9. (a) Let n be a positive integer. It is known that for any functions $f(x)$ and $g(x)$ with nth derivatives, if $h(x) = f(x)g(x)$, then $h^{(n)}(x) = \sum_{k=0}^{n} a_k f^{(k)}(x) g^{(n-k)}(x)$, where $a_0, a_1, \ldots, a_n$ are constants independent of $f(x)$ and $g(x)$, $f^{(0)}(x) = f(x)$ and $f^{(k)}(x) = \frac{d^k f(x)}{dx^k}$. Taking $f(x) = e^{\lambda x}$ and $g(x) = e^x$, where λ is a number independent of x,

(i) find $h^{(n)}(x)$ and $f^{(k)}(x)g^{(n-k)}(x)$ and hence

(ii) show that $a_k = C_k^n$ for $k = 0, 1, 2, \ldots, n$.

(b) Let $u(x) = x^m e^{-x}$,

$y(x) = e^x u^{(m)}(x)$,

where m is a positive integer.

(i) Show that $y(x)$ is a polynomial of degree m and find its coefficients.

(ii) Show that $x u'(x) + (x - m)u(x) = 0$.

Deduce that $x u^{(m+2)}(x) + (x+1)u^{(m+1)}(x) + (m+1)u^{(m)}(x) = 0$.

(iii) Using (ii), or otherwise, show that

$$x y''(x) + (1 - x)y'(x) + m y(x) = 0.$$

END OF PAPER

88–AL
P MATHS
PAPER I

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1988

純數學　試卷一

PURE MATHEMATICS PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. (a) Given a real polynomial $P(x)$, show that if α is a root of $P(x) - x = 0$, then α is also a root of $P(P(x)) - x = 0$.

(b) Let $P(x) = x^2 + ax + b$, where a and b are real.

(i) Using (a), or otherwise, resolve $P(P(x)) - x$ into two quadratic factors.

(ii) Find a relation between a and b which is a necessary and sufficient condition for all roots of $P(P(x)) - x = 0$ to be real.

(c) Using (b)(i), or otherwise, solve the equation

$$(x^2 - 3x + 1)^2 - 3(x^2 - 3x + 1) + 1 - x = 0.$$

2. A real number λ is said to be an *eigenvalue* of a 3×3 matrix M if there exists a non-zero column vector $\mathbf{x}$ such that $M\mathbf{x} = \lambda\mathbf{x}$.

(a) Show that any 3×3 matrix M has at most 3 eigenvalues.

(b) Show that if λ is an eigenvalue of M, then

$a_0 + a_1\lambda + \ldots + a_n\lambda^n$ is an eigenvalue of

$a_0 I + a_1 M + \ldots + a_n M^n$,

where $a_0, a_1, \ldots, a_n$ are real constants.

(c) (i) Find all eigenvalues of $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(ii) Using the above results, or otherwise, find ALL eigenvalues of

$$B = \begin{pmatrix} a_2 + a_0 & a_1 & a_2 \\ a_1 & 2a_2 + a_0 & a_1 \\ a_2 & a_1 & a_2 + a_0 \end{pmatrix},$$

where a_0, a_1 and a_2 are non-zero integers.

3. Given an infinite sequence $\{a_n\}$ of positive integers, two sequences $\{p_n\}$ and $\{q_n\}$ are defined by

$$p_1 = a_1, \quad p_2 = a_2 a_1 + 1, \quad p_k = a_k p_{k-1} + p_{k-2} \quad \text{for } k \geq 3;$$

$$q_1 = 1, \quad q_2 = a_2, \quad q_k = a_k q_{k-1} + q_{k-2} \quad \text{for } k \geq 3.$$

(a) For all positive integers n, prove that

$$p_{n+1} q_n - p_n q_{n+1} = (-1)^{n+1}.$$

(b) Let $b_k = \dfrac{p_k}{q_k}$, $k = 1, 2, 3, \ldots$.

(i) Show that $\{b_1, b_3, b_5, \cdots\}$ is a strictly increasing sequence and that $\{b_2, b_4, b_6, \cdots\}$ is a strictly decreasing sequence.

(ii) For all positive integers n, show that

$$b_{2n-1} < b_2$$

$$b_{2n} > b_1 .$$

(iii) Show that the two sequences $\{b_1, b_3, b_5, \cdots\}$ and $\{b_2, b_4, b_6, \cdots\}$ converge to the same limit.

4. (a) Let $N = \{0, 1, 2, \ldots\}$. A mapping $f : N^2 \to N^2$ is defined by $f((x, y)) = (x, y+1)$ for any $(x, y) \in N^2$.

Given $k \in N$, let $S_k = \{(x, y) \in N^2 : x + 2y = k\}$.

(i) Show that f is injective but not surjective.

(ii) Show that $S_{k+2} = f[S_k] \cup \{(k+2, 0)\}$.

Hence deduce that $n(S_{k+2}) = n(S_k) + 1$, where $n(S_k)$ denotes the number of elements in S_k.

(iii) Show, by mathematical induction, that for any $k \in N$,

$$n(S_k) = \begin{cases} \dfrac{k+2}{2} & \text{if } k \text{ is even,} \\[2ex] \dfrac{k+1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

(b) Let p be a non-negative integer. Consider the inequality $x + 2y \leq p$.

Show that the number of non-negative integral solutions (x, y) of the inequality is $\begin{cases} \dfrac{(p+2)^2}{4} & \text{if } p \text{ is even,} \\[2ex] \dfrac{(p+1)(p+3)}{4} & \text{if } p \text{ is odd.} \end{cases}$

5. Let a be a positive real number not equal to 1 and let p, q, r and s be four real numbers such that

$$p + q = r + s \text{ and } 0 < p - q < r - s .$$

(a) (i) Show that the function $f(x) = a^x + a^{-x}$ is strictly increasing for $x > 0$.

(ii) By considering the values of f at $\frac{1}{2}(p - q)$ and $\frac{1}{2}(r - s)$, deduce that

$$a^p + a^q < a^r + a^s .$$

(b) Let u and v be two distinct positive real numbers.

(i) Show that $u^p v^q + u^q v^p < u^r v^s + u^s v^r$.

Hence deduce that $(u^p + v^p)(u^q + v^q) < (u^r + v^r)(u^s + v^s)$.

(ii) Show that

$$(u^8 + v^8)(u^{80} + v^{80})(u^{900} + v^{900})(u^{1000} + v^{1000}) < 2^3(u^{1988} + v^{1988}) .$$

6. A mapping $f : \mathbf{R}^3 \to \mathbf{R}$ is said to be *linear* if

$$f(ax + by) = a f(\mathbf{x}) + b f(\mathbf{y})$$

for any $a, b \in \mathbf{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$.

(a) Given a vector $\mathbf{u} \in \mathbf{R}^3$, the mapping $\phi_{\mathbf{u}} : \mathbf{R}^3 \to \mathbf{R}$ is defined by $\phi_{\mathbf{u}}(\mathbf{x}) = \mathbf{u} \cdot \mathbf{x}$ for any $\mathbf{x} \in \mathbf{R}^3$.

(i) Show that $\phi_{\mathbf{u}}$ is linear.

(ii) Show that if $\phi_{\mathbf{u}}(\mathbf{x}) = 0$ for any $\mathbf{x} \in \mathbf{R}^3$, then $\mathbf{u} = \mathbf{0}$.

(iii) For any two vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbf{R}^3$, show that if $\phi_{\mathbf{u}} = \phi_{\mathbf{v}}$, then $\mathbf{u} = \mathbf{v}$.

(b) Given a linear mapping $f : \mathbf{R}^3 \to \mathbf{R}$, show that there exists one and only one vector $\mathbf{w} \in \mathbf{R}^3$ such that $f(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$ for any $\mathbf{x} \in \mathbf{R}^3$.

7. A fair die is thrown repeatedly. Denote by a_k $(k \geqslant 1)$ the sum of the scores in the first k throws. For any positive integer n, let $\mathrm{p}(n)$ be the probability that $a_k = n$ for some $k \geqslant 1$.

(a) (i) Find $\mathrm{p}(1)$, $\mathrm{p}(2)$ and $\mathrm{p}(3)$.

(ii) Express $\mathrm{p}(4)$ in terms of $\mathrm{p}(1)$, $\mathrm{p}(2)$ and $\mathrm{p}(3)$.

(iii) For $n > 6$, express $\mathrm{p}(n)$ in terms of $\mathrm{p}(n-1)$, $\mathrm{p}(n-2)$, ... and $\mathrm{p}(n-6)$.

(b) Define $\mathrm{p}(k) = 0$ for $k < 0$ and $\mathrm{p}(0) = 1$.

(i) Prove that for any positive integer n,

$$\mathrm{p}(n) + \frac{5}{6}\mathrm{p}(n-1) + \frac{4}{6}\mathrm{p}(n-2) + \frac{3}{6}\mathrm{p}(n-3) + \frac{2}{6}\mathrm{p}(n-4) + \frac{1}{6}\mathrm{p}(n-5) = 1.$$

(ii) Given that $\lim_{n\to\infty} \mathrm{p}(n)$ exists, find its value.

8. For $j = 1, 2, 3, 4$ or 5, let α_j be a complex number such that $|\alpha_j| = 1$.

(a) (i) Suppose $(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5) = \sum_{k=0}^{5} b_k z^k$ for all complex numbers z, where the b_k's are constants. Show that

$$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\bar{b}_4 = -b_1,$$

and $$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\bar{b}_2 = -b_3.$$

(ii) If $\sum_{j=1}^{5}\alpha_j = \sum_{j=1}^{5}\alpha_j^2 = 0$, show that

$$(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5) = z^5 - \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5.$$

(b) Using (a), or otherwise, show that the five points which represent the α_j's in the complex plane form a regular pentagon if and only if $\sum_{j=1}^{5}\alpha_j = \sum_{j=1}^{5}\alpha_j^2 = 0$.

9. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ be 5 vectors in $\mathbf{R}^3$.

(a) It is known that any 4 vectors in $\mathbf{R}^3$ are linearly dependent.

(i) By considering the vectors $\mathbf{v}_i - \mathbf{v}_5$, $i = 1, 2, 3, 4$, show that there exist real numbers $t_1, t_2, \ldots, t_5$, not all zero, such that

$$\sum_{i=1}^{5} t_i = 0 \quad \text{and} \quad \sum_{i=1}^{5} t_i \mathbf{v}_i = \mathbf{0} .$$

(ii) Let $\lambda_1, \lambda_2, \ldots, \lambda_5$ be five real numbers with $\lambda_i > 0$ and $\sum_{i=1}^{5} \lambda_i = 1$. For the t_i's in (i), define

$$\mu_i = \lambda_i - \frac{\lambda_r}{t_r} t_i , \quad i = 1, 2, \ldots, 5 ,$$

where $\left|\frac{t_r}{\lambda_r}\right| \geqslant \left|\frac{t_i}{\lambda_i}\right|$ for all i.

(1) Check that the μ_i's are well-defined, i.e. $t_r \neq 0$.

(2) Show that $\sum_{i=1}^{5} \mu_i = 1$, $\mu_i \geqslant 0$ and $\mu_r = 0$.

(3) Show that $\sum_{i=1}^{5} \mu_i \mathbf{v}_i = \sum_{i=1}^{5} \lambda_i \mathbf{v}_i$.

(b) Let $\mathbf{v} = \sum_{i=1}^{5} \alpha_i \mathbf{v}_i$, where $\alpha_i \geqslant 0$ and $\sum_{i=1}^{5} \alpha_i = 1$. Using (a), or otherwise, show that there exist real numbers $k_1, k_2, \ldots, k_5$ with $k_i \geqslant 0$, $\sum_{i=1}^{5} k_i = 1$ such that $\mathbf{v} = \sum_{i=1}^{5} k_i \mathbf{v}_i$ and at least one k_i equals zero.

END OF PAPER

88-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1988

純數學 試卷二

PURE MATHEMATICS PAPER II

2.00 pm–5.00 pm (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. (a) Find the following integrals:

(i) $\displaystyle\int \frac{dx}{x - \sqrt{x^2 - 1}}$,

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x}$.

(b) For $n = 0, 1, 2, \ldots$, it is known that $\displaystyle\lim_{k \to \infty} \int_0^k \frac{dx}{(x^2 + 1)^{n+1}}$ exists. Denote this limit by I_n . For $n \geqslant 1$, express I_n in terms of I_{n-1} , and hence show that

$$I_n = \frac{(2n-1)(2n-3)\ldots 1}{(2n)(2n-2)\ldots 2} \cdot \frac{\pi}{2} .$$

2. Given a positive integer n and a real number a , let

$$I_n(a) = \int_{\frac{1}{n}}^{n} \frac{dx}{(1 + x^a)(1 + x^2)} .$$

(a) (i) Find $I_n(0)$.

(ii) Using partial fractions, find $I_n(1)$ and $I_n(-1)$.

(b) (i) Using the substitution $x = \frac{1}{u}$, or otherwise, show that

$$I_n(a) = \int_{\frac{1}{n}}^{n} \frac{x^a}{(1 + x^a)(1 + x^2)} dx .$$

(ii) Deduce that $I_n(a)$ is independent of a .

(iii) Compute $\displaystyle\lim_{n \to \infty} I_n(a)$.

3. Consider the parabola $\Gamma : y^2 = 4ax \quad (a > 0)$.

(a) Find the equation of the tangent to Γ at the point $(at^2 , 2at)$.

(b) Let $P(at_1^2 , 2at_1)$ and $Q(at_2^2 , 2at_2)$ be two distinct points on Γ .

Find the coordinates of the point of intersection of the tangents at P and Q .

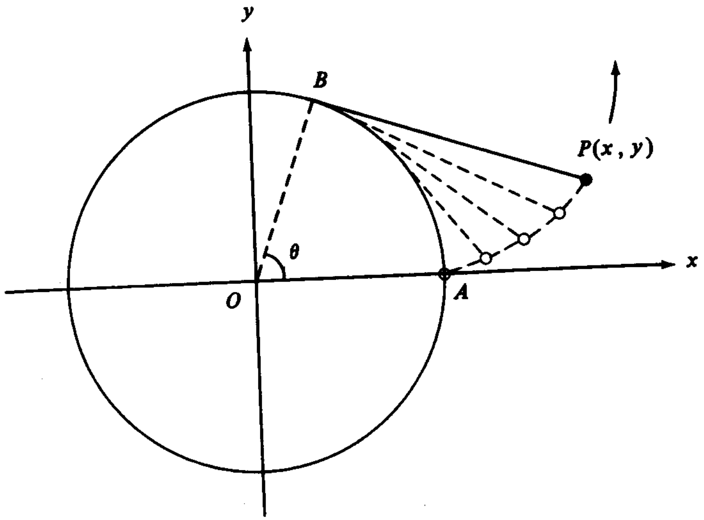
(c) Suppose P and Q are two variable points on Γ such that $\angle POQ = 90°$, where O is the origin.

(i) Find the locus of the point of intersection of the tangents at P and Q .

(ii) Find the locus of the mid-point of PQ .

(d) Consider the parabola $\Gamma' : (x + y)^2 = 8(x - y)$. Let P' and Q' be two variable points on Γ' such that $\angle P'OQ' = 90°$. Find the locus of the point of intersection of the tangents at P' and Q' .

4.



A long piece of thread is wound round the circumference of a circle, centred at O and with radius $OA = a$. A particle P is tied to the free end of the thread. Starting at A, the particle moves in a direction so that the thread unwinds and remains taut. Choose the line containing OA as the x-axis and the line through O and perpendicular to OA as the y-axis, as shown in the figure. Denote $\angle BOA$ by θ, where B is the point, on the circumference, at which the thread leaves the circle.

(a) Show that the locus of P is given by

$$x = a(\cos\theta + \theta\sin\theta)$$

$$y = a(\sin\theta - \theta\cos\theta)\,, \quad \text{where } \theta \geq 0\,.$$

(b) Show that $\frac{dy}{dx} = \tan\theta$.

Hence show that the thread is normal to the locus at any moment.

(c) Find the area bounded by the locus, the positive x and y axes and the thread when $\theta = \frac{\pi}{2}$.

5. (a) Show that if f is a polynomial of degree ≤ 2, then it satisfies the condition

$$f(x) - f(t) = (x - t)\,f'\left(\frac{x+t}{2}\right) \quad \text{for any } x, t \in \mathbf{R}\,.$$

(b) Let $g : \mathbf{R} \to \mathbf{R}$ be a function with a continuous third derivative. Suppose $g(x) - g(t) = (x - t)\,g'\left(\frac{x+t}{2}\right)$ for any $x, t \in \mathbf{R}$.

(i) Show that

$$g'''\left(\frac{x+t}{2}\right) = \frac{4\left(g''(x) - g''\left(\frac{x+t}{2}\right)\right)}{x - t} \quad \text{for } x \neq t\,.$$

(ii) Find $g'''(t)$ for any $t \in \mathbf{R}$.

Hence, or otherwise, show that g is a polynomial of degree ≤ 2.

6. Consider the function $f(x) = \begin{cases} x^2(x+1)^{\frac{2}{3}} & \text{for } x > 0\,, \\ -x^2(x+1)^{\frac{2}{3}} & \text{for } x \leq 0\,. \end{cases}$

(a) Find $f'(x)$ and $f''(x)$ for $x \neq 0, -1$.

Discuss the existence of $f'(x)$ and $f''(x)$ when $x = 0, -1$.

(b) Determine the values of x at which the graph of f has an extreme point or a point of inflexion.
[Note. Your working may be given in table form.]

(c) Use the above results to sketch the graph of f.

7. Let n be a positive integer and $i = \sqrt{-1}$.

(a) For any integers p and q , find

$$\int_0^{2\pi} (\cos px \cos qx + \sin px \sin qx)\,dx \ .$$

(b) If $f(x) = \sum_{p=0}^{n} a_p (\cos x + i \sin x)^p$, where the a_p's are real constants, show that

$$\int_0^{2\pi} |f(x)|^2\,dx = 2\pi \sum_{p=0}^{n} a_p^{\,2} \ .$$

(c) Let $g(x) = (1 + \cos x + i \sin x)^n$, $h(x) = (1 - \cos x - i \sin x)^n$.

(i) Show that $\int_0^{2\pi} |g(x)|^2\,dx = \int_0^{2\pi} |h(x)|^2\,dx$.

(ii) By considering the coefficient of x^n in the expansion of $(1 + x)^{2n}$, or otherwise, show that the common value of the integrals in (i) is $2\pi C_n^{2n}$.

8. Consider two lines

$$L_1 : \begin{cases} x + y = 0 \\ y + z = 0 \end{cases} \quad \text{and} \quad L_2 : \begin{cases} x = 1 + 2t \\ y = 1 \\ z = 1 + t \end{cases} .$$

(a) Show that L_1 and L_2 are not coplanar.

(b) Find the equation of the plane π_1 containing L_1 and parallel to L_2 .

(c) Find the equation of the plane π_2 which contains L_2 and is perpendicular to π_1 .

(d) Find the coordinates of the foot Q of the perpendicular from the point $P(1, 1, 1)$ to the plane π_1 .
Hence determine the shortest distance between L_1 and L_2 .

9. A real-valued function f defined on an interval I is said to be *Lipschitz-continuous* if there is a constant $k > 0$ such that

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2| \quad \text{for any } x_1 , x_2 \in I .$$

(a) Show that a function which is Lipschitz-continuous is also continuous.

Verify that the continuous function $g(x) = \sqrt{x}$ defined on the interval $[0, 1]$ is NOT Lipschitz-continuous.

(b) If f is continuously differentiable with $|f'(x)| \leq M$ for all $x \in I$, where M is a positive constant, show that f is Lipschitz-continuous.

(c) Let $f : [a, b] \to [a, b]$ be a continuous function.

(i) Show that the equation $x - f(x) = 0$ has a solution in $[a, b]$.
[Hint: You may use the fact that if h is a continuous function on $[a, b]$ such that $h(a) < 0$ and $h(b) > 0$, then there exists $x_0 \in (a, b)$ such that $h(x_0) = 0$.]

(ii) Assume further that f is Lipschitz-continuous with $0 < k < 1$. Show that the solution in (c)(i) is unique.

END OF PAPER