HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1987

純數學 試卷一

PURE MATHEMATICS PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. Let $\mathcal{M}$ be the set of 2×2 real matrices and I be the identity matrix of order 2.

(a) For any $A \in \mathcal{M}$, show that if $A^3 = I$, then $\det A = 1$.

(b) Let $B \in \mathcal{M}$ such that $B^2 + B + I = 0$.

(i) Show that $B^3 = I$ and $B^{-1} = -(B + I)$.

(ii) Simplify $I + B + B^2 + \ldots + B^{100}$.

(iii) If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $a + d = -1$.

(c) Find a matrix $M \in \mathcal{M}$ with integral entries such that

$$M \neq I \text{ and } M^3 = I.$$

2. (a) For any real numbers a, b and c such that $a^2 + b^2 + c^2 = 1$ and $c \neq 1$, let $z = \dfrac{a + ib}{1 - c}$.

(i) Show that $|z|^2 = \dfrac{1 + c}{1 - c}$.

(ii) Express each of a, b and c in terms of z and $\bar{z}$.

(b) Let $S = \{(a, b, c) : a, b, c \in \mathbf{R},\ a^2 + b^2 + c^2 = 1 \text{ and } c \neq 1\}$. A mapping $f : S \to \mathbf{C}$ is defined by

$$f((a, b, c)) = \frac{a + ib}{1 - c}.$$

(i) Show that f is a bijection.

(ii) Let $A = \{(a, b, c) \in S : a = b\}$.

Sketch the direct image $f[A]$ on the complex plane.

3. Let a and b be two positive real numbers not both equal to 1. For $n = 1, 2, 3, \ldots$, let $x_n = n\left(\dfrac{\sqrt[n]{a} + \sqrt[n]{b}}{2} - 1\right)$.

(a) (i) Find $\displaystyle\lim_{h \to 0} \frac{a^h - 1}{h}$.

(ii) Show that $\displaystyle\lim_{n \to \infty} x_n = \ln \sqrt{ab}$.

(iii) If $\displaystyle\lim_{n \to \infty} x_n = 0$, show that $x_n \neq 0$ for all n.

(b) For $n = 1, 2, 3, \ldots$, let $y_n = \left(\dfrac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n$. By expressing y_n in terms of n and x_n, find $\displaystyle\lim_{n \to \infty} \ln y_n$.

Hence, or otherwise, show that

$$\lim_{n \to \infty} y_n = \sqrt{ab}.$$

4. There are $n (n > 1)$ different boxes each of which can hold up to $n + 2$ books. Find the probability that

(a) no box is empty when n different books are put into the boxes at random,

(b) exactly one box is empty when n different books are put into the boxes at random,

(c) no box is empty when $n + 1$ different books are put into the boxes at random,

(d) no box is empty when $n + 2$ different books are put into the boxes at random,

(e) exactly one box is empty when $n + 1$ different books are put into the boxes at random.

5. (a) Given a real number α with $0 < \alpha < 1$, show that if $0 < x \leq 1$,

(i) $(1 + x)^{\alpha} < 1 + \alpha x$,

(ii) $(1 - x)^{\alpha} < 1 - \alpha x$.

(b) Show that for any positive integers n and k,

$$\left(\frac{n+1}{n}\right)\left((k+1)^{\frac{n}{n+1}} - k^{\frac{n}{n+1}}\right) < \frac{1}{\left(k^{\frac{1}{n+1}}\right)} < \left(\frac{n+1}{n}\right)\left(k^{\frac{n}{n+1}} - (k-1)^{\frac{n}{n+1}}\right)$$

(c) Show that $14\,998 < \frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \ldots + \frac{1}{\sqrt[3]{1\,000\,000}} < 15\,000$.

6. It is known that for any continuous function ϕ defined on $\mathbf{R}$, if $\phi(x + y) = \phi(x) + \phi(y)$ for any $x, y \in \mathbf{R}$, then $\phi(x) = \phi(1)x$ for any $x \in \mathbf{R}$.

(a) Suppose f is a non-constant continuous function defined for all positive real numbers such that

$$f(xy) = f(x) + f(y) \quad \text{for any } x, y > 0 .$$

The function g is defined by $g(t) = f(e^t)$ for $t \in \mathbf{R}$.

(i) Show that $g(t) = g(1)t$ for any $t \in \mathbf{R}$.

(ii) Deduce that $f(x) = \log_b x$ for any $x > 0$, where $b = e^{\frac{1}{f(e)}}$.

(b) Suppose h is a non-constant continuous function defined for all positive real numbers such that

$$h(xy) = h(x)h(y) \quad \text{for all } x, y > 0 .$$

Consider the function $H(x) = \log_e h(x)$ for $x > 0$.

(i) Show that $H(x)$ is well defined (i.e. $h(x) > 0$) for all positive x.

(ii) Using (a), or otherwise, show that $h(x) = x^c$ for all $x > 0$, where c is a real constant.

7. A relation R is defined on the set $A = \{ (m, n) : m, n = 0, 1, 2, \ldots \}$ by $(m', n')R(m'', n'')$ iff $m' + n'' = n' + m''$.

(a) Show that R is an equivalence relation.

(b) Let A/R be the quotient set defined by R and let $[m, n]$ denote the equivalence class containing (m, n). A function $f: A/R \to \mathbf{Z}$ is defined by $f([m, n]) = m - n$.

(i) Show that f is well defined.

(ii) Show that f is a bijective mapping.

(c) Given $(a, b) \in A$ with $a, b \neq 0$, a function $h: A/R \to A/R$ is defined by $h([m, n]) = [am + bn, bm + an]$.

(i) Show that $(f \circ h)([m, n]) = (a - b)(m - n)$.

(ii) Show that h is injective iff $a \neq b$.

(iii) Show that h is surjective iff $|a - b| = 1$.

8. Given two positive integers n and r, let

$$P(x) = x^r + (x + 1)^r + \ldots + (x + n)^r .$$

(a) When $P(x)$ is written in the form $P(x) = \sum_{t=0}^{r} a_t x^t$, show that

$$a_r = n + 1 ,\ a_t = C_t^r \left(1^{r-t} + 2^{r-t} + \ldots + n^{r-t}\right) \text{ for } t = 0, 1, 2, \ldots, r-1 .$$

(b) Let $S(0, n) = n + 1$ and $S(t, n) = \sum_{m=1}^{n} m^t$, where $t = 1, 2, \ldots$.

Show that $(n + 1)^r = \sum_{t=0}^{r-1} C_t^r \, S(t, n)$.

(c) Use (b) to find $S(1, n)$, $S(2, n)$ and $S(3, n)$.

9. (a) Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$ be three vectors in $\mathbf{R}^3$ such that

$$\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{c} = 1$$

and $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$.

(i) If $M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, show that $MM^T = I$, where M^T is the transpose of M and I is the identity matrix of order 3.

(ii) For any $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$, by considering the system of equations

$$\begin{cases} a_1u_1 + a_2u_2 + a_3u_3 = 0 \\ b_1u_1 + b_2u_2 + b_3u_3 = 0 \\ c_1u_1 + c_2u_2 + c_3u_3 = 0 , \end{cases}$$

show that if $\mathbf{u}\cdot\mathbf{a} = \mathbf{u}\cdot\mathbf{b} = \mathbf{u}\cdot\mathbf{c} = 0$, then $\mathbf{u} = \mathbf{0}$.

(iii) Use (ii) to deduce that for any $\mathbf{v} \in \mathbf{R}^3$,

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{a})\mathbf{a} + (\mathbf{v}\cdot\mathbf{b})\mathbf{b} + (\mathbf{v}\cdot\mathbf{c})\mathbf{c} .$$

[Hint : Put $\mathbf{u} = \mathbf{v} - [(\mathbf{v}\cdot\mathbf{a})\mathbf{a} + (\mathbf{v}\cdot\mathbf{b})\mathbf{b} + (\mathbf{v}\cdot\mathbf{c})\mathbf{c}]$.]

(b) Let $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ be a mapping such that $\phi(\mathbf{x})\cdot\phi(\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$.

Show that $\phi(\mathbf{x}) = x_1\phi(\mathbf{i}) + x_2\phi(\mathbf{j}) + x_3\phi(\mathbf{k})$, where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

Hence, or otherwise, show that ϕ is linear, i.e.

$$\phi(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\phi(\mathbf{x}) + \mu\phi(\mathbf{y})$$

for all $\lambda, \mu \in \mathbf{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$.

END OF PAPER

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HONG KONG ADVANCED LEVEL EXAMINATION 1987

純數學 試卷二
PURE MATHEMATICS PAPER II

2.00 pm–5.00 pm (3 hours)
This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. (a) Let $I_k = \int_0^{\frac{\pi}{2}} \cos^k x \, dx$, $k = 0, 1, 2, \ldots$.

Express I_{k+2} in terms of k and I_k .

Hence evaluate I_{2n} and I_{2n+1} for $n = 0, 1, 2, \ldots$.

(b) For any positive integer m, show that

$$\int_0^{\frac{\pi}{2}} \cos^{m+2} x \, dx < \int_0^{\frac{\pi}{2}} \cos^{m+1} x \, dx < \int_0^{\frac{\pi}{2}} \cos^m x \, dx .$$

Hence evaluate $\lim_{m \to \infty} \dfrac{\int_0^{\frac{\pi}{2}} \cos^{m+1} x \, dx}{\int_0^{\frac{\pi}{2}} \cos^m x \, dx}$.

(c) Using (a) and (b), or otherwise, evaluate

$$\lim_{n \to \infty} \frac{2 \times 2}{1 \times 3} \times \frac{4 \times 4}{3 \times 5} \times \cdots \times \frac{2n \times 2n}{(2n-1)(2n+1)} .$$

2. Consider the parabola Γ with parametric equations

$$\begin{cases} x = at^2 \\ y = 2at, \quad a > 0 . \end{cases}$$

(a) Show that the equation of the normal to Γ at the point $(at^2, 2at)$ is

$$tx + y - (at^3 + 2at) = 0 .$$

(b) If $t_1 \neq 0$, show that the normal to Γ at the point $(at_1^2, 2at_1)$ meets Γ again at a point $(at_2^2, 2at_2)$, where $t_2 \neq t_1$.

(c) Let $\{P_n(x_n, y_n)\}$ be a sequence of points on Γ such that P_nP_{n+1} is normal to Γ at P_n for all positive integers n .

(i) Show that $x_{n+1} - x_n = \dfrac{4a^2}{x_n} + 4a$.

(ii) Prove that $x_{n+1} - x_1 > 4na$ and $\lim_{n \to \infty} \dfrac{1}{x_n} = 0$.

(iii) Find $\lim_{n \to \infty} (x_{n+1} - x_n)$ and $\lim_{n \to \infty} (|y_{n+1}| - |y_n|)$.

3. For $n = 1, 2, 3, \ldots$, let $a_n = \sum_{r=0}^{n} \frac{1}{r!}$ and $b_n = \left(1 + \frac{1}{n}\right)^n$.

(a) Show that the sequence $\{a_n\}$ is convergent.

[Note: You may use without proof the fact that a monotonic increasing sequence which is bounded above converges.]

(b) Show that $b_n = 2 + \sum_{r=2}^{n} \frac{1}{r!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{r-1}{n}\right)$.

Hence deduce that $b_n \leqslant a_n$.

(c) If n is fixed and greater than 1, show by induction on r that, for $2 \leqslant r \leqslant n$,

$$1 - \frac{r(r-1)}{n} \leqslant \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{r-1}{n}\right).$$

Deduce that $\left(1 - \frac{1}{n}\right) a_n \leqslant b_n$.

Hence show that the sequence $\{b_n\}$ converges and $\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n$.

4. (a) For any non-negative integers m and n, let

$$B(m, n) = \int_0^1 x^m (1 - x)^n \, dx.$$

Show that $B(m, n) = \frac{n}{m+1} B(m+1, n-1)$ for any $m \geqslant 0$, $n \geqslant 1$.

Hence, or otherwise, deduce that

$$B(m, n) = \frac{m!\,n!}{(m + n + 1)!}.$$

(b) (i) Evaluate $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx$.

(ii) Using (b)(i) and (a), show that

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}.$$

5. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ be a polynomial of degree n with real coefficients $a_n, a_{n-1}, \ldots, a_0$, such that

$$\int_0^1 x^k f(x)\,dx = 0$$

for $k = 1, 2, \ldots, n$.

(a) Show that

(i) $\displaystyle\int_0^1 [f(x)]^2\,dx = a_0 \int_0^1 f(x)\,dx$,

(ii) $\dfrac{a_n}{k+n+1} + \dfrac{a_{n-1}}{k+n} + \ldots + \dfrac{a_0}{k+1} = 0$ for $k = 1, 2, \ldots, n$.

(b) Prove that $\dfrac{a_n}{t+n+1} + \dfrac{a_{n-1}}{t+n} + \ldots + \dfrac{a_0}{t+1}$ can be written as

$$\frac{C(t-1)(t-2)\ldots(t-n)}{(t+n+1)(t+n)\ldots(t+1)} ,$$

where C is a constant.

(c) Show that the constant C in (b) equals $(-1)^n (n+1) \displaystyle\int_0^1 f(x)\,dx$.

Hence, or otherwise, show that

$$\int_0^1 [f(x)]^2\,dx = (n+1)^2 \left\{\int_0^1 f(x)\,dx\right\}^2 .$$

6. (a) Let $\ell_1 : \begin{cases} x = a_1 + p_1 t \\ y = b_1 + q_1 t \\ z = c_1 + r_1 t \end{cases}$ and $\ell_2 : \begin{cases} x = a_2 + p_2 t \\ y = b_2 + q_2 t \\ z = c_2 + r_2 t \end{cases}$ be two given lines. Suppose ℓ_1 and ℓ_2 intersect.

(i) Show that $\begin{vmatrix} a_1 - a_2 & p_1 & p_2 \\ b_1 - b_2 & q_1 & q_2 \\ c_1 - c_2 & r_1 & r_2 \end{vmatrix} = 0$.

(ii) If ℓ_1 and ℓ_2 are distinct, find a vector normal to the plane containing ℓ_1 and ℓ_2 .

Hence, or otherwise, obtain the equation of this plane.

(b) Consider the lines

$$L_1 : \begin{cases} x = pt \\ y = qt \\ z = rt \end{cases} ,$$

$$L_2 : \begin{cases} x = qt \\ y = rt \\ z = pt \end{cases} ,$$

and $$L_3 : \begin{cases} x = rt \\ y = pt \\ z = qt \end{cases} ,$$

where p, q and r are distinct and non-zero. Find the equation of a plane containing L_1 and perpendicular to the plane which contains L_2 and L_3 when

(i) $pq + qr + rp \neq 0$,

(ii) $pq + qr + rp = 0$.

7. Let $f(x) = \dfrac{x\,|x|\,(x+7)}{x-1}$, where $x \in \mathbf{R}$ and $x \neq 1$.

(a) Find $f''(x)$ if $x \neq 0$.

(b) (i) Find the local maximum and minimum points and the asymptotes of the graph of $f(x)$.

(ii) Show that $(-1, 3)$ and $(0, 0)$ are the only points of inflexion.

(c) Sketch the graph of $f(x)$, indicating the extreme points, points of inflexion, asymptotes and intercepts.

8. (a) Let $g: \mathbf{R} \to \mathbf{R}$ be a function with a continuous second derivative and let $a \in \mathbf{R}$. Using integration by parts, or otherwise, show that

$$g(x) = g(a) + (x-a)g'(a) + \int_a^x (x-t)g''(t)\,dt$$

for every $x \in \mathbf{R}$.

(b) Let $f: \mathbf{R} \to \mathbf{R}$ be a function with a continuous second derivative and let $a \in \mathbf{R}$. A function $g: \mathbf{R} \to \mathbf{R}$ is defined by

$$g(x) = \int_a^x f(t)\,dt - \frac{(x-a)}{2}[f(x) + f(a)].$$

(i) Using (a), or otherwise, show that

$$g(x) = -\int_a^x \frac{(x-t)(t-a)}{2} f''(t)\,dt.$$

(ii) Suppose $|f''(t)| \leq M$ for some constant M and for all $t \in [0, 1]$. If $0 \leq a < b \leq 1$, show that

$$\left| \int_a^b f(t)\,dt - \frac{(b-a)}{2}[f(b) + f(a)] \right| \leq \frac{M}{12}(b-a)^3.$$

Deduce that for any positive integer n,

$$\left| \int_0^1 f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{2n}\left[f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right] \right| \leq \frac{M}{12n^2}.$$

[Hint: In the last part, you may divide the interval [0, 1] into n equal sub-intervals.]

9. (a) Let n be a positive integer. It is known that for any functions $f(x)$ and $g(x)$ with nth derivatives, if $h(x) = f(x)g(x)$, then $h^{(n)}(x) = \sum_{k=0}^{n} a_k f^{(k)}(x) g^{(n-k)}(x)$, where $a_0, a_1, \ldots, a_n$ are constants independent of $f(x)$ and $g(x)$, $f^{(0)}(x) = f(x)$ and $f^{(k)}(x) = \frac{d^k f(x)}{dx^k}$. Taking $f(x) = e^{\lambda x}$ and $g(x) = e^x$, where λ is a number independent of x,

(i) find $h^{(n)}(x)$ and $f^{(k)}(x) g^{(n-k)}(x)$ and hence

(ii) show that $a_k = C_k^n$ for $k = 0, 1, 2, \ldots, n$.

(b) Let $u(x) = x^m e^{-x}$,

$y(x) = e^x u^{(m)}(x)$,

where m is a positive integer.

(i) Show that $y(x)$ is a polynomial of degree m and find its coefficients.

(ii) Show that $x u'(x) + (x - m) u(x) = 0$.

Deduce that $x u^{(m+2)}(x) + (x+1) u^{(m+1)}(x) + (m+1) u^{(m)}(x) = 0$.

(iii) Using (ii), or otherwise, show that

$$x y''(x) + (1 - x) y'(x) + m y(x) = 0.$$

END OF PAPER

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PAPER I

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1988

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PURE MATHEMATICS PAPER I

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