Solutions	Marks	Remarks
1. (a) $AB^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 2 & -1 \end{pmatrix}$		
$= \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}$	1A	
$B^TA = \begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$		
$= \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix}$	1A	
(b) $\lvert AB^T \rvert = 9 \neq 0$. AB^T is invertible and	1M	
$(AB^T)^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$	1A	
$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$		
As $\lvert B^TA \rvert = 0$, B^TA is not invertible.	1M 5	
2. $\left[\prod_{k=1}^{n} (a^k + b^k) \right]^2 = \left[\prod_{k=1}^{n} (a^k + b^k) \right]\left[\prod_{k=1}^{n} (a^{n+1-k} + b^{n+1-k}) \right]$	2	
$= \prod_{k=1}^{n} (a^{n+1} + b^{n+1} + a^k b^{n+1-k} + a^{n+1-k} b^k)$	1A	
$> \prod_{k=1}^{n} (a^{n+1} + b^{n+1})$ as $a, b > 0$	1A	
$= (a^{n+1} + b^{n+1})^n$	1A 5	

Solutions	Marks	Remarks
1. (a) $\lim_{x\to\infty} x\left[\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}\right]=\lim_{x\to\infty} x\left[\frac{(1+\frac{1}{x})-(1-\frac{1}{x})}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}\right]$	1M	May use L'Hospital's rule
$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}$		
$=1$ (as $\lim_{x\to\infty}\frac{1}{x}=0$)	1A	
(b) $\lim_{n\to\infty}\frac{n}{1+nh+\frac{n(n-1)}{2}h^2}=\lim_{n\to\infty}\frac{1}{\frac{1}{n}+h+\frac{n-1}{2}h^2}$		
$=0$	1A	
Now $0<\frac{n}{(1+h)^n}=\frac{n}{1+nh+\frac{n(n-1)}{2}h^2+\ldots\text{ positive terms}}$		
$\leq\frac{n}{1+nh+\frac{n(n-1)}{2}h^2}$ for $n\geq 2$	1M	Accept L'Hospital's Rule
As $\lim_{n\to\infty}\frac{n}{1+nh+\frac{n(n-1)}{2}h^2}=0$, by the sandwich theorem,		
$\lim_{n\to\infty}\frac{n}{(1+h)^n}=0$	1A 5	

Solutions	Marks	Remarks
4. The system has infinitely many solutions only if the determinant of its coeff. matrix is zero.	1M	
$\begin{vmatrix}1&1&3\\4&h&-1\\6&7&5\end{vmatrix}=-13h+65$	1A	
$=0$		
iff $h=5$	1A	
Now $6x+7y+5z=2(x+y+3z)+4x+5y-z$	1M	
For the system to have infinitely many solutions,		
$2=2k+1$	1M	
$\therefore k=\frac{1}{2}$	1A 6	

Solutions	Marks	Remarks
5. The number of 4-digit numbers formed $=P^7_4=840$.	1A	
For a number to be divisible by 3, the sum of its digits must be divisible by 3.	1	
Now $1+2+3+4+5+6+7=28$		
For the sum to be 21, we have $21=28-7-2-4$	1M	For first given combination
There are $P^4_4=24$ numbers.	1A	
Similarly,		
$18=28-1-2-7=28-1-3-6=28-1-4-5$ $=28-2-3-5$		
There are $4\times P^4_4=96$ numbers.	1A	
$15=28-1-5-7=28-2-4-7=28-2-5-6$ $=28-3-4-6$		
There are 96 numbers.		
$12=28-3-6-7=28-4-5-7$		
There are 48 numbers.		
Altogether there are 264 numbers.	1A 6	
Alternatively: Possible combinations are: $\{1,2,3,6\}\{1,2,4,5\}\{1,2,5,7\}\{1,3,4,7\}$ $\{1,3,5,6\}\{1,4,6,7\}\{2,3,4,6\}\{2,3,6,7\}$ $\{2,4,5,7\}\{3,4,5,6\}\{3,5,6,7\}$	1A + 1A	← For any combination

Solutions	Marks	Remarks
6. $\frac{1}{2}(z+z^{-1})=\frac{1}{2}[(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)]$		OR $\frac{1}{2}(z+\bar{z})$
$=\cos\theta$	1A	
$\therefore \cos^n\theta=\frac{1}{2^n}(z+z^{-1})^n$	1A	
$=\frac{1}{2^n}\sum_{r=0}^{n}C^n_r z^{n-r}z^{-r}$	1A	
$=\frac{1}{2^n}\sum_{r=0}^{n}C^n_r z^{n-2r}$	1A	
$=\frac{1}{2^n}\sum_{r=0}^{n}C^n_r[\cos(n-2r)\theta+i\sin(n-2r)\theta]$	1A	
$=\frac{1}{2^n}\sum_{r=0}^{n}C^n_r\cos(n-2r)\theta$ as $\cos^n\theta$ is real.	1A 6	

Solutions	Marks	Remarks

7. (a) (i) For any z, z' and z'' in $\mathbb{C}$,

(1) $z\ S\ z$ as $Re(z) \le Re(z)$ — 1A

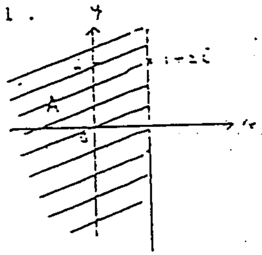
(2) if $z\ S\ z'$ and $z'\ S\ z''$,

then $Re(z) \le Re(z') \le Re(z'')$

$\Rightarrow z\ S\ z''$. — 1A

Hence S is both reflexive and transitive.

(ii) $z\ S\ (1 + 2i)$ iff $Re(z) \le 1$. — 1A



(b) For any z, z', z'' in $\mathbb{C}$,

(1) $z \sim z$ as $z\ S\ z$ and $z\ S\ z$ by (a)

(2) if $z \sim z'$, then $z\ S\ z'$ and $z'\ S\ z$

$\therefore$ $z'\ S\ z$ and $z\ S\ z'$ and hence $z' \sim z$ — 1A

(3) if $z \sim z'$ and $z' \sim z''$, then ($z\ S\ z'$ and $z'\ S\ z$)

and ($z'\ S\ z''$ and $z''\ S\ z'$)

i.e. ($z\ S\ z'$ and $z'\ S\ z''$) and ($z''\ S\ z'$ and $z'\ S\ z$)

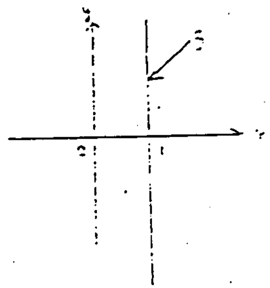
$\therefore$ $z\ S\ z''$ and $z''\ S\ z$ as S is transitive.

$\therefore$ $z \sim z''$ — 1A

Thus $\sim$ is an equivalence relation. — 1A

$z \sim (1 + 2i)$ iff $Re(z) = 1$ — 1A

The set B is the line $x = 1$.



Total: 7

Solutions	Marks	Remarks

8. (a)(i) For any $\theta, \phi \in \mathbb{R}$, $A(\theta)A(\phi)$

$= [I - (\sin\theta)S + (1 - \cos\theta)S^2][I - (\sin\phi)S + (1 - \cos\phi)S^2]$

$= I^2 - (\sin\theta + \sin\phi)S + [\sin\theta\sin\phi + (1-\cos\theta) + (1-\cos\phi)]S^2$

$- [\sin\theta(1-\cos\phi) + \sin\phi(1-\cos\theta)]S^3 + (1-\cos\theta)(1-\cos\phi)S^4$ — 1

$= I - (\sin\theta\cos\phi + \cos\theta\sin\phi)S + (1 + \sin\theta\sin\phi - \cos\theta\cos\phi)S^2$ (as $S^3 = -S$) — 1

$= I - \sin(\theta + \phi)S + (1 - \cos(\theta + \phi))S^2$

$= A(\theta + \phi)$ — 1

(ii) We shall prove by induction. The case where $n=1$ is trivial.

Assume $[A(\theta)]^k = A(k\theta)$ for some positive integer k.

Then $[A(\theta)]^{k+1} = [A(\theta)]^k A(\theta)$

$= A(k\theta)A(\theta)$ — 1

$= A((k + 1)\theta)$ by (i) — 1

Hence $[A(\theta)]^n = A(n\theta)$ $\forall n \ge 1$

(iii) For any $\theta \in \mathbb{R}$, $A(-\theta) = [A(\theta)]^{-1}$ — 1

as $A(-\theta)A(\theta) = A(-\theta + \theta)$

$= A(0)$

$= I$ — 1

Total: 7

(b)(i) $T^3 = T^2 T$

$$= \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$ — 1

$$= \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} = -T \quad \text{i.e. } T^3 + T = 0$$ — 1

(ii) As $T^3 + T = 0$, putting $S = T$ and $\theta = \frac{3\pi}{2}$ in (a).

$I + T + T^2 = A(\frac{3\pi}{2})$. — 2

(1) By (a)(iii) $(I + T + T^2)^{-1} = A(-\frac{3\pi}{2})$ — 1

$= I - T + T^2$ — 1

(2) $(I + T + T^2)^{1969} = A(\frac{3\pi}{2} \times 1969)$ — 1

$= A(1491 \times 2\pi + \frac{3\pi}{2})$

$= A(\frac{3\pi}{2})$

$= I + T + T^2$ — 1

Total: 8

Solutions	Marks	Remarks
9. (a) Let $f(x) = x^n + x + 1$.		
$f'(x) = nx^{n-1} + 1$		
$> 0 \quad \forall x$ if n is odd.	1	
$\therefore f(x)$ increases monotonically.		
Further $f(-1) < 0$ and $f(1) > 0$, $f(x)=0$ has exactly one real root.	1	
Suppose n is even.		
$f'(x) = 0$ iff $x = -\sqrt[n-1]{\frac{1}{n}}$		
$f''(x) = n(n-1)x^{n-2}$		
> 0 at $x = -\sqrt[n-1]{\frac{1}{n}}$		
As f is continuously differentiable, f attains its absolute minimum at $x = -\sqrt[n-1]{\frac{1}{n}}$	1	
$f(-\sqrt[n-1]{\frac{1}{n}}) = (\sqrt[n-1]{\frac{1}{n}})^n - \sqrt[n-1]{\frac{1}{n}} + 1$		
$> (\sqrt[n-1]{\frac{1}{n}})^n$		
> 0	1	
$\therefore f(x) > 0 \quad \forall x$.		
i.e. $f(x) = 0$ has no real root.	1 / 5	
(b) (i) If α is a root of (*),		
$\alpha^n + \alpha + 1 = 0 \Rightarrow \overline{\alpha^n + \alpha + 1} = 0$	1	
$\Rightarrow \overline{\alpha^n} + \overline{\alpha} + \overline{1} = 0$		
$\Rightarrow (\overline{\alpha})^n + \overline{\alpha} + 1 = 0$.	1	
$\therefore \overline{\alpha}$ is also a root of (*).		
Now $\{\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_n\} \subset \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ as the latter is the set of all roots.	1	
On the other hand, for any r, α_r is a root $\Rightarrow \overline{\alpha}_r$ is also a root, i.e. $\overline{\alpha}_r \in \{\alpha_1, \alpha_2, \dots, \alpha_n\}$.		
$\therefore \alpha_r = \overline{\overline{\alpha}}_r \in \{\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_n\}$		
Thus $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \{\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_n\}$	1	

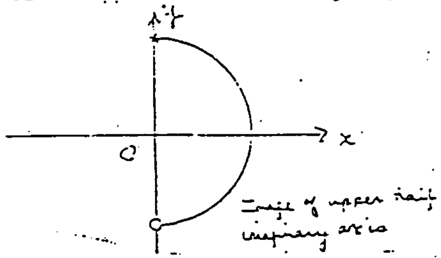
Solutions	Marks	Remarks
9. (b) (ii) For any integer k,		
$\overline{(\sum_{r=1}^{n}\alpha_r^k)} = (\sum_{r=1}^{n}(\overline{\alpha}_r)^k) = \sum_{r=1}^{n}\alpha_r^k$ by (i) ($\alpha_r \neq 0$)	1	
$\therefore \sum_{r=1}^{n}\alpha_r^k$ is real	1	
(iii) $\alpha_r \neq 0$		OR
(1) $\sum_{r=1}^{n}\frac{1}{\alpha_r} = \frac{\sum_{r=1}^{n}(\prod_{j\neq r}\alpha_j)}{\prod_{r=1}^{n}\alpha_r}$		put $y = \frac{1}{x}$ in (*)
$= \frac{(-1)^{n-1}}{(-1)^n}$	1 / 1	
$= -1$		
(2) From (*), $\alpha_r^{n-1} + 1 + \frac{1}{\alpha_r} = 0$.		
$\sum_{r=1}^{n}\alpha_r^{n-1} = -\sum_{r=1}^{n}(1 + \frac{1}{\alpha_r})$	1	
$= -n - (-1)$		
$= -n + 1$	1 / 10	

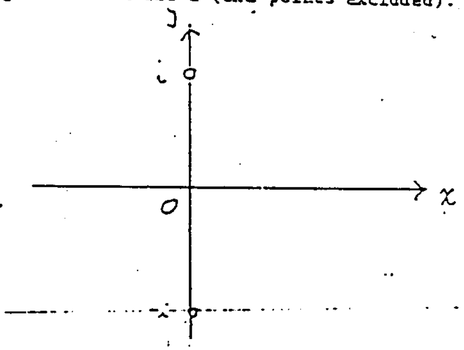
Solutions	Marks	Remarks
10. (a) $f(x) = e^{x-1} - x$		
$f'(x) = e^{x-1} - 1$	1	
$f''(x) = e^{x-1}$		
$f'(x) = 0$ iff $x = 1$ at which	1	
$f''(x) = e^{0} > 0$		
As $f(x)$ is continuously differentiable in $\mathbb{R}$, $f(x) \geq f(1)$		
$\Rightarrow e^{x-1} - x \geq e^{1-1} - 1 = 0$, i.e. $e^{x-1} \geq x \quad \forall x \in \mathbb{R}$	$\frac{1}{3}$	
(b) As $b_i \neq 0$, put $x_i = \frac{a_i}{b_i}$ $(i = 1, 2, \ldots, n)$ in (a).		
$e^{\left(\frac{a_i}{b_i} - 1\right)} \geq \frac{a_i}{b_i}$	1	
$\prod_{i=1}^{n} e^{\left(\frac{a_i}{b_i} - 1\right)} \geq \prod_{i=1}^{n} \frac{a_i}{b_i}$ as $\frac{a_i}{b_i} > 0$	1	
$\therefore e^{\left\{\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right) - n\right\}} \geq \prod_{i=1}^{n} \frac{a_i}{b_i}$	1	
If $\sum_{i=1}^{n} \frac{a_i}{b_i} \leq n$, $1 \geq e^{\left\{\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right) - n\right\}} \geq \frac{\prod_{i=1}^{n} a_i}{\prod_{i=1}^{n} b_i}$		
$\Rightarrow \prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i$ as $a_i, b_i > 0$.	$\frac{1}{4}$	
(c) (i) For $i = 1, 2, \ldots, n$, put $b_i = \frac{1}{n} \sum_{r=1}^{n} a_r = m$ (> 0) in (b).	1	
Then $\sum_{i=1}^{n} \frac{a_i}{b_i} = \frac{1}{m} \sum_{i=1}^{n} a_i = n \leq n$	1	
By (b), $\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} m = m^n$	1	
$= \left[\frac{1}{n} \sum_{i=1}^{n} a_i\right]^n$		
$\Rightarrow \left[\prod_{i=1}^{n} a_i\right]^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$	1	

Solutions	Marks	Remarks
10. (c) (ii) Consider the positive numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$.	1	
By (i), $\frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_i} \geq \left[\prod_{i=1}^{n} \frac{1}{a_i}\right]^{\frac{1}{n}}$	1	
$= \frac{1}{\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}}$		
$\geq \frac{1}{\frac{1}{n} \sum_{i=1}^{n} a_i}$	1	
$= \frac{1}{m}$		
$\therefore \sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{n}{m}$		
$\Rightarrow \sum_{i=1}^{n} \left(\frac{1}{a_i} - \frac{1}{m}\right) \geq 0.$	$\frac{1}{3}$	

Solutions	Marks	Remarks
11. (a) We first prove the uniqueness. Suppose ∃ integers a, b, c, d such that $a\sqrt{2} + b = c\sqrt{2} + d$.	1	
Then $(a - c)\sqrt{2} = d - b$.		
As $\sqrt{2}$ is irrational, $(a - c)$, $(d - b)$ are integers only if $a - c = d - b = 0$, i.e. $a = c$ and $b = d$.	1	
Hence the uniqueness.		
Next observe that the given statement is true for $n = 1$ with $a_1 = b_1 = 1$.	1	
Assume that for some $k \geq 1$, $(\sqrt{2} + 1)^k = a_k\sqrt{2} + b_k$, where a_k and b_k are positive integers with b_k odd and $b_k \geq a_k \geq 2^{k-1}$.		
$(\sqrt{2} + 1)^{k+1} = (a_k\sqrt{2} + b_k)(\sqrt{2} + 1)$ $= (a_k + b_k)\sqrt{2} + (2a_k + b_k) = a_{k+1}\sqrt{2} + b_{k+1}$, say	1	
Now $(a_k + b_k)$ and $(2a_k + b_k)$ are positive integers and $2a_k + b_k$ is odd as b_k is odd.	1	
Further $2a_k + b_k \geq a_k + b_k \geq 2a_k \geq 2^k$.	1	
Thus the statement is true ∀ positive n.		
To prove that a_n is odd for even n, first $a_1 = 1$ is odd.		
Assume that a_k is odd for some odd k,		
$(\sqrt{2} + 1)^{k+2} = (a_k\sqrt{2} + b_k)(2\sqrt{2} + 3)$ $= (3a_k + 2b_k)\sqrt{2} + (4a_k + 3b_k)$	1	
As a_k is odd, $3a_k + 2b_k$ is odd.	1	
The answer follows.	6	
(b) For $n = 1$, $(\sqrt{2} - 1)^1 = (-1)^2(1 \times \sqrt{2} - 1)$		
Suppose $(\sqrt{2} - 1)^k = (-1)^{k+1}(a_k\sqrt{2} - b_k)$, $k \geq 1$.		
$(\sqrt{2} - 1)^{k+1} = (-1)^{k+1}(a_k\sqrt{2} - b_k)(\sqrt{2} - 1)$	2	
$= (-1)^{k+1}[-(a_k + b_k)\sqrt{2} + (2a_k + b_k)]$		
$= (-1)^{k+2}(a_{k+1}\sqrt{2} - b_{k+1})$ by (a).	1	
Thus $(\sqrt{2} - 1)^n = (-1)^{n+1}(a_n\sqrt{2} - b_n)$ ∀ $n \geq 1$.		

Solutions	Marks	Remarks
11. (b) Now $0 < (\sqrt{2} - 1) < \frac{1}{2}$	1	
$\Rightarrow 0 < (\sqrt{2} - 1)^n < \frac{1}{2^n}$	1	
$\Rightarrow \lvert a_n\sqrt{2} - b_n \rvert < \frac{1}{2^n}$	1	
$\Rightarrow \left\lvert \sqrt{2} - \frac{b_n}{a_n} \right\rvert < \frac{1}{a_n 2^n}$		
$< \frac{1}{2^{2n-1}}$ by (a)	1	
	7	

Solutions	Marks	Remarks
12. (a) For any $z_1, z_2 \in \mathbb{C}\backslash\{-1\}$, $f(z_1) = f(z_2)$		
$\Rightarrow \dfrac{i(1-z_1)}{1+z_1} = \dfrac{i(1-z_2)}{1+z_2}$	1	
$\Rightarrow 1 - z_1 + z_2 - z_1 z_2 = 1 + z_1 - z_2 - z_1 z_2$		
$\Rightarrow z_1 = z_2$		
Hence f is injective.	1	
For any $w \in \mathbb{C}\backslash\{-i\}$, consider $w = \dfrac{i(1-z)}{1+z}$.		
Changing subject, we have $z = \dfrac{i-w}{i+w}$. (as $w \neq -i$)	1	
As $z \neq -1$ and $f(z) = w$, f is surjective and thus bijective.	1 / 4	
(b) (i) Let $z = ti$, $t \geq 0$, be any point on the upper half of the imaginary axis	1	OR May use $z + \bar{z} = 0$
$f(z) = \dfrac{i(1-ti)}{1+ti}$		iff $\dfrac{i-w}{i+w} + \overline{\left(\dfrac{i-w}{i+w}\right)} = 0$
$= \dfrac{2t + (1-t^2)i}{1+t^2}$	1	iff $w\bar{w} = 1$
$= x + iy$ where $x = \dfrac{2t}{1+t^2}$, $y = \dfrac{1-t^2}{1+t^2}$		
We see that $x^2 + y^2 = \dfrac{4t^2}{(1+t^2)^2} + \dfrac{1-2t^2+t^4}{(1+t^2)^2} = 1$.	1	etc.
As $x = \dfrac{2t}{1+t^2} \geq 0$, f(z) lies on right half of the unit circle (including the end point i)	1	
For any point $w = x + iy$ ($\neq -i$) on the right half of the unit circle, we have $x^2 + y^2 = 1$, $x \geq 0$.		
By (a), the pre-image of w is given by		
$z = \dfrac{i-w}{i+w}$		
$= \dfrac{i-(x+iy)}{i+(x+iy)}$		
$= \dfrac{-x+(1-y)i}{x+(1+y)i}$		
$= \dfrac{-x^2-y^2+1-2xi}{x^2+(1+y)^2}$		
$= \dfrac{2xi}{x^2+(1-y)^2}$ $(x \geq 0)$ as $x^2+y^2=1$		
∴ z lies on the upper half of the imaginary axis.	1	
	1	

Solutions	Marks	Remarks
12. (b) (ii) Let $z = t$, $t > 0$.	1	OR
$f(z) = \dfrac{i(1-t)}{1+t}$ lies on the imaginary axis.	1	May show tha
Further $-1 < \dfrac{1-t}{1+t} < 1$,		$z = \bar{z}$ iff
i.e. f(z) lies between $-i$ and i (end points excluded).	1	$i(w + \bar{w}) = 0$
For any $w = yi$, $-1 < y < 1$,		
$f^{-1}(w) = \dfrac{i-yi}{i+yi} = \dfrac{1-y}{1+y} > 0$.	1	
The image is exactly the part of the imaginary axis lying between $-i$ and i (end points excluded).		
	1	
	11	

Solutions	Marks	Remarks
13. (a) (i) $T(\underline{0}) = T(\underline{0}+\underline{0}) = T(\underline{0}) + T(\underline{0})$	1	
$\Rightarrow T(\underline{0}) = \underline{0}$	1	
(ii) For any $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^3$ and $\alpha, \beta, \gamma \in \mathbb{R}$,		
$T(\alpha\underline{x} + \beta\underline{y} + \gamma\underline{z}) = T(\alpha\underline{x} + \beta\underline{y}) + T(\gamma\underline{z})$		
$= \alpha T(\underline{x}) + \beta T(\underline{y}) + \gamma T(\underline{z})$	1	
(iii) For any linearly dependent $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^3$, $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$ (not all zero) such that		
$\alpha\underline{x} + \beta\underline{y} + \gamma\underline{z} = \underline{0}$	1	
$T(\alpha\underline{x} + \beta\underline{y} + \gamma\underline{z}) = \underline{0}$		
$\therefore \alpha T(\underline{x}) + \beta T(\underline{y}) + \gamma T(\underline{z}) = \underline{0}$		
i.e. $T(\underline{x}), T(\underline{y}), T(\underline{z})$ are linearly dependent.	1	
	5	
(b) To prove (1) $\Rightarrow$ (2), suppose T is injective. For any linearly independent $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^3$ and $\alpha, \beta, \gamma \in \mathbb{R}$,		
$\alpha T(\underline{x}) + \beta T(\underline{y}) + \gamma T(\underline{z}) = \underline{0}$	1	
$\Rightarrow T(\alpha\underline{x} + \beta\underline{y} + \gamma\underline{z}) = \underline{0}$	1	
$\Rightarrow \alpha\underline{x} + \beta\underline{y} + \gamma\underline{z} = \underline{0}$ by (a) and injectivity of T	1	
$\Rightarrow \alpha = \beta = \gamma = 0$ as $\underline{x}, \underline{y}, \underline{z}$ are linearly independent.	1	
Hence $T(\underline{x}), T(\underline{y}), T(\underline{z})$ are linearly independent.		
To prove (2) $\Rightarrow$ (3), observe that $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are linearly independent because if $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$ such that		
$\alpha\underline{e}_1 + \beta\underline{e}_2 + \gamma\underline{e}_3 = \underline{0}$,		
then $(\alpha, \beta, \gamma) = \underline{0}$		
i.e. $\alpha = \beta = \gamma = 0$	1	
$\therefore$ by (2), $T(\underline{e}_1), T(\underline{e}_2), T(\underline{e}_3)$ are linearly independent.	1	
To prove (3) $\Rightarrow$ (1), suppose $T(\underline{x}) = T(\underline{y})$ for some $\underline{x}, \underline{y} \in \mathbb{R}^3$.		
$\exists\, \alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that		
$\underline{x} = \sum_{i=1}^{3} \alpha_i \underline{e}_i$, $\underline{y} = \sum_{i=1}^{3} \beta_i \underline{e}_i$	1	
Now $T(\underline{x}) = T(\underline{y}) \Rightarrow T(\Sigma \alpha_i \underline{e}_i) = T(\Sigma \beta_i \underline{e}_i)$		
$\Rightarrow \Sigma \alpha_i T(\underline{e}_i) = \Sigma \beta_i T(\underline{e}_i)$	1	
$\Rightarrow \Sigma(\alpha_i - \beta_i) T(\underline{e}_i) = \underline{0}$	1	
$\Rightarrow \alpha_i = \beta_i \;\forall\, i$ as $T(\underline{e}_i)$ are linearly independent by assumption.	1	
	10	

89 II

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八九年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1989

Pure Mathematics (Paper II)

MARKING SCHEME

This is a restricted document.

It is meant for use by markers of this paper for marking purposes only.

Reproduction in any form is strictly prohibited.

Special Note for Teacher Markers

It is highly undesirable that this marking scheme should fall into the hands of students. They are likely to regard it as a set of model answers, which it certainly is not.

Markers should therefore resist pleas from their students to have access to this document. Making it available would constitute misconduct on the part of the marker and is, moreover, in breach of the 1977 Hong Kong Examinations Authority Ordinance.

89 II

Solutions	Marks	Remarks
1. f is continuously differentiable for $x > 0$ and		
$f'(x) = \dfrac{x^e \cdot e^x - e \cdot x^{e-1} \cdot e^x}{x^{2e}}$		
$= \dfrac{e^x(x-e)}{x^{e+1}}$	1A	
$f'(x) = 0$ iff $x = e$.	1A	
For $0 < x < e$, $f'(x) < 0 \Rightarrow f$ is strictly decreasing there; For $x > e$, $f'(x) > 0 \Rightarrow f$ is strictly increasing there	2	May consider $f''(x) = \dfrac{e^x}{x^2}\left[\left(1-\dfrac{e}{x}\right)^2 + \dfrac{e}{x^2}\right]$
$\therefore f(x) > f(e) = 1$ if $x \neq e$		
Now $f(\pi) = \dfrac{e^\pi}{\pi^e} > f(e) = 1$.		
$\Rightarrow e^\pi > \pi^e$ (as $\pi^e > 0$)	1A	
	5	
2. $\dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)}$		
$= \dfrac{1}{3}\left(\dfrac{1}{x+1} - \dfrac{x-2}{x^2-x+1}\right)$	1M+1A	1M for attempt to solve by partial fractions
$\therefore \dfrac{1}{3}\int\left(\dfrac{1}{x+1} - \dfrac{x-2}{x^2-x+1}\right)dx$		
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{3}\int \dfrac{x-2}{x^2-x+1}\,dx$	1A	For $\int \frac{1}{x+1}dx = \ln\lvert x+1\rvert$
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{3}\int\left(\dfrac{x-\frac{1}{2}}{x^2-x+1} - \dfrac{\frac{3}{2}}{x^2-x+1}\right)dx$	1M	
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{6}\ln\lvert x^2-x+1\rvert + \dfrac{1}{2}\int \dfrac{1}{x^2-x+1}\,dx$		
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{6}\ln\lvert x^2-x+1\rvert + \dfrac{1}{2}\int \dfrac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx$		
$= \dfrac{1}{6}\ln\left\lvert\dfrac{(x+1)^2}{x^2-x+1}\right\rvert + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right) + C$	1A	
	5	

Solutions	Marks	Remarks
(a) Consider x fixed and put $u = xt$. Then $du = x\,dt$. $t = \frac{1}{x} \rightarrow u = 1$; $t = x \rightarrow u = x^2$	1A	
$\therefore f(x) = \displaystyle\int_1^{x^2} \sin\sqrt{u}\,\frac{du}{x}$		
$= \dfrac{1}{x}\displaystyle\int_1^{x^2} \sin\sqrt{u}\,du$	1A	
(b) $\dfrac{df}{dx} = -\dfrac{1}{x^2}\displaystyle\int_1^{x^2}\sin\sqrt{u}\,du + \dfrac{1}{x}\cdot 2x\cdot\sin\sqrt{x^2}$	1A+1A	
$= 2\sin 1$ at $x = 1$ (as $\int_1^1 \sin\sqrt{u}\,du = 0$)	1A	Withheld if answer given as 0.0349
$= (1.683)$		
	5	
4. (a) The two curves intersect at $x = 0$ and $x = 1$.		
Area bounded by the curves is $\displaystyle\int_0^1 (\sqrt{x} - x^2)\,dx$	1A	
$= \left[\dfrac{2}{3}x^{\frac{3}{2}} - \dfrac{1}{3}x^3\right]_0^1 = \dfrac{1}{3}$	1A	
(b) $y = \ln\cos x \Rightarrow \dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} = -\tan x$	1A	
Arc length $= \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1 + (-\tan x)^2}\,dx$	1M	
$= \displaystyle\int_0^{\frac{\pi}{4}} \sec x\,dx$		
$= \left[\ln\lvert\sec x + \tan x\rvert\right]_0^{\frac{\pi}{4}}$	1A	
$= \ln(\sqrt{2}+1)$ units $(= 0.881)$	1A	
	6	

Solutions	Marks	Remarks
Differentiating $y(1+x^2)=1$ with respect to x, by Leibnitz rule, $\sum_{r=0}^{n} C_r^n y^{(n-r)}(1+x^2)^{(r)} = 0$.	1A	May use induct:
As $(1+x^2)' = 2x$, $(1+x^2)^{(2)} = 2$, $(1+x^2)^{(r)} = 0$ for $r \geq 3$,		
$(1+x^2)y^{(n)} + n\cdot 2x\cdot y^{(n-1)} + \frac{n(n-1)}{2}\cdot 2y^{(n-2)} = 0$ for $n \geq 2$.	1A	
Now $y^{(n)}(0) = -n(n-1)y^{(n-2)}(0)$ for $n \geq 2$	1A	
$y(0) = 1$		
$y'(0) = 0$		
$\therefore y^{(n)}(0) = 0$ if n is odd.	1A	
If n is even, $y^{(n)}(0) = -n(n-1)y^{(n-2)}(0)$		
$= (-1)^2(n)(n-1)(n-2)(n-3)y^{(n-4)}(0)$	1A	
= etc.		
$= (-1)^{\frac{n}{2}} n!$ $(y^{(0)}(0) = 1)$	1A 6	

Solutions	Marks	Remarks
6. (a) Let (r, θ) be the polar coordinates of a point on Γ .		
Then $x = r\cos\theta$, $y = r\sin\theta$.		
Substituting in Γ , $r^2\sin^2\theta = 1 + 2r\cos\theta$	1A	
$r^2\sin^2\theta - 2r\cos\theta - 1 = 0$		
$r = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta + 4\sin^2\theta}}{2\sin^2\theta}$		
$= \frac{\cos\theta + 1}{\sin^2\theta}$ or $\frac{\cos\theta - 1}{\sin^2\theta}$	1A	For either
i.e. $\frac{1}{1-\cos\theta}$ or $\frac{-1}{1+\cos\theta}$		
Either $r = \frac{1}{1-\cos\theta}$ or $r = \frac{-1}{1+\cos\theta}$ could be the required equation, depending on the restrictions on r.		
(b) Let $r = \frac{1}{1-\cos\theta}$ be the polar equation of Γ .		
Since PQ passes through O, let $P = (r_1, \theta)$, $Q = (r_2, \theta+\pi)$.	1	
We have $r_1 = \frac{1}{1-\cos\theta}$, $r_2 = \frac{1}{1-\cos(\theta+\pi)} = \frac{1}{1+\cos\theta}$	1A	
$\frac{8}{3} = r_1 + r_2$	1M	
$= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}$		
$= \frac{2}{\sin^2\theta}$		
$\sin\theta = \pm\frac{\sqrt{3}}{2}$.		
$\therefore P = (2, \frac{\pi}{3})$, $Q = (\frac{2}{3}, \frac{4\pi}{3})$	1A 6	

Solutions	Marks	Remarks
7. (a) Let $y = [\ln(e+h)]^{\frac{1}{h}}$		
$\ln y = \frac{1}{h}\ln(\ln(e+h))$	1	
$\lim_{h\to 0} \ln y = \lim_{h\to 0} \frac{1}{h}\ln(\ln(e+h))$		
$= \lim_{h\to 0} \frac{\frac{1}{(e+h)\ln(e+h)}}{1}$ (By L'Hospital's Rule)		
$= \frac{1}{e}$	1A	
$\ln \lim_{h\to 0} y = \frac{1}{e}$		
$\to \lim_{h\to 0} y = e^{\frac{1}{e}}$	1A	
(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{1+(\frac{k}{n})^2}\right)$		
$= \int_0^1 \frac{1}{1+x^2}dx$	2A	
$= [\tan^{-1}x]_0^1$	1A	
$= \frac{\pi}{4}$	1A 7	

Solutions	Marks	Remarks
(a) $I_0 = \int_0^1 e^{ax}dx = \frac{1}{a}e^{ax}\Big\|_0^1 = \frac{1}{a}(e^a - 1)$	1	
For $n \geq 1$, $I_n = \int_0^1 x^n e^{ax}dx$		
$= \frac{1}{a}\int_0^1 x^n de^{ax}$	1	
$= \frac{1}{a}x^n e^{ax}\Big\|_0^1 - \frac{n}{a}\int_0^1 x^{n-1}e^{ax}dx$	1	
$= \frac{e^a}{a} - \frac{n}{a}I_{n-1}$	1 / 4	
(b) We shall prove inductively.		
First $I_1 = \frac{e^a}{a} - \frac{1}{a}I_0$ $= -\frac{1}{a^2} + e^a[\frac{1}{a} - \frac{1}{a^2}]$	1	
Hence the statement is true for n = 1.		
Assume that for some $k \geq 1$,		
$I_k = \frac{(-1)^{k+1}k!}{a^{k+1}} + e^a\left[\frac{1}{a} + \sum_{r=1}^{k}\frac{(-1)^r k(k-1)\dots(k-r+1)}{a^{r+1}}\right]$		

Solutions	Marks	Remarks
(b) then $I_{k+1} = \frac{e^a}{a} - \frac{k+1}{a}I_k$	2	
$= \frac{e^a}{a} - \left\{\frac{k+1}{a} \times \frac{(-1)^{k+1}k!}{a^{k+1}} + e^a\left[\frac{k+1}{a} \times \frac{1}{a} + \sum_{r=1}^{k}\frac{(-1)^r(k+1)(k)(k-1)\dots(k-r+1)}{a^{r+2}}\right]\right\}$		
	1	
$= \frac{(-1)^{k+2}(k+1)!}{a^{k+2}} + e^a\left[\frac{1}{a} - \frac{k+1}{a^2} - \sum_{r=2}^{k+1}\frac{(-1)^{r-1}(k+1)(k)(k-1)\dots(k+1-r+1)}{a^{r+1}}\right]$		
	1	
$= \frac{(-1)^{k+2}(k+1)!}{a^{k+2}} + e^a\left[\frac{1}{a} + \sum_{r=1}^{k+1}\frac{(-1)^r(k+1)(k)\dots(k+1-r+1)}{a^{r+1}}\right]$		
	1	
Thus the statement is true for n = k + 1 and hence $\forall\, n \geq 1$.	6	
(c) Put $x = \log\sqrt{u}$.	1	
Then $u = e^{2x}$, $du = 2e^{2x}dx$. When $u = 1$, $x = 0$; when $u = e^2$, $x = 1$.	1	
$\int_1^{e^2}\left(\frac{\log u}{u}\right)^3 du = 16\int_0^1 x^3 e^{-4x}dx$	1	
$= 16I_3$ with $a = -4$	1	
$= 16\left[\frac{(-1)^4 \cdot 3 \cdot 2}{(-4)^4} + e^{-4}\left(\frac{1}{-4} + \frac{-3}{(-4)^2} + \frac{3 \cdot 2}{(-4)^3} - \frac{3 \cdot 2 \cdot 1}{(-4)^4}\right)\right]$		
$= \frac{3}{8} - \frac{71}{8}e^{-4}$ (= 0.2124)	1 / 5	

Solutions	Marks	Remarks
9. (a) Slope of the chord $= \dfrac{\dfrac{t_2^2}{1+t_2^3} - \dfrac{t_1^2}{1+t_1^3}}{\dfrac{t_2}{1+t_2^3} - \dfrac{t_1}{1+t_1^3}}$	1	
$= \dfrac{t_1^2t_2^2 - t_1 - t_2}{t_1t_2(t_1+t_2) - 1}$ (for $t_1 \neq t_2$)	1	
Equation of the chord is $y - \dfrac{t_1^2}{1+t_1^3} = \dfrac{t_1^2t_2^2 - t_1 - t_2}{t_1t_2(t_1+t_2) - 1}\left(x - \dfrac{t_1}{1+t_1^3}\right)$	1	
i.e. $(t_1^2t_2^2 - t_1 - t_2)x + [1 - t_1t_2(t_1+t_2)]y + t_1t_2 = 0$		
Letting $t_1, t_2 \to t$, the equation of the tangent at t is $(t^4 - 2t)x + (1 - 2t^3)y + t^2 = 0$	1 4	
(b) By (a), putting $x = \dfrac{t_3}{1+t_3^3}$, $y = \dfrac{t_3^2}{1+t_3^3}$, a necessary and sufficient condition for the three points to be collinear is $(t_1^2t_2^2 - t_1 - t_2)\dfrac{t_3}{1+t_3^3} + [1 - t_1t_2(t_1+t_2)]\dfrac{t_3^2}{1+t_3^3} + t_1t_2 = 0$	1	
$\Leftrightarrow t_1^2t_2^2t_3 - t_1t_3 - t_2t_3 + t_3^2 - t_1^2t_2t_3^2 - t_1t_2^2t_3^2 + t_1t_2 + t_1t_2t_3^2 = 0$		
$\Leftrightarrow t_1t_2t_3(t_1t_2 - t_1t_3 - t_2t_3 + t_3^2) + (t_1t_2 - t_1t_3 - t_2t_3 + t_3^2) = 0$	1	
$\Leftrightarrow (t_1t_2t_3 + 1)[t_1(t_2 - t_3) - t_3(t_2 - t_3)] = 0$		
$\Leftrightarrow (t_1t_2t_3 + 1)(t_1 - t_3)(t_2 - t_3) = 0$	1	
$\Leftrightarrow t_1t_2t_3 = -1$ as t_1, t_2, t_3 are distinct.	1 4	

Solutions	Marks	Remarks
9. (c) Equation of tangent at t is $(t^4 - 2t)x + (1 - 2t^3)y + t^2 = 0$		OR
Putting $x = \dfrac{T}{1+T^3}$, $y = \dfrac{T^2}{1+T^3}$, the tangent intersects the curve at $P(T)$	1	From (b),
iff $t^2T^3 + (1 - 2t^3)T^2 + (t^4 - 2t)T + t^2 = 0$		$t_1t_2t_3 = -1$.
iff $(T - t)(t^2T^2 - (t^3 - 1)T - t) = 0$	1	Letting $t_1, t_2 \to$ etc.
iff $(T - t)(T - t)(t^2T + 1) = 0$		
iff $T = t$ or $-\dfrac{1}{t^2}$	1	
$T = t$ is the point of contact.		
As $t \neq 0$ or ± 1, $-\dfrac{1}{t^2} \neq t$ or -1		
∴ the tangent meets the curve again at another point T, where $T = -\dfrac{1}{t^2}$.	1	
Let $P(t_1)$, $P(t_2)$, $P(t_3)$ be three distinct points on the curve and let the tangents at these points meet the curve again at $P(T_1)$, $P(T_2)$, $P(T_3)$ respectively, where		
$T_1 = -\dfrac{1}{t_1^2}$, $T_2 = -\dfrac{1}{t_2^2}$, $T_3 = -\dfrac{1}{t_3^2}$.	1	
By (b), $t_1t_2t_3 = -1$.		
∴ $T_1T_2T_3 = -\dfrac{1}{t_1^2t_2^2t_3^2} = -1$	1	
By (b) again, $P(T_1)$, $P(T_2)$, $P(T_3)$ are collinear.	1 7	

Solutions	Marks	Remarks

10. (a) (i) As $x \to \pm\infty$, $f(x) \to \pm\infty$ respectively.

∴ the graph of $f(x)$ does not have any horizontal asymptote. On the other hand, $x^2 + 1$ does not vanish for any real x, there is no vertical asymptote. — 1, 1

Now $\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(1 + \frac{8}{x^2+1}\right) = 1$

and $\lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \frac{8x}{x^2+1} = 0$ — 1

∴ $y = x$ is an asymptote and is also the only one of the graph of $f(x)$.

(ii) $f'(x) = \frac{(x^2+1)(3x^2+9) - x(x^2+9)(2x)}{(x^2+1)^2} = \frac{(x^2-3)^2}{(x^2+1)^2}$ — 1

$f'(x) = 0$ iff $x = \sqrt{3}$ or $-\sqrt{3}$ — 1

$f''(x) = \frac{(x^2+1)^2(2)(x^2-3)(2x) - (x^2-3)^2(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{16x(x^2-3)}{(x^2+1)^3}$ — 1

$f''(x) = 0$ iff $x = 0$ or $\sqrt{3}$ or $-\sqrt{3}$ — 1

Consider the following table:

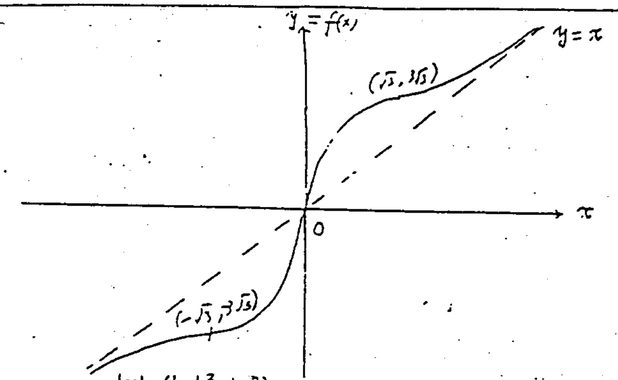
	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$	$0 < x < \sqrt{3}$	$x = \sqrt{3}$	$x > \sqrt{3}$
$f'(x)$	+	0	+	+	+	0	+
$f''(x)$	−	0	+	0	−	0	+
$f(x)$	↗ ⌒	pt. of inflexion	↗ ◡	pt. of inflexion	↗ ⌒	pt. of inflexion	↗ ◡

∴ the graph of $f(x)$ has inflexion points $(-\sqrt{3}, -3\sqrt{3})$, $(0, 0)$ and $(\sqrt{3}, 3\sqrt{3})$ — 1+1

Since f is continuously differentiable, the only possible extreme values occur at x where $f'(x) = 0$. Thus f has no extreme point. — $\frac{1}{10}$

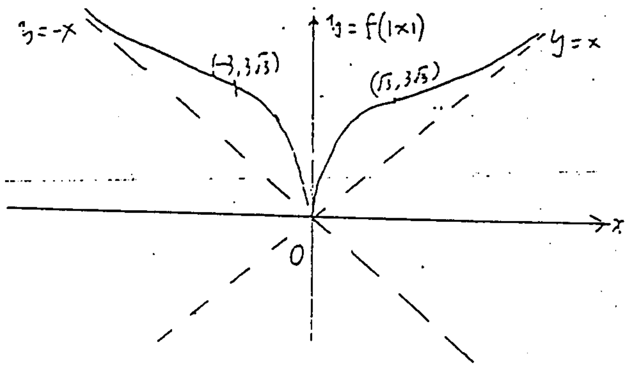
Solutions	Marks	Remarks

10. (b) (i) — 2



(ii) $f(|x|) = \frac{|x|(|x|^2 + 9)}{|x|^2 + 1}$

$= \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0 \end{cases}$ — 1



— $\frac{2}{5}$

Solutions	Marks	Remarks

11. (a) $\int_a^b (x-a)f'(x)\,dx = (x-a)f(x)\Big|_a^b - \int_a^b f(x)\,dx$ [1]

$= (b-a)f(b) - \int_a^b f(x)\,dx$

$= \int_a^b f(b)\,dx - \int_a^b f(x)\,dx$

$= \int_a^b [f(b) - f(x)]\,dx$ [1]

Total: 2

(b) $\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\tfrac{k}{n}) - f(x)]\,dx = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(\tfrac{k}{n})\,dx - \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx$

$= \sum_{k=1}^{n} \frac{1}{n} f(\tfrac{k}{n}) - \int_0^1 f(x)\,dx$ [1]

$= E_n$

If $|f'(x)| \le M \quad \forall x \in [0, 1]$,

$|E_n| = \left| \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\tfrac{k}{n}) - f(x)]\,dx \right|$

$= \left| \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \tfrac{k-1}{n}) f'(x)\,dx \right|$ by (a) [1]

$\le \sum_{k=1}^{n} \left| \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \tfrac{k-1}{n}) f'(x)\,dx \right|$

$\le \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |f'(x)|\, |x - \tfrac{k-1}{n}|\,dx$ [1]

$\le \sum_{k=1}^{n} M \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \tfrac{k-1}{n})\,dx$ [1]

$= \sum_{k=1}^{n} M \left[\tfrac{1}{2} (x - \tfrac{k-1}{n})^2 \right]_{\frac{k-1}{n}}^{\frac{k}{n}}$

$= \frac{M}{2} \cdot \frac{1}{n}$ [1]

Total: 5

Solutions	Marks	Remarks

11. (c) For $1 \le k \le n$,

$\int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\tfrac{k}{n}) - f(x)]\,dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}} f'(x)(x - \tfrac{k-1}{n})\,dx$ by (a) [1]

$= f'(\xi_k) \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \tfrac{k-1}{n})\,dx$

for some $\xi_k \in [\tfrac{k-1}{n}, \tfrac{k}{n}]$ by Eint with $h(x) = x - \tfrac{k-1}{n} \ge 0$ on $[\tfrac{k-1}{n}, \tfrac{k}{n}]$ and $f'(x)$, $h(x)$ are continuous. [1]

$= f'(\xi_k) \left[\tfrac{1}{2} (x - \tfrac{k-1}{n})^2 \right]_{\frac{k-1}{n}}^{\frac{k}{n}}$

$= \frac{f'(\xi_k)}{2n^2}$ [1]

$E_n = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\tfrac{k}{n}) - f(x)]\,dx$

$= \sum_{k=1}^{n} f'(\xi_k) \frac{1}{2n^2}$ where $\xi_k \in [\tfrac{k-1}{n}, \tfrac{k}{n}]$ [1]

$\therefore \lim_{n\to\infty} n E_n = \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} f'(\xi_k) \frac{1}{n}$

$= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} f'(\xi_k) (\tfrac{k}{n} - \tfrac{k-1}{n})$ [1]

$= \frac{1}{2} \int_0^1 f'(x)\,dx$ by definition of definite integral [2]

$= \frac{1}{2} [f(1) - f(0)]$ [1]

Total: 8

Solutions	Marks	Remarks
12. (a) Let R be any point on ℓ with position vector $\vec{r} = \vec{r}_0 + t\vec{a}$		
and let R' be the projection of R on π.		
The unit vector normal to π is $\frac{1}{\sqrt{\vec{n}\cdot\vec{n}}}\vec{n}$.	1	<u>Note</u>
The vector $\overrightarrow{R'R}$ is given by $[(\vec{r}-\vec{r}_0)\cdot\frac{1}{\vec{n}\cdot\vec{n}}\vec{n}]\vec{n}$	2	Candidates may use coordinate geometry method
$\therefore$ the vector $\overrightarrow{R_0R'}$ is given by		
$(\vec{r}-\vec{r}_0) - [(\vec{r}-\vec{r}_0)\cdot\frac{1}{\vec{n}\cdot\vec{n}}\vec{n}]\vec{n}$	1	
$= t\vec{a} - t\frac{\vec{a}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$	1	
$\therefore$ equation of the projection of ℓ on π is		
$\vec{r} = \vec{r}_0 + t(\vec{a} - \frac{\vec{a}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}),\quad t\in\mathbb{R}$	1 / 6	
(b) (i) Putting $x=-1-2t$, $y=3+3t$, $z=1+t$ in π_1,		
$4(-1-2t)+(3+3t)-2(1+t)-4=0$		
$t=-1$		
$\therefore P_1=(1,0,0)$	1	
Similarly, from ℓ_2 and π_1,		
$4(2-8t)+19t-2(2+4t)-4=0$		
$\Rightarrow t=0$		
$\therefore P_2=(2,0,2)$	1	
$\overrightarrow{P_1P_2}=\vec{i}+2\vec{k}$	1	
The directions of ℓ_1 and ℓ_2 are given by the vectors		
$-2\vec{i}+3\vec{j}+\vec{k}$ and $-8\vec{i}+19\vec{j}+4\vec{k}$ respectively	1	
$(\vec{i}+2\vec{k})\cdot(-2\vec{i}+3\vec{j}+\vec{k})=0$		
and $(\vec{i}+2\vec{k})\cdot(-8\vec{i}+19\vec{j}+4\vec{k})=0$		
$\therefore$ the line segment P_1P_2 is perpendicular to ℓ_1 and ℓ_2.	1	
(ii) $\ell_1: \vec{r}=(-\vec{i}+3\vec{j}+\vec{k})+t(-2\vec{i}+3\vec{j}+\vec{k})$		
$\ell_2: \vec{r}=(2\vec{i}+2\vec{k})+t(-8\vec{i}+19\vec{j}+4\vec{k})$		
By (a), set $\vec{r}_0=\vec{i}$, $\vec{a}=-2\vec{i}+3\vec{j}+\vec{k}$, $\vec{n}=4\vec{i}+\vec{j}-2\vec{k}$	1	
$\ell_1': \vec{r}=\vec{i}+t[(-2\vec{i}+3\vec{j}+\vec{k})+\frac{1}{3}(4\vec{i}+\vec{j}-2\vec{k})]$		
$=\vec{i}+\frac{1}{3}t(-2\vec{i}+10\vec{j}+\vec{k})$	1	
Similarly, $\ell_2': \vec{r}=2\vec{i}+2\vec{k}+2t(-2\vec{i}+10\vec{j}+\vec{k})$	1	
Hence, $\ell_1' \parallel \ell_2'$	1 / 9	

Solutions	Marks	Remarks
(a) (i) $\frac{d}{dx}[G(x)e^{-bx}] = G'(x)e^{-bx} - bG(x)e^{-bx}$	1	
$\le (a+bG(x))e^{-bx} - bG(x)e^{-bx}$		
(as $e^{-bx}>0$)		
$= ae^{-bx}\quad \forall x\ge 0$	1	
(ii) As $G(x)$ is continuously differentiable, for every		
$x\ge 0$, $\int_0^x \frac{d}{dt}[G(t)e^{-bt}]dt \le \int_0^x ae^{-bt}dt$	1	
$G(x)e^{-bx} - G(0) \le -\frac{a}{b}(e^{-bx}-1)$	1	
$\therefore\ G(x)\le G(0)e^{bx}+\frac{a}{b}(e^{bx}-1)$ (as $e^{-bx}>0$)	1 / 5	
(b) (i) As $f(x)=f(0)+\int_0^x f'(t)dt$,	1	
$\lvert f(x)\rvert \le \lvert f(0)\rvert + \left\lvert\int_0^x f'(t)dt\right\rvert$	1	
$\le \lvert f(0)\rvert + \int_0^x \lvert f'(t)\rvert dt$		
$\le \lvert f(0)\rvert + M\int_0^x \lvert f(t)\rvert dt$ for $x\ge 0$	1	
(ii) $\frac{d}{dx}\int_0^x \lvert f(t)\rvert dt = \lvert f(x)\rvert$		
$\le \lvert f(0)\rvert + M\int_0^x \lvert f(t)\rvert dt$	1	
We see that the function $\int_0^x \lvert f(t)\rvert dt$ satisfies		
the conditions for $G(x)$ in (a) with $a=\lvert f(0)\rvert$ and $b=M>0$.		
$\therefore \int_0^x \lvert f(t)\rvert dt \le e^{Mx}\int_0^0 \lvert f(t)\rvert dt + \frac{\lvert f(0)\rvert}{M}(e^{Mx}-1)$	1	
$M\int_0^x \lvert f(t)\rvert dt + \lvert f(0)\rvert \le \lvert f(0)\rvert e^{Mx}$		
i.e. $\lvert f(x)\rvert \le \lvert f(0)\rvert e^{Mx}$ by (i)	1 / 6	
(c) As $\lvert h'(x)\rvert = \lvert\sin(h(x))\rvert$		
$\le \lvert h(x)\rvert\quad \forall x\ge 0$	1	
Conditions in (b) are satisfied with $M=1$.	1	
$\therefore\ \lvert h(x)\rvert \le \lvert h(0)\rvert e^{x}$	1	
$=0\quad \forall x\ge 0$	1 / 4	