89-AL
P MATHS
PAPER I

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION
1989

PURE MATHEMATICS PAPER I

9.00 am-12.00 noon (3 hours)
This paper must be answered in English

This paper consists of two sections BOTH of which are to be answered.

INSTRUCTIONS FOR SECTION A

1. Answer ALL questions. Write your answers in the spaces provided in this question booklet.
2. Write your Candidate Number, Centre Number and Seat Number in the spaces provided on this cover.
3. Graph paper and supplementary sheets will be supplied on request. Write your Candidate Number on each sheet and fasten them with string INSIDE this booklet.

INSTRUCTIONS FOR SECTION B

Answer any FOUR questions. Write your answers in the separate answer book provided.

Candidate Number	
Centre Number	
Seat Number	

Question Number	Marker's Use Only	Examiner's Use Only
	Marker No.	Examiner No.
	Marks	Marks
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Total Marks	
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Page total

SECTION A (40 marks)
Answer ALL questions in this section.

1. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & -1 \end{pmatrix}$.

   (a) Find AB^T and B^TA, where B^T denotes the transpose of B.

   (b) For each of the matrices AB^T and B^TA, determine whether it is invertible, and find its inverse if it exists.

   (5 marks)

Page total

2. Let a and b be two positive numbers and let n be a positive integer. Making use of the equality $\prod_{k=1}^{n}(a^k+b^k)=\prod_{k=1}^{n}(a^{n+1-k}+b^{n+1-k})$, or otherwise, show that

$$\left[\prod_{k=1}^{n}(a^k+b^k)\right]^2>(a^{n+1}+b^{n+1})^n .$$

(5 marks)

Page total

3. (a) Evaluate $\lim_{x\to\infty} x\left[\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}\right]$.

(b) Let h be a positive constant.

Evaluate $\lim_{n\to\infty}\dfrac{n}{1+nh+\frac{n(n-1)}{2}h^2}$.

Hence, or otherwise, show that $\lim_{n\to\infty}\dfrac{n}{(1+h)^n}=0$.

(5 marks)

4. Find the constants h and k such that the system of equations

$$\begin{cases} x + y + 3z = k \\ 4x + hy - z = 1 \\ 6x + 7y + 5z = 2 \end{cases}$$

has infinitely many solutions.

(6 marks)

5. From the figures 1, 2, 3, 4, 5, 6 and 7, how many 4-digit numbers can be formed with no figure being used more than once in each number?

Of these 4-digit numbers formed, how many are divisible by 3?

(6 marks)

Page total

6. Let $z = \cos\theta + i\sin\theta$. By expressing $\cos\theta$ in terms of z , or otherwise, show that for any positive integer n ,

$$\cos^n\theta = \frac{1}{2^n}\sum_{r=0}^{n} C_r^n \cos(n-2r)\theta \ .$$

(6 marks)

Page total

7. Let S be a relation defined on the set C of complex numbers by $z \, S \, z'$ iff $\mathrm{Re}(z) \leqslant \mathrm{Re}(z')$, where $\mathrm{Re}(z)$ denotes the real part of z .

(a) (i) Show that S is both reflexive and transitive.

(ii) Indicate the set $A = \{z \in C : z \, S \, (1+2i)\}$ on the Argand plane.

(b) A relation $\sim$ is defined on C by $z \sim z'$ iff $z \, S \, z'$ and $z' \, S \, z$.

(i) Show that $\sim$ is an equivalence relation.

(ii) Indicate the set $B = \{z \in C : z \sim (1+2i)\}$ on the Argand plane.

(7 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section.
Each question carries 15 marks.
You may retain this part of the question paper by detaching pp.8–10 at the end of the examination.

8. (a) Let S be a square matrix such that $S^3 + S = 0$.

Define a matrix $A(\theta) = I - (\sin\theta)S + (1 - \cos\theta)S^2$ for $\theta \in \mathbb{R}$.

For $\theta, \phi \in \mathbb{R}$, show that

(i) $A(\theta)A(\phi) = A(\theta + \phi)$,

(ii) $[A(\theta)]^n = A(n\theta)$ for any positive integer n,

(iii) the inverse of $A(\theta)$ exists.

(7 marks)

(b) Let $T = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$.

(i) Verify that $T^3 + T = 0$.

(ii) Using (a), or otherwise, express the following in the form $I + \alpha T + \beta T^2$ (where $\alpha, \beta \in \mathbb{R}$):

(1) $(I + T + T^2)^{-1}$,

(2) $(I + T + T^2)^{1989}$.

(8 marks)

9. Given an integer $n > 2$, consider the equation $x^n + x + 1 = 0$ ....(*)

(a) Show that (*) has exactly one real root if n is odd and no real root if n is even.

(5 marks)

(b) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of (*).

(i) Show that if α is a root of (*), then $\bar{\alpha}$ is also a root of (*).

Deduce that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$.

(ii) Prove that $\sum_{r=1}^{n} \alpha_r{}^k$ is real for any integer k.

(iii) Evaluate

(1) $\sum_{r=1}^{n} \frac{1}{\alpha_r}$,

(2) $\sum_{r=1}^{n} \alpha_r{}^{n-1}$.

(10 marks)

10. (a) By determining the least value of the function $f(x) = e^{x-1} - x$, or otherwise, show that $e^{x-1} > x$ for all $x \in \mathbb{R}$.

(3 marks)

(b) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive numbers.

Show that $e^{\left\{\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right) - n\right\}} > \prod_{i=1}^{n} \frac{a_i}{b_i}$.

Hence, or otherwise, show that if $\sum_{i=1}^{n} \frac{a_i}{b_i} < n$, then $\prod_{i=1}^{n} a_i < \prod_{i=1}^{n} b_i$.

(4 marks)

(c) Using the result in (b), show that for any positive numbers $a_1, a_2, \ldots, a_n$,

$$\left[\prod_{i=1}^{n} a_i\right]^{\frac{1}{n}} < \frac{1}{n}\sum_{i=1}^{n} a_i .$$

Hence, or otherwise, show that

$$\sum_{i=1}^{n}\left[\frac{1}{a_i} - \frac{1}{m}\right] > 0, \text{ where } m = \frac{1}{n}\sum_{i=1}^{n} a_i .$$

(8 marks)

11. (a) Prove that for any positive integer n, there exist unique positive integers a_n and b_n such that

$$(\sqrt{2} + 1)^n = a_n\sqrt{2} + b_n .$$

Show also that

(i) b_n is odd for all n,

(ii) a_n is odd if n is odd.

(5 marks)

(b) For a_n and b_n as determined in (a), show that

(i) $(\sqrt{2} - 1)^n = (-1)^{n+1}(a_n\sqrt{2} - b_n)$,

(ii) $b_n > a_n > 2^{n-1}$.

Hence, or otherwise, show that $\left|\sqrt{2} - \frac{b_n}{a_n}\right| < \frac{1}{(2^{2n-1})}$ and evaluate $\lim_{n \to \infty} \frac{b_n}{a_n}$.

(10 marks)

12. The mapping $f : \mathbf{C} \setminus \{-1\} \to \mathbf{C} \setminus \{-i\}$ is defined by $f(z) = \frac{i(1-z)}{1+z}$.

(a) Show that f is bijective. (4 marks)

(b) Find and sketch the image, under f, of each of the following:

(i) the upper half of the imaginary axis (including the origin),

(ii) the positive real axis. (11 marks)

13. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a mapping satisfying $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for any $x, y \in \mathbf{R}^3$ and $\alpha, \beta \in \mathbf{R}$.

(a) Show that

(i) $T(0) = 0$,

(ii) $T(\alpha x + \beta y + \gamma z) = \alpha T(x) + \beta T(y) + \gamma T(z)$ for any $\alpha, \beta, \gamma \in \mathbf{R}$ and $x, y, z \in \mathbf{R}^3$,

(iii) if x, y and z are linearly dependent, then $T(x)$, $T(y)$ and $T(z)$ are also linearly dependent. (5 marks)

(b) Prove that the following three statements are equivalent:

(1) T is an injective mapping.

(2) If x, y and z are any three linearly independent vectors in $\mathbf{R}^3$, then $T(x)$, $T(y)$ and $T(z)$ are linearly independent.

(3) $T(e_1)$, $T(e_2)$ and $T(e_3)$ are linearly independent, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

[Hint: You may prove (1) ⇒ (2) ⇒ (3) ⇒ (1).] (10 marks)

END OF PAPER

89-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION

1989

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PURE MATHEMATICS PAPER II

2.00 pm-5.00 pm (3 hours)

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	Marker No.	Examiner No.
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INSTRUCTIONS FOR SECTION B

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Checker's Use Only	
Total Marks	
Checker's Initial	

12. The mapping $f : \mathbf{C}\setminus\{-1\} \to \mathbf{C}\setminus\{-i\}$ is defined by $f(z) = \dfrac{i(1-z)}{1+z}$.

(a) Show that f is bijective. (4 marks)

(b) Find and sketch the image, under f, of each of the following:

(i) the upper half of the imaginary axis (including the origin),

(ii) the positive real axis. (11 marks)

13. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a mapping satisfying $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for any $x, y \in \mathbf{R}^3$ and $\alpha, \beta \in \mathbf{R}$.

(a) Show that

(i) $T(0) = 0$,

(ii) $T(\alpha x + \beta y + \gamma z) = \alpha T(x) + \beta T(y) + \gamma T(z)$ for any $\alpha, \beta, \gamma \in \mathbf{R}$ and $x, y, z \in \mathbf{R}^3$,

(iii) if x, y and z are linearly dependent, then $T(x)$, $T(y)$ and $T(z)$ are also linearly dependent. (5 marks)

(b) Prove that the following three statements are equivalent:

(1) T is an injective mapping.

(2) If x, y and z are any three linearly independent vectors in $\mathbf{R}^3$, then $T(x)$, $T(y)$ and $T(z)$ are linearly independent.

(3) $T(e_1)$, $T(e_2)$ and $T(e_3)$ are linearly independent, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

[Hint: You may prove (1) ⇒ (2) ⇒ (3) ⇒ (1).] (10 marks)

END OF PAPER

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SECTION A (40 marks)

Answer ALL questions in this section.

1. Let $f(x) = \frac{e^x}{x^e}$ for $x > 0$. Find the least value of $f(x)$.

   Hence show that $e^\pi > \pi^e$.

   (5 marks)

Page total

2. Evaluate $\int \frac{1}{x^3+1}\,dx$.

   (5 marks)

Page total

3. Let $f(x) = \int_{\frac{1}{x}}^{x} \sin\sqrt{xt}\, dt$ for $x > 0$.

(a) Show that $f(x) = \frac{1}{x}\int_{1}^{x^2} \sin\sqrt{u}\, du$.

(b) Find $f'(1)$.

(5 marks)

Page total

4. (a) Find the area of the region bounded by the parabolas $y^2 = x$ and $x^2 = y$.

(b) Compute the arc length of the curve $y = \ln\cos x$, where $0 < x < \frac{\pi}{4}$.

(6 marks)

5. Let $y(1+x^2)=1$.

Show that for $n > 2$, $(1+x^2)y^{(n)}+2nxy^{(n-1)}+n(n-1)y^{(n-2)}=0$, where $y^{(0)}=y$, and $y^{(k)}=\frac{d^k y}{dx^k}$ for $k > 1$.

Hence evaluate $y^{(n)}(0)$ for $n > 0$.

(6 marks)

Page total

6. Consider the parabola $\Gamma : y^2 = 1 + 2x$. Let the origin O be the pole, and the positive x-axis be the initial line of a polar coordinate system.

(a) Find the polar equation of the curve in the form $r = f(\theta)$.

(b) PQ is a chord of length $\frac{8}{3}$, passing through O and with P lying in the first quadrant. Find the polar coordinates of P and Q.

(6 marks)

Page total

Page total

7. Evaluate

(a) $\lim_{h \to 0} [\ln (e + h)]^{\frac{1}{h}}$,

(b) $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$.

(7 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section.
Each question carries 15 marks.
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8. For any non-negative integer n , let $I_n = \int_0^1 x^n e^{ax}\, dx$, where a is a non-zero constant.

(a) Evaluate I_0 and express I_n in terms of I_{n-1} for $n \geq 1$.

(4 marks)

(b) For $n \geq 1$, show that

$$I_n = \frac{(-1)^{n+1} n!}{a^{n+1}} + e^a \left[\frac{1}{a} + \sum_{r=1}^{n} \frac{(-1)^r n(n-1)\ldots(n-r+1)}{a^{r+1}} \right] .$$

(6 marks)

(c) Using the above results, or otherwise, evaluate $\int_1^{e^2} \left(\frac{\ln u}{u}\right)^3 du$.

(5 marks)

9. Consider the curve defined by the parametric equations

$$\begin{cases} x = \dfrac{t}{1 + t^3} \\ y = \dfrac{t^2}{1 + t^3} \end{cases} , \quad t \neq -1 .$$

Let $P(t)$ be the point on the curve corresponding to the parameter t .

(a) Show that the equation of the chord joining the points $P(t_1)$ and $P(t_2)$ is

$$(t_1{}^2 t_2{}^2 - t_1 - t_2)x + [1 - t_1 t_2(t_1 + t_2)]y + t_1 t_2 = 0 .$$

Deduce the equation of the tangent at the point $P(t)$.

(4 marks)

(b) Let $P(t_1)$, $P(t_2)$ and $P(t_3)$ be three distinct points on the curve. Show that a necessary and sufficient condition for these three points to be collinear is $t_1 t_2 t_3 = -1$.

(4 marks)

(c) Show that when $t \neq 0$ or ± 1 , the tangent at the point $P(t)$ intersects the curve again at another point $P(T)$, where $T = -\frac{1}{t^2}$.

Hence, or otherwise, deduce that if the tangents at three collinear points on the curve intersect the curve again, then these points of intersection are also collinear.

(7 marks)

10. Consider the function $f(x) = \dfrac{x(x^2+9)}{x^2+1}$, $x \in \mathbb{R}$.

(a) (i) Show that $y = x$ is the only asymptote of the graph of $f(x)$.

(ii) Show that $f(x)$ does not have any extreme value.

Find all the points of inflexion of the graph of $f(x)$. (10 marks)

(b) Use the above results to sketch the graphs of

(i) $f(x)$,

(ii) $f(|x|)$ for $x \in \mathbb{R}$. (5 marks)

11. Let $f(x)$ be a function continuously differentiable on the interval $[0, 1]$.

For any integer $n > 1$, let $E_n = \dfrac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) - \int_0^1 f(x)\,dx$.

(a) If $0 \le a < b \le 1$, show that $\int_a^b (x-a)f'(x)\,dx = \int_a^b [f(b) - f(x)]\,dx$. (2 marks)

(b) Verify that $E_n = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\frac{k}{n}) - f(x)]\,dx$.

Hence use (a) to show that if there exists a positive constant M such that $|f'(x)| \le M$ for every $x \in [0, 1]$, then $|E_n| \le \dfrac{M}{2n}$. (5 marks)

(c) Let k be any integer with $1 \le k \le n$. Show that

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\tfrac{k}{n}) - f(x)]\,dx = \frac{f'(\xi_k)}{2n^2} \quad \text{..(*)}$$

for some $\xi_k \in [\frac{k-1}{n}, \frac{k}{n}]$.

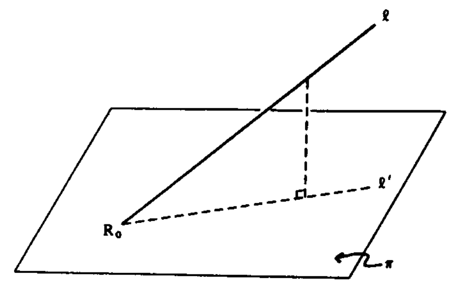
Deduce that $\lim_{n\to\infty} nE_n = \frac{1}{2}[f(1) - f(0)]$.

[Hint: In proving (*), you may assume that if $g(x)$ and $h(x)$ are continuous functions on the interval $[c, d]$, and if $h(x) \ge 0 \ \forall x \in [c, d]$, then $\int_c^d g(x)h(x)\,dx = g(x_0)\int_c^d h(x)\,dx$ for some $x_0 \in [c, d]$.] (8 marks)

12. (a) The position vector of a point $R(x, y, z)$ is given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

In the figure, $R_0(x_0, y_0, z_0)$ is a point on the plane $\pi : \mathbf{r}\cdot\mathbf{n} = p$.

The line $\ell : \mathbf{r} = \mathbf{r}_0 + t\mathbf{a}$, $t \in \mathbb{R}$, where $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, passes through R_0 and does not lie on π .



Show that the projection of ℓ on π is given by $\ell' : \mathbf{r} = \mathbf{r}_0 + t\left(\mathbf{a} - \dfrac{\mathbf{a}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right)$, $t \in \mathbb{R}$. (6 marks)

(b) Consider the lines $\ell_1 : \begin{cases} x = -1 - 2t \\ y = 3 + 3t \\ z = 1 + t \end{cases}$, $t \in \mathbb{R}$

and $\ell_2 : \begin{cases} x = 2 - 8t \\ y = 19t \\ z = 2 + 4t \end{cases}$, $t \in \mathbb{R}$

and the plane $\pi_1 : 4x + y - 2z - 4 = 0$.

(i) Let P_1 and P_2 be the points at which π_1 intersects ℓ_1 and ℓ_2 respectively.

Find P_1 and P_2 and show that the line segment P_1P_2 is perpendicular to both ℓ_1 and ℓ_2 .

(ii) Show that the projections of ℓ_1 and ℓ_2 on π_1 are parallel. (9 marks)

13. (a) Let $G(x)$ be a function continuously differentiable on $\mathbb{R}$ such that $G'(x) \leq a + bG(x)$ for every $x \geq 0$, where a and b are constants and $b \neq 0$.

(i) Show that $\frac{d}{dx}[G(x)e^{-bx}] \leq ae^{-bx}$ for every $x \geq 0$.

(ii) Deduce that for $x \geq 0$, $G(x) \leq G(0)e^{bx} + \frac{a}{b}(e^{bx} - 1)$.

(5 marks)

(b) Let $f(x)$ be a function continuously differentiable on $\mathbb{R}$ such that $|f'(x)| \leq M|f(x)|$ for every $x \geq 0$, where M is a positive constant.

(i) Show that

$$|f(x)| \leq |f(0)| + M\int_0^x |f(t)|\,dt$$

for every $x \geq 0$.

(ii) By putting $G(x) = \int_0^x |f(t)|\,dt$ in (a), or otherwise, show that

$$|f(x)| \leq |f(0)|e^{Mx}$$

for every $x \geq 0$.

(6 marks)

(c) Let $h(x)$ be a function continuously differentiable on $\mathbb{R}$ such that $h'(x) = \sin(h(x))$ for every $x \geq 0$ and $h(0) = 0$. Using (b), or otherwise, show that $h(x) = 0$ for every $x \geq 0$.

(4 marks)

END OF PAPER

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INSTRUCTIONS FOR SECTION B

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