \quad \dots (*) \end{aligned}$$

"If" part:

$$\begin{aligned} \underline{v} &\in S \\ \Rightarrow \underline{v} &= (\underline{v} \cdot \underline{w}_1) \underline{w}_1 + (\underline{v} \cdot \underline{w}_2) \underline{w}_2 \quad (\text{from (a)}) \\ \Rightarrow \underline{v} &= \underline{w} \quad (\text{by definition of } \underline{w}) \\ \Rightarrow \underline{v} \cdot \underline{v} &= \underline{w} \cdot \underline{w} \quad (\text{by } (*)) \end{aligned}$$

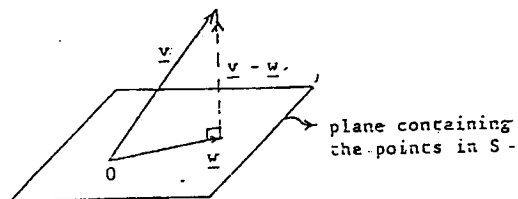
"Only if" part:

$$\begin{aligned} \underline{w} \cdot \underline{w} &= \underline{v} \cdot \underline{v} \\ \Rightarrow \underline{v} &= \underline{w} \quad (\text{by } (*)) \end{aligned}$$

$$\Rightarrow \underline{v} = (\underline{v} \cdot \underline{w}_1) \underline{w}_1 + (\underline{v} \cdot \underline{w}_2) \underline{w}_2 \quad (\because \underline{w} = (\underline{v} \cdot \underline{w}_1) \underline{w}_1 + (\underline{v} \cdot \underline{w}_2) \underline{w}_2)$$

$$\Rightarrow \underline{v} \in S$$

$S$  is the set of points lying on a plane containing the vectors  $\underline{w}_1$ ,  $\underline{w}_2$  and passing through the origin.



$$\begin{aligned} 10. (a) f(z) &= r(\cos\theta + i \sin\theta) + \frac{1}{r(\cos\theta + i \sin\theta)} \\ &= r(\cos\theta + i \sin\theta) + \frac{1}{r}(\cos\theta - i \sin\theta) \\ &= \left(r + \frac{1}{r}\right)\cos\theta + (i \sin\theta)\left(r - \frac{1}{r}\right) \end{aligned}$$

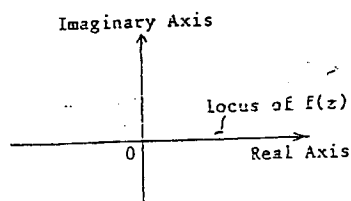
By comparing the real parts and imaginary parts,

$$u = \left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad v = \left(r - \frac{1}{r}\right)\sin\theta$$

$$(b) (i) |z| = 1 \Rightarrow r = 1$$

$$\therefore u = 2\cos\theta \quad \text{and} \quad v = 0$$

Hence, the locus of  $f(z)$  is the real axis.



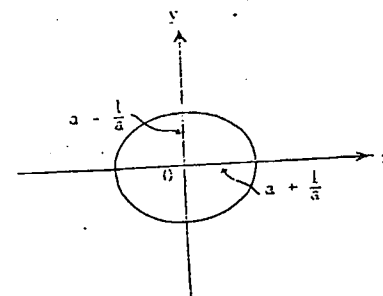
$$(ii) |z| = a \Rightarrow r = a$$

$$\therefore \begin{cases} u = \left(a + \frac{1}{a}\right)\cos\theta \\ v = \left(a - \frac{1}{a}\right)\sin\theta \end{cases}$$

$$\frac{u^2}{\left(a + \frac{1}{a}\right)^2} + \frac{v^2}{\left(a - \frac{1}{a}\right)^2} = 1$$

Hence, the locus of  $f(z)$  is an ellipse centred at origin and with lengths of semi-axes

$$a + \frac{1}{a} \quad \text{and} \quad a - \frac{1}{a}.$$



$$(c) f(3) = 3 + \frac{1}{3} = \frac{10}{3}$$

$$f\left(\frac{1}{3}\right) = \frac{1}{3} + \frac{1}{\frac{1}{3}} = \frac{10}{3}$$

$\therefore f$  is not injective.

$$\forall w \in \mathbb{C}, \exists \frac{w + \sqrt{w^2 - 4}}{2} \in \mathbb{C}$$

$$\text{Suppose } \frac{w + \sqrt{w^2 - 4}}{2} = 0$$

$$w = -\sqrt{w^2 - 4}$$

$$w^2 = w^2 - 4$$

$$0 = -4, \text{ which is false.}$$

Hence,  $\frac{w + \sqrt{w^2 - 4}}{2} \neq 0$

$$\frac{w + \sqrt{w^2 - 4}}{2} \in \mathbb{C} \setminus \{0\}$$

$$\begin{aligned} f\left(\frac{w + \sqrt{w^2 - 4}}{2}\right) &= \frac{w + \sqrt{w^2 - 4}}{2} + \frac{1}{\frac{w + \sqrt{w^2 - 4}}{2}} \\ &= \frac{w + \sqrt{w^2 - 4}}{2} + \frac{2(w - \sqrt{w^2 - 4})}{w^2 - (w^2 - 4)} \\ &= \frac{w + \sqrt{w^2 - 4}}{2} + \frac{w - \sqrt{w^2 - 4}}{2} \\ &= w \end{aligned}$$

$\therefore f$  is surjective.

$$(d) \forall z_1, z_2 \in E, f_E(z_1) = f_E(z_2)$$

$$\Rightarrow z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2}$$

$$\Rightarrow (z_1 - z_2) + \frac{z_2 - z_1}{z_1 z_2} = 0$$

$$\Rightarrow (z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right) = 0$$

$$\Rightarrow z_1 = z_2 \quad \text{and} \quad z_1 z_2 = 1$$

$$\Rightarrow z_1 = z_2 \quad (\because |z_1| < 1, |z_2| < 1, \therefore |z_1 z_2| < 1 \text{ and})$$

$$\Rightarrow f \text{ is injective} \quad \text{hence } z_1 z_2 \neq 1$$

For  $-2 \in \mathbb{C}$ , if  $f_E(z) = -2$  for some  $z \in \mathbb{C} \setminus \{0\}$

( $\because f$  is surjective

$\therefore z$  exists)

$$z + \frac{1}{z} = -2$$

$$z^2 + 2z + 1 = 0$$

$$(z + 1)^2 = 0$$

$$z = -1$$

$$|z| = 1 \nless 1.$$

The pre-image of  $-2$  under  $f_E$  does not exist.

$\therefore f_E$  is not surjective.

$$11. (a) (i) \quad u_n \geq v_n > 0 \text{ and } u_n = \frac{u_{n-1} + v_{n-1}}{2}, \quad v_n = \frac{2u_{n-1}v_{n-1}}{u_{n-1} + v_{n-1}}$$

$$\therefore u_n > 0 \text{ and } v_n > 0, \quad v_n = 0.1, 2, \dots$$

$$\text{Consider } u_n - v_n = \frac{u_{n-1} + v_{n-1}}{2} - \frac{2u_{n-1}v_{n-1}}{u_{n-1} + v_{n-1}}$$

$$= \frac{(u_{n-1} - v_{n-1})^2}{2(u_{n-1} + v_{n-1})}$$

$$\geq 0 \quad (\because u_{n-1}, v_{n-1} > 0)$$

$$\therefore u_n \geq v_n$$

$$\begin{aligned} (ii) \quad u_n - u_{n-1} &= \frac{u_{n-1} + v_{n-1}}{2} - u_{n-1} \\ &= \frac{v_{n-1} - u_{n-1}}{2} \end{aligned}$$

$$\leq 0 \quad (\text{by (a)(i)})$$

$$\therefore u_n \leq u_{n-1}$$

$$\text{Consider } u_n v_n = \left(\frac{u_{n-1} + v_{n-1}}{2}\right) \left(\frac{2u_{n-1}v_{n-1}}{u_{n-1} + v_{n-1}}\right)$$

$$= u_{n-1} v_{n-1}$$

$$\frac{u_n}{u_{n-1}} = \frac{v_{n-1}}{v_n}$$

$$\therefore \frac{u_n}{u_{n-1}} \leq 1$$

$$\therefore v_{n-1} \leq v_n$$

Hence,  $\{u_n\}$  is monotonic decreasing while  $\{v_n\}$  is monotonic increasing.

(iii) From (i) & (ii),  $u_0 \geq u_1 \geq u_2 \geq \dots \geq u_n \geq v_n \geq v_{n-1}$

$$\geq \dots \geq v_2 \geq v_1 \geq v_0$$

$\therefore \{u_n\}$  is monotonic decreasing and is bounded from

below by  $v_1$  while  $\{v_n\}$  is monotonic increasing

and is bounded from above by  $u_0$ .

Hence,  $\lim_{n \rightarrow \infty} u_n$  and  $\lim_{n \rightarrow \infty} v_n$  exist.

$$(b) (i) \quad u_n - v_n = \frac{u_{n-1} + v_{n-1} - 2u_{n-1}v_{n-1}}{2} = \frac{u_{n-1} - v_{n-1}}{2}$$

$$= \frac{1}{2} \frac{(u_{n-1} - v_{n-1})^2}{u_{n-1} + v_{n-1}} \leq \frac{u_{n-1} - v_{n-1}}{2}$$

$$\leq \frac{1}{2^n} (u_0 - v_0)$$

$$\therefore \frac{u_{n-1} - v_{n-1}}{u_{n-1} + v_{n-1}} = 1 - \frac{2v_{n-1}}{u_{n-1} + v_{n-1}} \leq 1$$

$$\leq \frac{1}{2^n} (u_0 - v_0)$$

(ii) From (a)(i),  $u_n \geq v_n$

From (b) (i),  $0 \leq u_n - v_n \leq \frac{1}{2^n} (u_0 - v_0)$

As  $\lim_{n \rightarrow \infty} \frac{1}{2^n} (u_0 - v_0) = 0$ , we have  $\lim_{n \rightarrow \infty} (u_n - v_n) = 0$

From (a) (iii),  $\lim_{n \rightarrow \infty} v_n$  exists.

Hence,  $\lim_{n \rightarrow \infty} (u_n - v_n) + \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} u_n$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n$$

(iii) Consider  $u_n v_n = u_{n-1} v_{n-1}$

$$= u_{n-2} v_{n-2}$$

$$= \dots$$

$$= \dots$$

$$= u_0 v_0$$

$$\therefore \lim_{n \rightarrow \infty} u_n v_n = u_0 v_0$$

From (b) (ii),  $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n$

$$\therefore \lim_{n \rightarrow \infty} u_n v_n = (\lim_{n \rightarrow \infty} u_n)^2$$

$$\text{i.e. } \lim_{n \rightarrow \infty} u_n = \sqrt{u_0 v_0} \quad (\text{Positive root is adopted since } u_n \geq u_0 > 0)$$

12. (a) (i)  $f(x) = x^p - px$

$$f'(x) = px^{p-1} - p$$

$$= p(x^{p-1} - 1)$$

$$\forall x \in (0, 1), f'(x) < 0 \quad (\because p > 1)$$

$$\forall x \in (1, \infty), f'(x) > 0 \quad (\because p > 1)$$

i.e.  $f$  is strictly decreasing on  $(0, 1)$  and strictly increasing on  $(1, \infty)$ .

$\therefore$  When  $x=1$ ,  $f(x)$  attains its absolute minimum value and the absolute minimum value is  $(1-p)$ .

(ii) From (a)(i),  $f(x) \geq f(1), \forall x > 0$

$$x^p - px \geq 1-p$$

$$x^p - 1 \geq p(x-1)$$



(b) (i) For  $x = \alpha\gamma$

From (a) (ii)  $(\alpha\gamma)^p - 1 \geq p(\alpha\gamma - 1)$

$$\alpha^{p-1}\gamma^p - \frac{1}{\alpha} \geq p(\gamma - \frac{1}{\alpha}) \quad \dots (1) \quad (\because \alpha > 0)$$

For  $x = \beta\delta$

from (a) (ii)  $(\beta\delta)^p - 1 \geq p(\beta\delta - 1)$

$$\beta^{p-1}\delta^p - \frac{1}{\beta} \geq p(\delta - \frac{1}{\beta}) \quad \dots (2) \quad (\because \beta > 0)$$

$$(1)-(2): \alpha^{p-1}\gamma^p - \beta^{p-1}\delta^p - (\frac{1}{\alpha} - \frac{1}{\beta}) \geq p(\gamma - \delta) - p(\frac{1}{\alpha} - \frac{1}{\beta})$$

$$\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p - 1 \geq p - p$$

$$\alpha^{p-1}\gamma^p + \beta^{p-1}\delta^p \geq 1$$

From (a), Equality holds

$$\Leftrightarrow x = 1$$

$$\Leftrightarrow \alpha\gamma = \beta\delta = 1$$

(ii) Put  $\alpha = \frac{a+b}{a} > 0$ ,  $\beta = \frac{a+b}{b} > 0$ ,  $\gamma = \frac{c}{c+d} > 0$ ,  $\delta = \frac{d}{c+d} > 0$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} = 1 \quad \text{and} \quad \gamma + \delta = 1$$

From (b) (i),

$$(\frac{a+b}{a})^{p-1}(\frac{c}{c+d})^p + (\frac{a+b}{b})^{p-1}(\frac{d}{c+d})^p \geq 1$$

$$(\frac{a+b}{a})^{p-1}c^p + (\frac{a+b}{b})^{p-1}d^p \geq (c+d)^p \quad (\because c+d > 0)$$

(c) Put  $a = (\sum_{i=1}^n a_i^p)^{\frac{1}{p}} > 0$ ,  $b = (\sum_{j=1}^n b_j^p)^{\frac{1}{p}} > 0$ ,

$$c = a_1 > 0, d = b_1 > 0$$

From (b) (ii)  $(\frac{a+b}{a})^{p-1}a_1^p + (\frac{a+b}{b})^{p-1}b_1^p \geq (a_1 + b_1)^p$

$$\therefore \sum_{i=1}^n ((\frac{a+b}{a})^{p-1}a_i^p + (\frac{a+b}{b})^{p-1}b_i^p) \geq \sum_{i=1}^n (a_i + b_i)^p$$

$$(\frac{a+b}{a})^{p-1} \sum_{i=1}^n a_i^p + (\frac{a+b}{b})^{p-1} \sum_{i=1}^n b_i^p \geq \sum_{i=1}^n (a_i + b_i)^p$$

$$(\frac{a+b}{a})^{p-1}a^p + (\frac{a+b}{b})^{p-1}b^p \geq \sum_{i=1}^n (a_i + b_i)^p$$

$$(a+b)^{p-1}(a+b) \geq \sum_{i=1}^n (a_i + b_i)^p$$

$$(a+b)^p \geq \sum_{i=1}^n (a_i + b_i)^p$$

$$a+b \geq (\sum_{i=1}^n (a_i + b_i)^p)^{\frac{1}{p}}$$

$$\text{i.e. } (\sum_{i=1}^n a_i^p)^{\frac{1}{p}} + (\sum_{i=1}^n b_i^p)^{\frac{1}{p}} \geq (\sum_{i=1}^n (a_i + b_i)^p)^{\frac{1}{p}}$$

Equality holds  $\Leftrightarrow \alpha\gamma = \beta\delta = 1$  (from (b))

$$\Leftrightarrow (\frac{a+b}{a})(\frac{c}{c+d}) = (\frac{a+b}{b})(\frac{d}{c+d}) = 1$$

$$\Leftrightarrow (\frac{a+b}{a})(\frac{a_1}{a_1+b_1}) = (\frac{a+b}{b})(\frac{b_1}{a_1+b_1}) = 1, \forall i$$

$$\Leftrightarrow \frac{a_1}{a} = \frac{b_1}{b}, \forall i$$

$$\Leftrightarrow \frac{a_1}{b_1} = \frac{a}{b}, \forall i$$

$$\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a}{b}$$

13. (a) (i)

$$M_\theta M_\phi = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi & -\cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

$$= M_{\theta+\phi}$$

(ii) From (i)  $M_\theta M_{\theta+\phi} = M_\phi$

$$= I$$

$$\therefore (M_\theta)^{-1} = M_{(-\theta)}$$

(b) (i) (1)  $\forall (x, y) \in \mathbb{R}^2$ ,

$$M_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\ = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore (x, y) \sim (x, y)$$

i.e.  $\sim$  is reflexive.

(2)  $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ ,

$$(x_1, y_1) \sim (x_2, y_2) \Rightarrow M_\theta \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ for some } \theta \in \mathbb{R}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M_{\theta-\alpha} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ (by (a)(ii))}$$

$$\Rightarrow (x_2, y_2) \sim (x_1, y_1) \quad (\because \theta - \alpha \in \mathbb{R})$$

$\Rightarrow \sim$  is symmetric

(3)  $\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$

$$(x_1, y_1) \sim (x_2, y_2) \text{ and } (x_2, y_2) \sim (x_3, y_3)$$

$$\Rightarrow M_\theta \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ and } M_\phi \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \text{ for some } \theta, \phi \in \mathbb{R}$$

$$\Rightarrow M_\phi M_\theta \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$$\Rightarrow M_{\phi+\theta} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$$\Rightarrow (x_1, y_1) \sim (x_3, y_3) \quad (\because \phi + \theta \in \mathbb{R})$$

$\Rightarrow \sim$  is transitive

$\therefore$  From (1), (2) & (3),  $\sim$  is an equivalence relation.

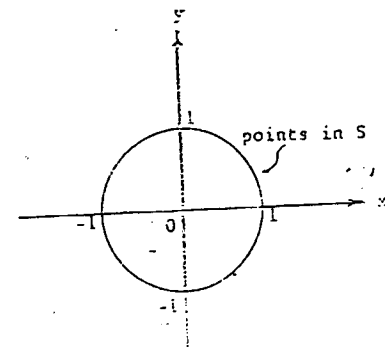
(ii)  $(x, y) \in S$

$$\Rightarrow (x, y) \sim (1, 0)$$

$$\Rightarrow \exists \theta \in \mathbb{R} \text{ s.t. } \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x \cos \theta + y \sin \theta = 1 \\ -x \sin \theta + y \cos \theta = 0 \end{cases}$$

$$\Rightarrow (x \cos \theta + y \sin \theta)^2 + (-x \sin \theta + y \cos \theta)^2 = 1 \\ \Rightarrow x^2 + y^2 = 1$$



(c) (i)  $\forall [x_1, y_1], [x_2, y_2] \in \mathbb{R}^2/\sim$

$$[x_1, y_1] = [x_2, y_2] \Rightarrow (x_1, y_1) \sim (x_2, y_2)$$

$$\Rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \text{where } \theta \in \mathbb{R}$$

$$\Rightarrow \begin{cases} x_1 \cos \theta + y_1 \sin \theta = x_2 \\ -x_1 \sin \theta + y_1 \cos \theta = y_2 \end{cases}$$

$$\Rightarrow (x_1 \cos \theta + y_1 \sin \theta)^2 + (-x_1 \sin \theta + y_1 \cos \theta)^2 \\ = x_2^2 + y_2^2$$

$$\Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$$

$$\Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

$$\Rightarrow f([x_1, y_1]) = f([x_2, y_2])$$

$\Rightarrow f$  is well-defined.

$$(ii) (1) \forall [x_1, y_1], [x_2, y_2] \in \mathbb{R}^2$$

$$f([x_1, y_1]) = f([x_2, y_2])$$

$$\Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

$$\Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$$

$$\Rightarrow \exists \theta \in \mathbb{R} \text{ such that } \begin{cases} x_1 \cos \theta + y_1 \sin \theta = x_2 \\ -x_1 \sin \theta + y_1 \cos \theta = y_2 \end{cases}$$

$$\Rightarrow \exists \theta \in \mathbb{R} \text{ such that } M_{\theta} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\Rightarrow [x_1, y_1] \sim [x_2, y_2]$$

$$\Rightarrow [x_1, y_1] = [x_2, y_2]$$

$\Rightarrow f$  is injective

$$(2) \forall t \in \mathbb{R}_+, \exists (t, 0) \in \mathbb{R}^2$$

$$\text{i.e. } t \in \mathbb{R}_+ \Rightarrow (t, 0) \in \mathbb{R}^2$$

$$\Rightarrow [t, 0] \in \mathbb{R}^2 / \sim$$

$$\text{As } f([t, 0]) = \sqrt{t^2 + 0} = t$$

$$= t \quad (\because t > 0)$$

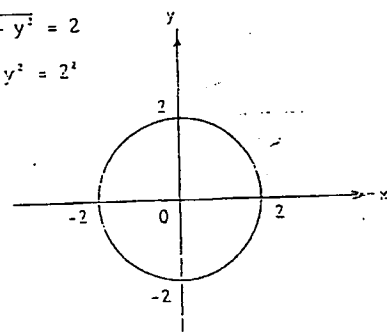
$f$  is surjective.

Hence,  $f$  is bijective.

$$(iii) (x, y) \in T \Rightarrow f([x, y]) = 2$$

$$\Rightarrow \sqrt{x^2 + y^2} = 2$$

$$\Rightarrow x^2 + y^2 = 2^2$$



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PURE MATHEMATICS

1990 PAPER II

### SECTION A

$$1. \text{ Let } f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{1}{x^2}(1 - \ln x)$$

$$\forall x \in (e, \infty), f'(x) < 0$$

$\therefore f(x)$  is strictly decreasing on  $(e, \infty)$ .

Hence, if  $b > a > e$ ,  $f(a) > f(b)$

$$\frac{\ln a}{a} > \frac{\ln b}{b}$$

$$b \ln a > a \ln b \quad (\because a, b > 0)$$

$$\ln a^b > \ln b^a$$

$\exp(\ln a^b) > \exp(\ln b^a)$  ( $\because$  exponential function is strictly increasing)

$$a^b > b^a$$

$$2. \cot kx - \cot(k+1)x = \frac{\cos kx \sin(k+1)x - \cos(k+1)x \sin kx}{\sin kx \sin(k+1)x}$$

$$= \frac{\sin[(k+1)x - kx]}{\sin kx \sin(k+1)x} = \frac{\sin x}{\sin kx \sin(k+1)x}$$

$$\therefore \frac{1}{\sin x \sin 2x} + \frac{1}{\sin 2x \sin 3x} + \dots + \frac{1}{\sin nx \sin(n+1)x}$$

$$= \frac{1}{\sin x} \left[ \frac{\sin x}{\sin x \sin 2x} + \frac{\sin x}{\sin 2x \sin 3x} + \dots + \frac{\sin x}{\sin nx \sin(n+1)x} \right]$$

$$= \frac{1}{\sin x} [\cot x - \cot 2x + \cot 2x - \cot 3x + \dots + \cot nx - \cot(n+1)x]$$

$$= \frac{1}{\sin x} [\cot x - \cot(n+1)x]$$

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$$= \frac{1}{\sin x} \left( \frac{\cos x \sin(n+1)x - \sin x \cos(n+1)x}{\sin x \sin(n+1)x} \right)$$

$$= \frac{\sin nx}{\sin x \sin(n+1)x}$$

$$3. \int_a^b f(x)g(x)dx = \int_a^b f(x)[K-g(a-x)]dx$$

$$= K \int_a^b f(x)dx - \int_a^b f(x)g(a-x)dx$$

$$\text{Let } I = \int_a^b f(x)g(a-x)dx$$

$$\text{Put } y = a-x$$

$$\begin{cases} x = a \\ y = 0 \end{cases} \quad \begin{cases} x = 0 \\ y = a \end{cases}$$

$$\therefore I = \int_a^0 f(a-y)g(y)(-dy)$$

$$= \int_0^a f(a-y)g(y)dy$$

$$= \int_0^a f(a-x)g(x)dx$$

$$= \int_0^a f(x)g(x)dx \quad (\because f(x) = f(a-x))$$

$$\text{Hence, } \int_a^b f(x)g(x)dx = K \int_a^b f(x)dx - \int_a^b f(x)g(x)dx$$

$$2 \int_a^b f(x)g(x)dx = K \int_a^b f(x)dx$$

$$\int_a^b f(x)g(x)dx = \frac{1}{2} K \int_a^b f(x)dx$$

$$\text{Let } f(x) = \sin x \cos^2 x \text{ and } g(x) = x$$

$$\therefore f(\pi-x) = \sin(\pi-x) \cos^2(\pi-x) \text{ and } g(x) + g(\pi-x) = x + (\pi-x)$$

$$= \sin x (-\cos x)^2$$

$$= \sin x \cos^2 x$$

$$= f(x)$$

$$\text{Hence, } \int_0^\pi x \sin x \cos^2 x dx = \frac{\pi}{2} \int_0^\pi \sin x \cos^2 x dx$$

$$= \frac{-\pi}{2} \int_0^\pi \cos^2 x \cos x dx$$

$$= \frac{-\pi}{2} \left( \frac{1}{5} \cos^5 x \right)$$

$$= \frac{\pi}{5}$$

$$4. (a) \lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{\tan x} \right) = \lim_{x \rightarrow 0} \frac{\tan x - x}{x \tan x}$$

$$= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{\tan x + x \sec^2 x} \quad (\text{by L'Hospital's Rule})$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{\sec^2 x + \sec^2 x + 2x \sec^2 x \tan x} \quad (\text{by L'Hospital's Rule})$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{2 \sec^2 x (1 + x \tan x)}$$

$$= \lim_{x \rightarrow 0} \frac{\tan x}{1 + x \tan x}$$

$$= 0$$

$$(b) \int \frac{dx}{\sqrt{x^2 + 4x + 2}} = \int \frac{dx}{\sqrt{(x+2)^2 - 2}}$$

$$= \int \frac{d(x+2)}{\sqrt{(x+2)^2 - 2}}$$

$$= 2n \left| \frac{x+2 + \sqrt{x^2 + 4x + 2}}{\sqrt{2}} \right| + c$$

where c is an arbitrary constant.

$$5. (a) \frac{d}{dx} \int_a^x f(t) dt = \left( \frac{d}{dx} \int_a^x f(t) dt \right) \frac{dx^n}{dx} = f(x) \cdot nx^{n-1}$$

$$(b) F(x) = \int_1^{x^2} e^{-t^2} dx = \int_1^{x^2} e^{-t^2} dx - \int_0^{x^2} e^{-t^2} dx$$

$$F'(x) = 2xe^{-(x^2)^2} - 3x^2 e^{-(x^2)^2}$$

$$= 2xe^{-x^4} - 3x^2 e^{-x^4}$$

$$F'(1) = 2e^{-1} - 3e^{-1} = -e^{-1}$$

$$\frac{d}{dx} \int_0^x f(t) dt = f(x)$$

6. Let the equation of the plane be

$$(x) : l(x-1) + m(y-1) + n(z-3) = 0, \text{ where } l, m, n \text{ are constants.}$$

$\therefore (1, -1, 2)$  lies on (1) and hence on (2).

$$-2m - n = 0$$

$$2m + n = 0$$

Direction ratios of (1) is 3:2:2

$$\therefore 3l + 2m + 2n = 0$$

As  $l^2 + m^2 + n^2 = 1$ , the system of equations

$$\begin{cases} l(x-1) + m(y-1) + n(z-3) = 0 \\ 3l + 2m + 2n = 0 \\ 2m + n = 0 \end{cases}$$

has non-zero solution.

$$\therefore \begin{vmatrix} (x-1) & (y-1) & (z-3) \\ 0 & 2 & 1 \\ 3 & 2 & 2 \end{vmatrix} = 0$$

$$\text{i.e. } -2x + 3y - 6z + 13 = 0$$

$\therefore$  The equation of the required plane is

$$2x + 3y - 6z + 13 = 0$$

7. (a)  $\int \ln(1+x^2) dx = x \ln(1+x^2) - \int x d \ln(1+x^2)$  (by parts)

$$= x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx$$

$$= x \ln(1+x^2) - \int (2 - \frac{2}{1+x^2}) dx$$

$$= x \ln(1+x^2) - 2x + 2 \tan^{-1} x + c, \text{ where } c \text{ is an arbitrary constant.}$$

$$(b) \ln u_n = \ln \left( \frac{1}{n} \prod_{k=1}^{2n} (n^2 + k^2)^{\frac{1}{n}} \right)$$

$$= \ln \left( \frac{1}{n} \right) + \frac{2n}{n} \ln(n^2 + k^2)^{\frac{1}{n}}$$

$$= -4 \ln n + \frac{2n}{n} \ln(n^2 + k^2)$$

$$= \frac{1}{n} (-4n \ln n + \sum_{k=1}^{2n} \ln(n^2 + k^2))$$

$$= \frac{1}{n} \sum_{k=1}^{2n} [\ln(n^2 + k^2) - 2 \ln n]$$

$$= \frac{1}{n} \sum_{k=1}^{2n} \ln \left( \frac{n^2 + k^2}{n^2} \right)$$

$$= \frac{1}{n} \sum_{k=1}^{2n} \ln \left( 1 + \frac{k^2}{n^2} \right)$$

$$\therefore \lim_{n \rightarrow \infty} \ln u_n = \int_0^1 \ln(1+x^2) dx$$

$$= [x \ln(1+x^2) - 2x + 2 \tan^{-1} x]_0^1$$

$$= 2 \ln 5 - 4 + 2 \tan^{-1} 2$$

$$\ln(\lim_{n \rightarrow \infty} u_n) = 2 \ln 5 - 4 + 2 \tan^{-1} 2 \quad (\because \ln x \text{ is continuous})$$

$$\lim_{n \rightarrow \infty} u_n = e^{2 \ln 5 - 4 + 2 \tan^{-1} 2}$$

$$= \frac{25 e^{2 \tan^{-1} 2}}{e^4}$$

## SECTION B

8. (a) (i)  $I_0 = \int_0^1 \frac{x}{(1+x)^2} dx$

$$= \int_0^1 \left[ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right] dx$$

$$= [\ln(1+x) + \frac{1}{1+x}]_0^1$$

$$= \ln 2 - \frac{1}{2}$$

(ii)  $\forall x \in [0, 1], 0 \leq \frac{x^{n+1}}{(1+x)^2} \leq \frac{x^n}{(1+x)^2}$

$$\therefore 0 \leq \int_0^1 \frac{x^{n+1}}{(1+x)^2} dx \leq \int_0^1 \frac{x^n}{(1+x)^2} dx$$

$$0 \leq I_n \leq I_{n-1}$$

$\therefore \{I_n\}$  is a monotonic decreasing sequence, and

bounded from below by 0.

Hence,  $\lim_{n \rightarrow \infty} I_n$  exists and let it be  $l$ .

Consider  $I_n + 2I_{n-1} + I_{n-2} = \int_0^1 \frac{x^{n+1}}{(1+x)^2} dx + 2 \int_0^1 \frac{x^n}{(1+x)^2} dx$

$$+ \int_0^1 \frac{x^{n-1}}{(1+x)^2} dx$$

$$= \int_0^1 \frac{x^{n-1}(x^2 + 2x + 1)}{(1+x)^2} dx$$

$$= \int_0^1 x^{n-1} dx$$

$$= \frac{1}{n}$$

As  $\lim_{n \rightarrow \infty} I_n = \lim_{n \rightarrow \infty} I_{n-1} = \lim_{n \rightarrow \infty} I_{n-2}$ ,

$$4l = \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$l = \lim_{n \rightarrow \infty} \frac{1}{4n}$$

$$= 0$$

(b) (i)  $I_0 = \int_0^1 x \left( \frac{1-(-x)^m}{1-x} \right) \left( \frac{1-(-x)^n}{1-x} \right) dx$

$$= \int_0^1 x \sum_{i=1}^m (-x)^{i-1} \sum_{j=1}^n (-x)^{j-1} dx \quad \left( \begin{array}{l} \text{from the formula in} \\ \text{summing a G.P.} \end{array} \right)$$

$$= \int_0^1 x \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j-2} x^{i+j-2} dx$$

$$= \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j-2} \int_0^1 x^{i+j-1} dx$$

$$= \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j-2} \left[ \frac{x^{i+j}}{i+j} \right]_0^1$$

$$= \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j-2} \cdot \frac{1}{i+j}$$

$$= \sum_{i=1}^m \sum_{j=1}^n \frac{(-1)^{i+j}}{i+j} \quad \left( \because (-1)^{i+j} = (-1)^{i+j-2} (-1)^2 \right)$$

$$= (-1)^{i+j-2}$$

(ii) Consider  $I_0 = \int_0^1 x \left( \frac{1-(-x)^n}{1+x} \right) \left( \frac{1-(-x)^n}{1+x} \right) dx$

$$= \int_0^1 x \frac{[1 - 2(-x)^n + x^{2n}]}{(1+x)^2} dx$$

$$= I_0 - 2(-1)^n I_n + I_{2n}$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n \frac{(-1)^{i+j}}{i+j} = \lim_{n \rightarrow \infty} \int_0^1 x \left( \frac{1-(-x)^n}{1+x} \right) \left( \frac{1-(-x)^n}{1+x} \right) dx$$

$$= \lim_{n \rightarrow \infty} (I_0 - 2(-1)^n I_n + I_{2n})$$

$$= \lim_{n \rightarrow \infty} I_0 \quad \left( \because \lim_{n \rightarrow \infty} I_n = \lim_{n \rightarrow \infty} I_{2n} = 0 \right)$$

$$= I_0$$

$$= \ln 2 - \frac{1}{2} \quad \text{(from (a) (i))}$$

9. (a) Equation of the line joining the points P and Q is

$$y - \frac{c}{t_1} = \frac{\frac{c}{t_1} - \frac{c}{t_2}}{ct_1 - ct_2}(x - ct_1)$$

$$= \frac{-1}{t_1 t_2}(x - ct_1)$$

i.e.  $x + t_1 t_2 y - c(t_1 + t_2) = 0$

Set  $t_1 = t_2$ , the equation of tangent at Q is

$$x + t_2^2 y - 2ct_2 = 0$$

Set  $t_2 = t_1$ , the equation of tangent at P is

$$x + t_1^2 y - 2ct_1 = 0$$

(b) (i) Solving  $\begin{cases} x + t_1^2 y - 2ct_1 = 0 & \dots (1) \\ x + t_2^2 y - 2ct_2 = 0 & \dots (2) \end{cases}$

$$(1) - (2) : (t_1^2 - t_2^2)y - 2c(t_1 - t_2) = 0$$

$$y = \frac{2c}{t_1 + t_2} \quad (\because t_1 \neq t_2)$$

Put into (1) :

$$x + \frac{2ct_1^2}{t_1 + t_2} - 2ct_1 = 0$$

$$x = \frac{2ct_1 t_2}{t_1 + t_2}$$

$$\therefore R = \left( \frac{2ct_1 t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2} \right)$$

(ii) Let  $R = (x, y)$

$$x = \frac{2ct_1 t_2}{t_1 + t_2}$$

$$y = \frac{2c}{t_1 + t_2}$$

$$x = \frac{2ck}{t_1 + t_2}, \text{ where } k \text{ is a constant}$$

$$y = \frac{2c}{t_1 + t_2}$$

i.e.  $x = ky$

$$x - ky = 0 \quad \dots (*)$$

$$\text{Mid-point of PQ} = \left( \frac{1}{2}(ct_1 + ct_2), \frac{1}{2}\left(\frac{c}{t_1} + \frac{c}{t_2}\right) \right)$$

$$= \left( \frac{c}{2}(t_1 + t_2), \frac{c}{2}\left(\frac{t_1 + t_2}{t_1 t_2}\right) \right)$$

$$= \left( \frac{c}{2}(t_1 + t_2), \frac{c}{2k}(t_1 + t_2) \right)$$

$$\text{As } \frac{c}{2}(t_1 + t_2) - k\left[\frac{c}{2k}(t_1 + t_2)\right] = 0, \text{ mid-point of}$$

PQ lies on (\*).

Hence, the locus of R is a straight line passing through the mid-point of PQ.

(iii) Let  $A\left(\frac{c}{2}\cos\theta, c\sin\theta\right)$  be a point on the ellipse

$$4x^2 + y^2 = c^2$$

Equation of tangent at A to  $4x^2 + y^2 = c^2$  is

$$4\left(\frac{c}{2}\cos\theta\right)x + y\sin\theta = c^2$$

$$2x\cos\theta + y\sin\theta = c \quad \dots (**)$$

As PQ is a tangent to the ellipse  $4x^2 + y^2 = c^2$ , by comparing with (\*\*), we have such that

$$\frac{1}{2\cos\theta} = \frac{t_1 t_2}{\sin\theta} = \frac{c(t_1 + t_2)}{c}$$

$$\cos\theta = \frac{1}{2(t_1 + t_2)}$$

$$\therefore \begin{cases} \cos\theta = \frac{1}{2(t_1 + t_2)} \\ \sin\theta = \frac{t_1 t_2}{t_1 + t_2} \end{cases}$$

If  $R = (x, y)$ , then  $\begin{cases} \cos\theta = \frac{y}{4c} \\ \sin\theta = \frac{x}{2c} \end{cases}$

$$\therefore \left(\frac{x}{2c}\right)^2 + \left(\frac{y}{4c}\right)^2 = 1$$

$$\frac{x^2}{LC^2} + \frac{y^2}{loc^2} = 1$$

∴ The locus of R lies on an ellipse with centre at the origin and with equation

$$\frac{x^2}{LC^2} + \frac{y^2}{loc^2} = 1$$

$$10. (a) \begin{cases} x = r \cos \theta = e^{\frac{\pi}{4}} \cos \theta \\ y = r \sin \theta = e^{\frac{\pi}{4}} \sin \theta \\ \frac{dx}{d\theta} = e^{\frac{\pi}{4}} \cos \theta - e^{\frac{\pi}{4}} \sin \theta \\ \frac{dy}{d\theta} = e^{\frac{\pi}{4}} \sin \theta + e^{\frac{\pi}{4}} \cos \theta \end{cases}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{e^{\frac{\pi}{4}} \sin \theta + e^{\frac{\pi}{4}} \cos \theta}{e^{\frac{\pi}{4}} \cos \theta - e^{\frac{\pi}{4}} \sin \theta} \\ &= \frac{\tan \theta + 1}{1 - \tan \theta} \\ &= \frac{\tan \theta + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{4} \tan \theta} \end{aligned}$$

$$= \tan\left(\theta + \frac{\pi}{4}\right)$$

(b) From (a), slope of the tangent at P is  $\tan\left(\theta + \frac{\pi}{4}\right)$ .

Hence, the inclination of the tangent at P to the initial line is  $\theta + \frac{\pi}{4}$ .

As  $\theta$  is the inclination of OP to the initial line, the tangent at P always makes an angle  $\frac{\pi}{4}$  with the line OP.

(c) If the tangent at Q is perpendicular to the x-axis,

$$\theta + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{4}$$

when  $\theta = \frac{\pi}{4}$ ,  $r = e^{\frac{\pi}{4}}$

∴ The polar coordinates of Q is  $\left(e^{\frac{\pi}{4}}, \frac{\pi}{4}\right)$

and the rectangular coordinates of Q is  $\left(e^{\frac{\pi}{4}} \cos \frac{\pi}{4}, e^{\frac{\pi}{4}} \sin \frac{\pi}{4}\right)$

$$= \left(\frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}, \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}\right)$$

(d) (i) Area of the shaded region

$$\begin{aligned} &= \text{area of } \triangle OSQ - \int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2} \cdot e^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{2} e^{\frac{\pi}{2}} d\theta \\ &= \frac{1}{4} e^{\frac{\pi}{2}} - \left[\frac{1}{4} e^{\frac{\pi}{2}} \theta\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{4} \end{aligned}$$

(ii) Length of arc PQ =  $\int_0^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

$$\begin{aligned} &= \int_0^{\frac{\pi}{4}} \sqrt{e^{2\theta} + e^{2\theta}} d\theta \\ &= \int_0^{\frac{\pi}{4}} \sqrt{2} e^{\theta} d\theta \\ &= [\sqrt{2} e^{\theta}]_0^{\frac{\pi}{4}} \\ &= \sqrt{2} (e^{\frac{\pi}{4}} - 1) \end{aligned}$$



11. (a) Since  $a_{n+1} = \sin(a_n)$  and  $0 < a_1 < 1 < \frac{\pi}{2}$ , it is obvious that  $0 < a_n < 1$ , for all  $n=1, 2, \dots$

Further,  $a_{n+1} = \sin(a_n) < a_n$  ( $\because 0 < a_n < 1$ )

i.e.  $(a_n)$  is a monotonic decreasing sequence.

As  $(a_n)$  is monotonic decreasing and is bounded from below by 0,  $\lim_{n \rightarrow \infty} a_n$  exists and let it be  $z$ .

$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$  and sine function is continuous,

$\therefore z = \sin z$

Consider  $f(x) = x - \sin x$

$$f'(x) = 1 - \cos x$$

$$> 0, \quad \forall x \in (0, 1)$$

i.e.  $f(x)$  is strictly increasing on  $(0, 1)$ .

Also,  $f(0) = 0$

Hence,  $f(x) > 0, \quad \forall x \in (0, 1)$

The only real-root of the equation  $f(x) = 0$  in the interval  $[0, 1]$  is 0.

Therefore,  $z = 0$ .

$$(b) (i) \lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{2x - 2 \sin x \cos x}{2x \sin^2 x + 2x^2 \cos x \sin x} \quad (\text{by L'Hospital's Rule})$$

$$= \lim_{x \rightarrow 0} \frac{2 - 2 \cos^2 x + 2 \sin^2 x}{2 \sin^2 x + 4x \sin x \cos x + 2x \sin x \cos x + 2x^2 \cos^2 x - 2x^2 \sin^2 x}$$

(by L'Hospital's Rule)

$$= \lim_{x \rightarrow 0} \frac{4 \sin^2 x}{2 \sin^2 x + 8x \sin x \cos x - 2x^2 \cos^2 x - 2x^2 \sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{4}{2 + \frac{8x}{\sin x} \cos x + 2\left(\frac{x}{\sin x}\right)^2 \cos^2 x - 2x^2}$$

$$= \frac{4}{2 - 8 + 2}$$

$$= \frac{1}{3}$$

$$(ii) \lim_{n \rightarrow \infty} \left( \frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{\sin^2 a_n} - \frac{1}{a_n^2} \right)$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1}{\sin^2 a_n} - \frac{1}{a_n^2} \right) \quad (\text{by (a)})$$

$$= \lim_{n \rightarrow \infty} \frac{a_n^2 - \sin^2 a_n}{a_n^2 \sin^2 a_n}$$

$$= \frac{1}{3} \quad (\text{from (b)(i)})$$

$$(c) \text{ Put } x_n = \frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}$$

$$\text{From (b)(ii), } \lim_{n \rightarrow \infty} x_n = \frac{1}{3}$$

$$\frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{a_{i+1}^2} - \frac{1}{a_i^2} \right)$$

$$= \frac{1}{n} \left( \frac{1}{a_{n+1}^2} - \frac{1}{a_1^2} \right)$$

$$= \frac{1}{na_{n+1}^2} - \frac{1}{na_1^2}$$

$$\frac{1}{na_{n+1}^2} = \frac{1}{n} \sum_{i=1}^n x_i + \frac{1}{na_1^2}$$

As  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i$  and  $\lim_{n \rightarrow \infty} \frac{1}{na_1^2}$  exist,

$\lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=1}^n x_i + \frac{1}{na_1^2} \right)$  exists.

Hence  $\lim_{n \rightarrow \infty} \left( \frac{1}{na_{n+1}^2} \right)$  exists.

$$\lim_{n \rightarrow \infty} \frac{1}{na_{n+1}^2} = \frac{1}{3} \neq 0 \quad (\because \lim_{n \rightarrow \infty} \frac{1}{na_1^2} = 0)$$

$\therefore \lim_{n \rightarrow \infty} na_{n+1}^2$  exists and equal to 3.

Consider  $\lim_{n \rightarrow \infty} na_n^2 = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)(n+1)a_{n+1}^2$

$\therefore \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$  and  $\lim_{n \rightarrow \infty} na_{n+1}^2 = 3$

$\therefore \lim_{n \rightarrow \infty} (n+1)a_{n+1}^2$  exists and equal to 3.

Hence,  $\lim_{n \rightarrow \infty} na_n^2$  exists and is also equal to 3.

12. (a)  $f(x) = (2x-1)x^{\frac{1}{3}}$

$f'(x) = 2x^{\frac{1}{3}} + \frac{2}{3}x^{-\frac{2}{3}}(2x-1)$

$= 2x^{\frac{1}{3}} + \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}}$

$= \frac{10}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}}$

$= \frac{10x-2}{3x^{\frac{4}{3}}}$

$= \frac{2(5x-1)}{3x^{\frac{4}{3}}}$

$f''(x) = \frac{20}{9}x^{-\frac{1}{3}} + \frac{2}{9}x^{-\frac{5}{3}}$

$= \frac{2(10x+1)}{9x^{\frac{4}{3}}}$

(b)  $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \rightarrow 0} \frac{(2h-1)h^{\frac{1}{3}}}{h}$

$= \lim_{h \rightarrow 0} (2h^{\frac{2}{3}} - \frac{1}{h^{\frac{1}{3}}})$

As  $\lim_{h \rightarrow 0} \frac{1}{h^{\frac{1}{3}}}$  does not exist and  $\lim_{h \rightarrow 0} 2h^{\frac{2}{3}} = 0$ ,  $f'(0)$  does not exist.

(c) (i)  $f'(x) = 0$  or  $x = \frac{1}{5}$

(ii)  $f'(x) > 0$ ,  $\frac{2(5x-1)}{3x^{\frac{4}{3}}} > 0$

$\therefore x < 0$  or  $x > \frac{1}{5}$

(iii)  $f'(x) < 0$ ,  $\frac{2(5x-1)}{3x^{\frac{4}{3}}} < 0$

$\therefore 0 < x < \frac{1}{5}$

(iv)  $f''(x) = 0$  when  $x = \frac{-1}{10}$

(v)  $f''(x) > 0$ ,  $\frac{2(10x+1)}{9x^{\frac{4}{3}}} > 0$

$\therefore x > \frac{-1}{10}$  and  $x \neq 0$

(vi)  $f''(x) < 0$ ,  $\frac{2(10x+1)}{9x^{\frac{4}{3}}} < 0$

$x < \frac{-1}{10}$

(d) When  $x = \frac{1}{5}$ ,  $y = \frac{-3}{5}(\frac{1}{5})^{\frac{1}{3}}$

From (c) (i) & (v),  $f'(\frac{1}{5}) = 0$  and  $f''(\frac{1}{5}) > 0$

$\therefore (\frac{1}{5}, \frac{-3}{5}(\frac{1}{5})^{\frac{1}{3}})$  is a minimum point.

When  $x = 0$ ,  $y = 0$

As  $x$  is slightly less than 0,  $f'(x) > 0$  (from (c) (ii))

As  $x$  is slightly greater than 0,  $f'(x) < 0$  (from (c) (iii))

$\therefore (0,0)$  is a maximum point.

When  $x = \frac{-1}{10}$ ,  $y = \frac{-5}{6}(\frac{-1}{10})^{\frac{1}{3}}$

As  $x$  is slightly less than  $\frac{-1}{10}$ ,  $f''(x) < 0$  (from (c) (v))

As  $x$  is slightly greater than  $\frac{-1}{10}$ ,  $f''(x) > 0$  (from (c) (vi))

$\therefore (\frac{-1}{10}, \frac{-5}{6}(\frac{-1}{10})^{\frac{1}{3}})$  is an inflexion point.

(e)  $f(x) \rightarrow \infty$  only when  $x \rightarrow \infty$

$\therefore$  There is no vertical asymptote.

Suppose there exists a slant asymptote  $y = mx + c$

$$m = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$$

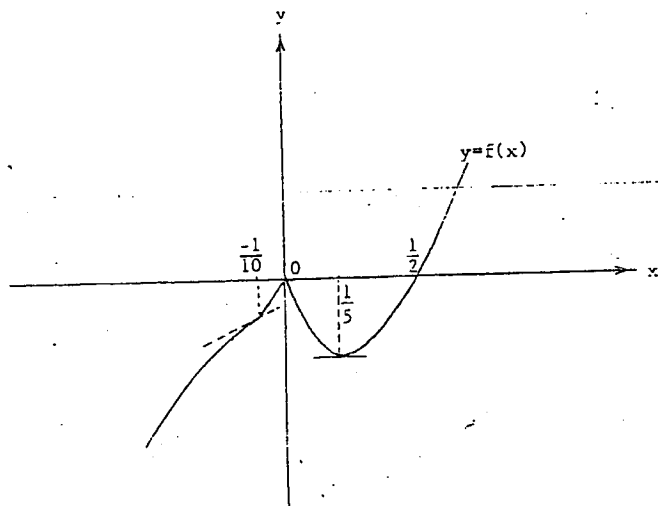
$$= \lim_{x \rightarrow \infty} \left( 2x^{\frac{2}{3}} - \frac{1}{x^{\frac{1}{3}}} \right)$$

As  $\lim_{x \rightarrow \infty} 2x^{\frac{2}{3}} = \infty$  and  $\lim_{x \rightarrow \infty} \frac{1}{x^{\frac{1}{3}}} = 0$ ;  $m$  does not exist.

$\therefore$  There is no slant asymptote.

There is no asymptote.

(f)



13. (a)  $f(x) = \frac{1}{\sqrt{1+x^2}}$

$$f'(x) = \frac{-x}{(1+x^2)^{\frac{3}{2}}}$$

$$= \frac{-x}{1+x^2} \cdot \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{-x f(x)}{1+x^2}$$

$$(1+x^2)f'(x) + x^2 f(x) = 0 \quad \dots (*)$$

Differentiate both sides of (\*) with respect to  $x$  by  $n$  times,

$$(1+x^2)f^{(n+1)}(x) + C_1^n(2x)f^{(n)}(x) + C_1^n(2)f^{(n-1)}(x) \\ + x f^{(n)}(x) + C_1^n f^{(n-1)}(x) = 0$$

$$(1+x^2)f^{(n+1)}(x) + (2n+1)x f^{(n)}(x) + n^2 f^{(n-1)}(x) = 0$$

(b) (i)  $P_{n+1}(x) = (1+x^2)^{n+\frac{1}{2}} f^{(n+1)}(x)$

$$P_n'(x) = (1+x^2)^{n+\frac{1}{2}} f^{(n+1)}(x) + (n+\frac{1}{2})(2x)(1+x^2)^{n-\frac{1}{2}} f^{(n)}(x) \\ = (1+x^2)^{n+\frac{1}{2}} f^{(n+1)}(x) + (2n+1)x(1+x^2)^{n-\frac{1}{2}} f^{(n)}(x)$$

$$\therefore (1+x^2)P_n'(x) - (2n+1)xP_n(x)$$

$$= (1+x^2)^{n+\frac{3}{2}} f^{(n+1)}(x) + (2n+1)x(1+x^2)^{n+\frac{1}{2}} f^{(n)}(x) \\ - (2n+1)x(1+x^2)^{n+\frac{1}{2}} f^{(n)}(x)$$

$$= (1+x^2)^{n+\frac{3}{2}} f^{(n+1)}(x)$$

$$= P_{n+1}(x)$$

Induce on  $n$ ,

(ii) when  $n = 0$

$$P_0(x) = (1+x^2)^{\frac{1}{2}} f^{(0)}(x) \\ = 1$$

$$= (-1)^0 0! x^0$$

It is true for  $n = 0$ .

$$(2) \text{ Assume } P_k(x) = (-1)^k k! x^k + a_{k-1} x^{k-1} + \dots + a_1 x + a_0$$

$$\therefore P_k'(x) = (-1)^k k! k x^{k-1} + a_{k-1} (k-1) x^{k-2} + \dots + a_1$$

$$P_{k+1}(x) = (1+x^2) [(-1)^k k! k x^{k-1} + a_{k-1} (k-1) x^{k-2} + \dots + a_1] \\ - (2k+1)x [(-1)^k k! x^k + a_{k-1} x^{k-1} + \dots + a_1 x + a_0]$$

$$= x^{k+1} [(-1)^k k! k - (-1)^k k! (2k+1)] \\ + x^k [a_{k-1} (k-1) - (2k+1) a_{k-1}] + \dots$$

$$= x^{k+1} [(-1)^k k! (-k)] + \dots$$

$$= (-1)^{k+1} (k+1)! x^{k+1} + \dots$$

$\therefore$  It is also true for  $n = k+1$ .

Hence,  $P_n(x)$  is a polynomial of degree  $n$  with leading coefficient  $(-1)^n n!$ .

$$(ii) P_{n+1}(x) + (2n+1)xP_n(x) - n^2(1+x^2)P_{n-1}(x) \\ = (1+x^2)^{n+\frac{1}{2}} f^{(n+1)}(x) + (2n+1)x(1+x^2)^{n+\frac{1}{2}} f^{(n)}(x) \\ + n^2(1+x^2)(1+x^2)^{n-\frac{1}{2}} f^{(n-1)}(x) \\ = (1+x^2)^{n+\frac{1}{2}} [(1+x^2)f^{(n+1)}(x) + (2n+1)xf^{(n)}(x) \\ + n^2 f^{(n-1)}(x)]$$

$$= 0$$

(by (a))

When  $x = 0$ , we have

$$P_{n+1}(0) + n^2 P_{n-1}(0) = 0$$

$$P_{n+1}(0) = -n^2 P_{n-1}(0)$$

$$\therefore P_n(0) = -(n-1)^2 P_{n-2}(0)$$

$$\text{When } n \text{ is even, } P_n(0) = (-1)(n-1)^2 P_{n-2}(0)$$

$$= (-1)^2 (n-1)^2 (n-3)^2 P_{n-4}(0)$$

$$= \dots$$

$$= (-1)^{\frac{n}{2}} (n-1)^2 (n-3)^2 \dots 1^2 P_0(0)$$

$$= (-1)^{\frac{n}{2}} (n-1)^2 (n-3)^2 \dots 1^2$$

$$(\because P_0(0) = 1)$$

$$\text{When } n \text{ is odd, } P_n(0) = (-1)^{\frac{n-1}{2}} (n-1)^2 (n-3)^2 \dots 2^2 P_1(0)$$

$$= (-1)^{\frac{n-1}{2}} (n-1)^2 (n-3)^2 \dots 2^2 \cdot (1+0^2)^{\frac{1}{2}} f'(0)$$

$$= 0 \quad (\because f'(0) = 0)$$

$$(iii) \text{ From (b) (i), } P_n'(x) = \frac{1}{1+x^2} [P_{n+1}(x) + (2n+1)xP_n(x)]$$

$$= \frac{1}{1+x^2} [-n^2(1+x^2)P_{n-1}(x)] \quad (\text{by (b)(ii)})$$

$$= -n^2 P_{n-1}(x)$$

$$P_n^{(r)}(x) = \frac{d^{r-1}}{dx^{r-1}} [-n^2 P_{n-1}(x)]$$

$$= \frac{d^{r-2}}{dx^{r-2}} [-n^2 P_{n-1}'(x)]$$

$$= \frac{d^{r-2}}{dx^{r-2}} [(-n^2) [-(n-1)^2] P_{n-2}(x)]$$

$$= \frac{d^{r-1}}{dx^{r-1}} [(-n^2) [-(n-1)^2] P_{n-2}'(x)]$$

$$= \frac{d^{r-1}}{dx^{r-1}} [(-n^2) [-(n-1)^2] [-(n-2)^2] P_{n-3}(x)]$$

$$= \dots$$

$$= (-n^2) [-(n-1)^2] [-(n-2)^2] \dots [-(n-(r-1))^2] P_{n-r}(x)$$

$$= (-1)^r [n(n-1)(n-2) \dots (n-r+1)]^2 P_{n-r}(x)$$

(iv) From Maclaurin's Expansion,

$$P_n(x) = \sum_{r=0}^n \frac{P_n^{(r)}(0)}{r!} x^r$$

case (i) when  $n$  is even, let  $n = 2m$

$$\therefore P_n(x) = \sum_{r=0}^m \frac{P_{2m}^{(2r)}(0)}{(2r)!} x^{2r} + \sum_{r=1}^m \frac{P_{2m}^{(2r-1)}(0)}{(2r-1)!} x^{2r-1}$$

$$\text{From (b)(iii), } P_{2m}^{(2r-1)}(0) = (-1)^{2r-1} [2m(2m-1)\dots(2m-2r+2)]^2 P_{2m-2r+1}(0)$$

As  $2m-2r+1$  is odd, from (b)(ii),  $P_{2m-2r+1}(0) = 0$

$$\text{Hence } P_n(x) = \sum_{r=0}^m \frac{P_{2m}^{(2r)}(0)}{(2r)!} x^{2r}$$

As  $x^{2r} = (-x)^{2r}$ ,  $P_n(x)$  is an even function.

case (ii), when  $n$  is odd, let  $n = 2m + 1$

$$\therefore P_n(x) = \sum_{r=0}^m \frac{P_{2m+1}^{(2r)}(0)}{(2r)!} x^{2r} + \sum_{r=0}^m \frac{P_{2m+1}^{(2r+1)}(0)}{(2r+1)!} x^{2r+1}$$

$$\text{Consider } P_{2m+1}^{(2r)}(0) = (-1)^{2r} [(2m+1)(2m)\dots(2m-2r)]^2 P_{2m+1-2r}(0)$$

As  $2m+1-2r$  is odd, from (b)(ii)  $P_{2m+1-2r}(0) = 0$

$$\therefore P_n(x) = \sum_{r=0}^m \frac{P_{2m+1}^{(2r+1)}(0)}{(2r+1)!} x^{2r+1}$$

$$\begin{aligned} \text{Since } P_{2m+1}^{(2r+1)}(0) &= (-1)^{2r+1} [(2m+1)(2m)\dots(2m-2r+1)]^2 P_{2m-2r}(0) \\ &= (-1)^{2r+1} [(2m+1)(2m)\dots(2m-2r+1)]^2 (-1)^{m-r} (2m-2r-1)!! \\ &\neq 0 \end{aligned}$$

$$\text{and } (-x)^{2r+1} = (-1)^{2r+1} x^{2r+1}$$

$$= -x^{2r+1}$$

$P_n(x)$  is an odd function.