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I		
1. (a) (i) If $U \in \mathcal{F}$, $\quad U^TU = I$		
$U^T = U^{-1}$ $\quad \Rightarrow \lvert U^T\rvert\lvert U\rvert = 1$		
$\lvert U^T\rvert = \lvert U^{-1}\rvert$ $\quad \lvert U^TU\rvert = \lvert U^T\rvert\lvert U\rvert$	1	det
$\lvert U\rvert = \frac{1}{\lvert U\rvert}$ $\quad \lvert U^T\rvert = \lvert U\rvert$		
$\lvert U\rvert^2 = 1$ $\quad \lvert U^{-1}\rvert = \frac{1}{\lvert U\rvert}$		
$\therefore \lvert U\rvert = 1$ as $\lvert U\rvert > 0$.	1	apply $\lvert U\rvert >$
(ii) <u>'If' part</u>		
If $U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$,		
$U^{-1} = \frac{1}{\lvert U\rvert}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	1	U^{-1}
$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$		
$= U^T$		
$\therefore U \in \mathcal{F}$	1	
<u>'Only if' part</u>		
Suppose $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{F}$, then $U^T = U^{-1}$		
$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{\lvert U\rvert}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$	1	
$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ as $\lvert U\rvert = 1$ by (i)	1	
$\Rightarrow a = d$ and $c = -b$. ①		
$U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$		
and $\lvert U\rvert = a^2 + b^2 = 1$ .... ②	1 7	

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(b) (i) Let $U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathcal{J}$

$UBU^T = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$

$\Rightarrow \quad B = U^{-1}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}(U^T)^{-1}$ 1

$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ 1

$= \begin{pmatrix} a^2p + b^2q & ab(p - q) \\ ab(p - q) & b^2p + a^2q \end{pmatrix}$

$\therefore \quad x = y$ 1

i.e. $B = B^T$

(ii) If $B = B^T$, let $B = \begin{pmatrix} w & x \\ x & z \end{pmatrix}$.

For $U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathcal{J}$, $a^2 + b^2 = 1$

$UBU^{-1} = \begin{pmatrix} aw + bx & ax + bz \\ -bw + ax & -bx + az \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$= \begin{pmatrix} a^2w + 2abx + b^2z & (a^2-b^2)x + ab(z-w) \\ (a^2-b^2)x + ab(z-w) & b^2w - 2abx + a^2z \end{pmatrix}$ 1

If $x = 0$, $B = \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}$, then $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 作 satisfies the given condition. 1

If $x \neq 0$, equating $(a^2 - b^2)x + ab(z - w)$ to zero.

$\boxed{\frac{a^2 - b^2}{ab} = \frac{w - z}{x} = c}$, say 1

$\frac{a^2 - (1 - a^2)}{a\sqrt{1 - a^2}} = c$ $X \neq 0$ 所以要分[illegible]

$(4 + c^2)a^4 - (4 + c^2)a^2 + 1 = 0$

$a^2 = \frac{(4 + c^2) \pm \sqrt{c^2(4 + c^2)}}{2(4 + c^2)}$

Putting $a = [\frac{(4 + c^2) \pm \sqrt{c^2(4 + c^2)}}{2(4 + c^2)}]^{\frac{1}{2}}$, $0 < a < 1$

$b = \sqrt{1 - a^2}$, ... 1

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2\. (a) (i) For any given a, b, c, (E) has a unique solution iff

$\begin{vmatrix} 1 & 1 & -1 \\ -k & -1 & k \\ k^2 & 1 & -k \end{vmatrix} \neq 0$ 1

$k(k - 1)^2 \neq 0$ 1

$\therefore \; k \neq 0$ and $k \neq 1$ 1

(ii) For $k = 0$, the system becomes

$\Delta < 0$

$\frac{\Delta_i}{\Delta} = 0$ by substitution

$\begin{cases} x + y - z = a \\ -y = b \\ y = c \end{cases}$ 1

$\therefore$ for any values of a, b, c with $b = -c$, the system is consistent. 1

(iii) For $a = b = c = 0$ and $k = 1$, the system reduces to

$x + y - z = 0$ 1

(1, 0, 1) and (1, -1, 0) are two linearly independent solutions as one is not a scalar multiple of the other. 1 + 1

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(b) Assuming, for contradiction, that (E) is consistent, then

$x + y - z = a$

$-kx - y + kz = b$

$k^2x + y - kz = c$

for some $x, y, z \in R$ 1

$ax_0 + by_0 + cz_0 = (x+y-z)x_0 + (-kx-y+kz)y_0 + (k^2x+y-kz)z_0$ 2

$= (x_0-ky_0+k^2z_0)x + (x_0-y_0+z_0)y + (-x_0+ky_0-kz_0)z$ 1

$= 0 + 0 + 0$

$= 0$ (as (x_0, y_0, z_0) satisfies the 2nd system) 2

This contradicts the assumption that $(a,b,c) \cdot (x_0,y_0,z_0) \neq 0$.

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Alternatively

(b) As (x_0 , y_0 , z_0) satisfies the 2nd system,

$$\begin{cases} x_0 - ky_0 + k^2z_0 = 0 \\ x_0 - y_0 + z_0 = 0 \\ -x_0 + ky_0 - kz_0 = 0 \end{cases}$$ 1

For any real x, y, z,

$x(x_0 - ky_0 + k^2z_0) + y(x_0 - y_0 + z_0) + z(-x_0 + ky_0 - kz_0) = 0$ 2

$(x + y - z)x_0 + (-kx - y + kz)y_0 + (k^2x + y - kz)z_0 = 0$. 1

Since $(a, b, c) \cdot (x_0, y_0, z_0) \neq 0$, the system

$x + y - z = a$

$-kx - y + kz = b$

$k^2x + y - kz = c$

cannot hold simultaneously. 2

i.e. (E) is not solvable.

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3. (a) (i) <u>'If' part</u>

If $\underline{x}_k$ $(1 \leq k \leq n)$ is a linear combination of the other vectors, let

$\underline{x}_k = \lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \ldots + \lambda_{k-1}\underline{x}_{k-1} + \lambda_{k+1}\underline{x}_{k+1} + \ldots + \lambda_n\underline{x}_n$ 1

Then $\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \ldots - \underline{x}_k + \ldots + \lambda_n\underline{x}_n = \underline{0}$

Since the coefficient of $\underline{x}_k \neq 0$, the vectors

$\underline{x}_1, \underline{x}_2, \ldots , \underline{x}_n$ are linearly dependent. 1

<u>'Only if' part</u>

If $\underline{x}_1 , \underline{x}_2 , \ldots , \underline{x}_n$ are linearly dependent,

$\exists$ $\lambda_1 , \lambda_2 , \ldots , \lambda_n$, not all zero, such that

$\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \ldots + \lambda_n\underline{x}_n = \underline{0}$ 1

Let $\lambda_k \neq 0$ for some k.

Then

$\lambda_k\underline{x}_k = -[\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \ldots + \lambda_{k-1}\underline{x}_{k-1} + \lambda_{k+1}\underline{x}_{k+1} + \ldots + \lambda_n\underline{x}_n]$

or

$\underline{x}_k = -\frac{1}{\lambda_k}[\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \ldots + \lambda_{k-1}\underline{x}_{k-1} + \lambda_{k+1}\underline{x}_{k+1} + \ldots + \lambda_n\underline{x}_n]$ 1

(ii) Let $\underline{x}_{i_1} , \underline{x}_{i_2} , \ldots , \underline{x}_{i_k}$ be k of the vectors which are linearly dependent, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

$\exists$ $u_1 , u_2 , \ldots , u_k$, not all zero, such that

$u_1\underline{x}_{i_1} + u_2\underline{x}_{i_2} + \ldots + u_k\underline{x}_{i_k} = \underline{0}$ 1

Let $\lambda_\ell = \begin{cases} u_j & \text{if } \ell = i_j . \\ 0 & \text{otherwise.} \end{cases}$

Then $\exists$ $\lambda_1 , \lambda_2 , \ldots , \lambda_n$, not all zero, such that

$\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \ldots + \lambda_n\underline{x}_n = \underline{0}$.

$\therefore$ the n given vectors are linearly dependent. 1

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(b) (i) If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then the system

$$\begin{cases} \boxed{a_1}x + \boxed{b_1}y + \boxed{c_1}z = 0 \\ \boxed{a_2}x + \boxed{b_2}y + \boxed{c_2}z = 0 \\ \boxed{a_3}x + \boxed{b_3}y + \boxed{c_3}z = 0 \end{cases}$$

|k|≠0 ⇒ trivial soln (0,0,0)

|k|=0 ⇒ non-trivial

has a non-trivial solution $(\lambda_1, \lambda_2, \lambda_3)$. 2

i.e. $\exists\ \lambda_1, \lambda_2, \lambda_3$, not all zero, such that

$\lambda_1(a_1, a_2, a_3) + \lambda_2(b_1, b_2, b_3) + \lambda_3(c_1, c_2, c_3) = (0,0,0)$ 1

$\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ are linearly dependent.

(ii) If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

non-homogeneous equation

$\vec{x}_1, \vec{x}_2, \vec{x}_3$ are three linearly independent vector

has a unique solution $(\lambda_1, \lambda_2, \lambda_3)$ 2

a linear combination of vectors

⇒ linearly

i.e. $\underline{x}_4 = \lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \lambda_3\underline{x}_3$

by (a)(i), $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, $\underline{x}_4$ are linearly dependent. 1

For any four vectors $\underline{x}_1 = (a_1, a_2, a_3)$, $\underline{x}_2 = (b_1, b_2, b_3)$, $\underline{x}_3 = (c_1, c_2, c_3)$, $\underline{x}_4 = (d_1, d_2, d_3)$ in $\mathbb{R}^3$,

either $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

or $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ 1

In each case, the four vectors are linearly dependent 1

by (b)(i) and (ii)

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4. (a) (i) Suppose f is injective.

Since fof = f , for any $x \in X$

$$\begin{aligned} & fof(x) = f(x) \quad \ldots\ldots\ldots\ldots 1 \\ \Rightarrow\ & f(f(x)) = f(x) \\ \Rightarrow\ & f(x) = x \quad \ldots\ldots\ldots\ldots 1 \end{aligned}$$

(∵ inject)

i.e. $f = i_X$

Suppose f is surjective.

For any $y \in X$, let $y = f(x)$ for some $x \in X$ 1

∵ surject

$$\begin{aligned} f(y) &= f(f(x)) \\ &= fof(x) \quad \ldots\ldots\ldots\ldots 1 \\ &= f(x) \\ &= y \quad \ldots\ldots\ldots\ldots 1 \end{aligned}$$

i.e. $f = i_X$

(ii) If $X = \{a, b\}$, consider $f: X \to X$ defined by

$f(a) = f(b) = a$. or f(a) = f(b) = b

not injective

not surjective

Clearly fof = f but

f is neither injective

nor surjective. 2

If X contains more than two elements; let a, b be two distinct elements of X . Consider f defined by

$f(a) = f(b) = a$ and $f(x) = x$ for other $x \in X$.

f is neither injective nor surjective but it satisfies

fof = f and f is non-constant. 2

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(b) $\mathcal{P}(E) \neq \emptyset$.

For any $A \subset E$,
$$\begin{aligned} hoh(A) &= h(h(A)) \quad \ldots\ldots\ldots\ldots 1 \\ &= h(A \cap B) \\ &= (A \cap B) \cap B \\ &= A \cap B \\ &= h(A) \quad \ldots\ldots\ldots\ldots 1 \end{aligned}$$

∴ hoh = h .

If h is injective or surjective, by (a)(i),

$h = i_{\mathcal{P}(E)}$ 1

$$\begin{aligned} \therefore\ E &= h(E) \quad 1 \\ &= E \cap B \\ &= B \quad \ldots\ldots\ldots\ldots 1 \end{aligned}$$

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5. (a) (i) Putting $x = 1$ in $(1 + x)^{2n+1} = \sum_{r=0}^{2n+1} C_r^{2n+1} x^r$, I	1	
$2^{2n+1} = \sum_{r=0}^{2n+1} C_r^{2n+1}$.		
Since $C_r^{2n+1} = C_{2n+1-r}^{2n+1}$,	1	
$\sum_{r=1}^{n+1} C_{n+r}^{2n+1} = \frac{1}{2} \cdot 2^{2n+1} = 2^{2n}$ ……………………	1	
[illegible]		
(ii) $(1 + x)^m(1 + \frac{1}{x})^n = \frac{1}{x^n} (1 + x)^{m+n}$		
$= \frac{1}{x^n} \sum_{r=0}^{m+n} C_r^{m+n} x^r$	1	
For $m - n \leqslant k \leqslant m$, the coefficient of x^k is C_{n+k}^{m+n}.	1	
On the other hand,		
$(1 + x)^m(1 + \frac{1}{x})^n = (\sum_{s=0}^{m} C_s^m x^s)(\sum_{r=0}^{n} C_r^n \frac{1}{x^r})$	1	
The coefficient of $x^k = \sum_{r=0}^{m-k} C_{k+r}^{m} C_r^{n}$.		
Hence the result.	1	
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5. (b) (i) When B tosses n coins, the probability that he will obtain r heads $(r = 0, 1, \ldots, n)$ is I		
$C_r^n \frac{1}{2^r} \cdot \frac{1}{2^{n-r}} = \frac{C_r^n}{2^n}$.	1	
When A and B toss their coins, the probability that A will obtain k $(1 \leqslant k \leqslant n + 1)$ more heads than		
B is $\sum_{r=0}^{n+1-k} \frac{C_{k+r}^{n+1}}{2^{n+1}} \cdot \frac{C_r^n}{2^n} = \frac{1}{2^{2n+1}} \sum_{r=0}^{n+1-k} C_{k+r}^{n+1} C_r^n$ ……	2	
$r \leq n+1-k$ ~~[illegible]~~ $= \frac{1}{2^{2n+1}} C_{n+k}^{2n+1}$	1	
(ii) The probability that A will obtain more heads than B is $\sum_{k=1}^{n+1} \frac{1}{2^{2n+1}} C_{n+k}^{2n+1}$ …by (ai)…………	2	
$= \frac{1}{2^{2n+1}} \cdot 2^{2n}$		
$= \frac{1}{2}$ …………………………………	1	
	7	

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6(a) Let $f(x) = 2^{p-1}(1 + x^p) - (1 + x)^p - (1 - x)^p$		
$f'(x) = p[(2x)^{p-1} - (1 + x)^{p-1} + (1 - x)^{p-1}]$	1	
For $0 \leqslant x \leqslant 1$, $p \geqslant 2$,		
$(\frac{1-x}{1+x})^{p-1} \leqslant \frac{1-x}{1+x}$ and $(\frac{2x}{1+x})^{p-1} \leqslant \frac{2x}{1+x}$		
$(\frac{1-x}{1+x})^{p-1} + (\frac{2x}{1+x})^{p-1} \leqslant \frac{1-x}{1+x} + \frac{2x}{1+x}$		
$= 1$	1	
i.e. $(1 - x)^{p-1} + (2x)^{p-1} \leqslant (1 + x)^{p-1}$		
$\therefore$ $f'(x) \leqslant 0$ and f is decreasing in $[0, 1]$.	1	
$f(x) \geqslant f(1)$		
$= 0$		
i.e. $(1 + x)^p + (1 - x)^p \leqslant 2^{p-1}(1 + x^p)$	1 4	
(b) $h'(\theta) = pr\sin\theta[(1 + r^2 - 2r\cos\theta)^{\frac{p}{2}-1} - (1 + r^2 + 2r\cos\theta)^{\frac{p}{2}-1}]$ $\geqslant 0$	1	
For $r > 0$, $\boxed{p \geqslant 2}$ $h'(\theta) \leqslant 0$ if $x \in [0, \frac{\pi}{2}]$	1	
and $h'(\theta) \geqslant 0$ if $x \in [\frac{\pi}{2}, \pi]$.		
$\therefore$ $h(\theta)$ is decreasing in $[0, \frac{\pi}{2}]$ and increasing in $[\frac{\pi}{2}, \pi]$.		
☆ $h(\theta) \leqslant h(0) = h(\pi)$ if $\theta \in [0, \pi]$	1	
Noting that $h(\theta + n\pi) = h(\theta)$ $\forall n \in I$,		
$h(\theta) \leqslant h(0)$ for any $\theta \in \mathbb{R}$.	1 4	
(c) The inequality is trivial if either z_1 or z_2 equals zero.		
Without loss of generality, suppose ① $\lvert z_1\rvert \geqslant \lvert z_2\rvert > 0$.		
We shall prove ② $\left\lvert 1 + \frac{z_2}{z_1}\right\rvert^p + \left\lvert 1 - \frac{z_2}{z_1}\right\rvert^p \leqslant 2^{p-1}(1 + \left\lvert\frac{z_2}{z_1}\right\rvert^p)$	1	
③ Let $\frac{z_2}{z_1} = r\,\mathrm{cis}\,\theta$, $0 < r \leqslant 1$.	1	
$\left\lvert 1 + \frac{z_2}{z_1}\right\rvert^2 = \lvert 1 + r\,\mathrm{cis}\,\theta\rvert^2$		
$= (1 + r\cos\theta)^2 + (r\sin\theta)^2$		
$= 1 + r^2 + 2r\cos\theta$	1	
$\therefore$ $\left\lvert 1 + \frac{z_2}{z_1}\right\rvert^p + \left\lvert 1 - \frac{z_2}{z_1}\right\rvert^p = (1+r^2+2r\cos\theta)^{\frac{p}{2}} + (1+r^2-2r\cos\theta)^{\frac{p}{2}}$		
$\leqslant (1 + r)^p + (1 - r)^p$ by (b)	1	
$\leqslant 2^{p-1}(1 + r^p)$ by (a)	1	
$= 2^{p-1}(1 + \left\lvert\frac{z_2}{z_1}\right\rvert^p)$	1 6	

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7. (a) Putting $z = x+iy$, $q = q_0+iq_1$, $r = r_0+ir_1$, $q_0, q_1, r_0, r_1 \in \mathbb{R}$,		
(*) becomes $x^2+y^2+(q_0+iq_1)(x+iy)+(r_0+ir_1)(x-iy)+s = 0 + 0i$.	1	
i.e. $\begin{cases} x^2 + y^2 + (q_0+r_0)x + (r_1-q_1)y + s = 0 \quad \ldots\ldots(i) \\ (q_1+r_1)x + (q_0-r_0)y = 0 \quad \ldots\ldots\ldots\ldots(ii). \end{cases}$	1 1	
As $s < 0$, (i) is a circle with positive radius. $(\frac{q_0+r_0}{2})^2 + (\frac{r_1-q_1}{2})^2 - s >$	1	
☆ Further the origin lies inside (i) $\sqrt{(\frac{q_0+r_0}{2})^2+(\frac{r_1-q_1}{2})^2 - s} > \sqrt{(\frac{q_0+r_0}{2})^2+(\frac{r_1-q_1}{2})^2}$ $(s<0)$	1	
If $q_1 + r_1 = q_0 - r_0 = 0$, (i) and (ii) (and therefore (*))		
has infinitely many solutions.	1	
Otherwise (ii) is a straight line through the origin	1	
intersecting (i) at two points. $\therefore$ (*) has 2 solutions.	1 8	
(b) $[tu_1\bar{u}_2+(1-t)v_1\bar{v}_2][(u_1+zv_1)(\bar{u}_1+\bar{z}\bar{v}_1)+(u_2+zv_2)(\bar{u}_2+\bar{z}\bar{v}_2)]$		
$= (u_1+zv_1)(\bar{u}_2+\bar{z}\bar{v}_2)$	1	
☆ Expanding and using $\lvert u_1\rvert^2 + \lvert u_2\rvert^2 = \lvert v_1\rvert^2 + \lvert v_2\rvert^2 = 1$, we obtain	1	
$t(v_1\bar{v}_2 - u_1\bar{u}_2)\lvert z\rvert^2 + Pz + Q\bar{z} + (1-t)(u_1\bar{u}_2 - v_1\bar{v}_2) = 0$, $P, Q \in \overset{\mathbb{C}}{\mathbb{R}}$.	1	
Since $u_1\bar{u}_2 \neq v_1\bar{v}_2$, $t > 0$, this can be written as		
$\lvert z\rvert^2 + pz + q\bar{z} + \frac{t-1}{t} = 0$	1	
where $\underbrace{\frac{t-1}{t}}_{=s} < 0$ as $0 < t < 1$.	1	
By (a), this equation has at least two solutions.	1 6	

[illegible]

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3. (a) If $d_1(x)$ and $d_2(x)$ divide each other, let		
$d_1(x) = h_1(x)\, d_2(x)$	1	
and $d_2(x) = h_2(x)\, d_1(x)$ for some polynomials $h_1(x)$, $h_2(x)$.		
Then $\deg d_1(x) = \deg h_1(x) + \deg d_2(x)$	1	
$= \deg h_1(x) + [\deg h_2(x) + \deg d_1(x)]$		
$\therefore \deg h_1(x) = \deg h_2(x) = 0$.		
$h_1(x) = k \neq 0$ (as $d_1(x)$ and $d_2(x) \neq 0$)	1	
$d_1(x) = k\, d_2(x)$.		
	3	
(b) (i) If $s(x)$ divides $f(x)$ and $g(x)$,		
let $f(x) = h_1(x)s(x)$, $g(x) = h_2(x)s(x)$ for some polynomials $h_1(x)$, $h_2(x)$.	1	
$\therefore p(x) = m(x)f(x) + n(x)g(x)$		
$= [m(x)h_1(x) + n(x)h_2(x)]s(x)$		
i.e. $s(x)$ divides $p(x)$ $\forall\, p(x) \in A$.	1	
(ii) Let $p(x) = m(x)f(x) + n(x)g(x)$ and suppose $p(x) \nmid f(x)$.		By Euclidean Algorithm
Let $f(x) = q(x)p(x) + r(x)$, where $r(x) \neq 0$ and $\deg r(x) < \deg p(x)$.	1	
Then $r(x) = f(x) - q(x)p(x)$		
$= f(x) - q(x)[m(x)f(x) + n(x)g(x)]$	1	
$= [1 - q(x)m(x)]\underset{\uparrow \text{non-zero}}{f(x)} + [-n(x)q(x)]\underset{\uparrow \text{non-zero}}{g(x)}$		
$\therefore r(x) \in A. \leftarrow$ non-zero	1	
(iii) Let $d_1(x) = m_1(x)f(x) + n_1(x)g(x)$ and suppose $d_1(x) \nmid f(x)$.		
Let $f(x) = h(x)d_1(x) + r(x)$, where $r(x) \neq 0$ and $\deg r(x) < \deg d_1(x)$.		
Then by (ii), $r(x) \in A$.	1	
But $\deg r(x) < \deg d_1(x)$ contradicts the definition		
of $d_1(x)$.	1	
$\therefore d_1(x)$ divides $f(x)$.		
Similarly $d_1(x)$ divides $g(x)$.		
$\therefore d_1(x)$ is a common divisor of $f(x)$ and $g(x)$.	1	
By (i) $d_1(x)$ is a G.C.D. of $f(x)$ and $g(x)$.	1	
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I		
(c) By (b), $d_1(x) = m_1(x)f(x) + n_1(x)g(x)$ is a G.C.D. of $f(x)$ and $g(x)$.	1	
Let $d(x)$ be a G.C.D. of $f(x)$ and $g(x)$.		
Since $d_1(x)$ and $d(x)$ divide each other, by (a), $d(x) = kd_1(x)$		
$= k(m_1(x)f(x) + n_1(x)g(x))$ for some non zero k.		
$= m_0(x)f(x) + n_0(x)g(x)$, where $m_0(x) = km_1(x)$		
$n_0(x) = kn_1(x)$	1	
	2	

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(a) (i) For $0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq \tan\theta \leq 1$. 1

$0 \leq \tan^{n+1}\theta \leq \tan^n\theta$

$\therefore\ 0 \leq I_{n+1} \leq I_n$ for $n \geq 0$ 1

(ii) For $n \geq 2$, $I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta(\sec^2\theta - 1)d\theta$ 1

$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \, d\tan\theta - I_{n-2}$ 1

$= \frac{\tan^{n-1}\theta}{n-1}\Big|_0^{\frac{\pi}{4}} - I_{n-2}$

$= \frac{1}{n-1} - I_{n-2}$

$\therefore\ I_n + I_{n-2} = \frac{1}{n-1}$ 1

(iii) For $n \geq 2$, $\frac{1}{n-1} = I_n + I_{n-2}$

$\geq 2I_n$ by (i)

$\therefore\ \frac{1}{2(n-1)} \geq I_n$ 1

Further $\frac{1}{n+1} = I_{n+2} + I_n$

$\leq 2I_n$

$\therefore\ \frac{1}{2(n+1)} \leq I_n$ 1

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1. (b) For $n \geq 1$,

$I_{2n+1} = \frac{1}{2n} - I_{2n-1}$ 1

$= \frac{1}{2n} - \frac{1}{2n-2} + I_{2n-3}$ $\frac{1}{2}(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2}$

$= \dots$

$= \frac{1}{2n} - \frac{1}{2n-2} + \frac{1}{2n-4} - \dots + (-1)^{n-1}\frac{1}{2} + (-1)^n I_1$ 1

$= \frac{(-1)^{n-1}}{2}(a_n - 2I_1)$ 1

Now $I_1 = \int_0^{\frac{\pi}{4}} \tan\theta \, d\theta$

$= -\int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \, d\cos\theta$

$= [-\ln\cos\theta]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\ln 2$ 1

$\therefore\ I_{2n+1} = \frac{(-1)^{n-1}}{2}(a_n - \ln 2)$ 1

From (a)(iii) $\frac{1}{2(2n+2)} \leq I_{2n+1} \leq \frac{1}{2(2n)}$,

Since both $\lim\limits_{n\to\infty} \frac{1}{2(2n+2)}$ and $\lim\limits_{n\to\infty} \frac{1}{2(2n)}$ equal zero 1

$\lim\limits_{n\to\infty} I_{2n+1} = 0$

i.e. $\lim\limits_{n\to\infty} a_n = \ln 2$ 1

7

$\because\ a_n = \frac{2I_{2n+1}}{(-1)^{n-1}} + \ln 2$

$\therefore\ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left[\frac{2I_{2n+1}}{(-1)^{n-1}} + \ln 2\right]$.

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II

2. (a) For $C \neq 0$, let the equations of the two lines be

$y - m_1x - C_1 = 0$ and $y - m_2x - C_2 = 0$.

Then the given equation can be written as

$*(y - m_1x - C_1)(y - m_2x - C_2) = 0$ 1

$m_1m_2x^2 - (m_1+m_2)xy + y^2 + (m_1C_2+m_2C_1)x - (C_1+C_2)y + C_1C_2 = 0$

$C \neq 0$ $C \neq 0$; Comparing coefficients, we have

$m_1m_2 = \frac{A}{C}$, $m_1 + m_2 = -\frac{B}{C}$ 1

(i) If $A + C = 0$, $m_1m_2 = -1$.
$A = -C$
$\therefore \alpha = \frac{\pi}{2}$ 1

(ii) If $A + C \neq 0$, $\tan^2\alpha = (\frac{m_1 - m_2}{1 + m_1m_2})^2$ 1

$$= \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m m)^2}$$

$$= \frac{(-\frac{B}{C})^2 - \frac{4A}{C}}{(1 + \frac{A}{C})^2}$$

$$= \frac{B^2 - 4AC}{(A+C)^2}$$ 1

*Note: Candidates may also consider $Ax^2 + Bxy + Cy^2 = 0$.
For $C = 0$,

Case 1

$Ax^2 + Bxy = 0$
$x(Ax + By) = 0$

If $B \neq 0$, then the pair of straight lines are given by

$(y - m_1x - C_1)(x - C_2) = 0$

$-m_1x^2 + xy - (C_1 - m_1C_2)x - C_2y + C_1C_2 = 0$

$\therefore m_1 = -\frac{A}{B}$

$x - C_2 = 0$ is a vertical line,

If $A + C = 0$, then $A = 0$ and therefore $m_1 = 0$.

i.e. the two lines intersect at right angles. 1

If $A + C \neq 0$, then $A \neq 0$.

$\tan^2\alpha = \tan^2(\frac{\pi}{2} - \beta)$, (where $\tan\beta = m_1$)

$= \cot^2\beta = \frac{1}{m_1^2} = \frac{B^2}{A^2} = \frac{B^2 - 4AC}{(A+C)^2}$ (as $C = 0$) 1

Case 2

$C = 0$
$B = 0$
$A \neq 0$

If $B = 0$ (in which case $A \neq 0$ and $A + C \neq 0$), then both straight lines are vertical.

$\therefore \alpha = 0 \rightarrow \tan^2\alpha = 0 = \frac{B^2 - 4AC}{(A+C)^2}$ (as $B = C = 0$) 1

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II

2. (b) Let $y = mx + c$ be the equation of a line through P.

Substituting in (E) 1

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0.$ 1

For tangency, $4a^4m^2c^2 = 4a^2(c^2 - b^2)(b^2 + a^2m^2)$

$c^2 - b^2 - a^2m^2 = 0$ 1

Since $y = mx + c$ passes through (h, k)

$c = k - mh$

$(k - mh)^2 - b^2 - a^2m^2 = 0.$

$(h^2 - a^2)m^2 - 2hkm + (k^2 - b^2) = 0$ 1

If the tangents are perpendicular

$\frac{k^2 - b^2}{h^2 - a^2} = -1$ 1

$h^2 + k^2 = a^2 + b^2$

$\therefore$ P lies on the circle $x^2 + y^2 = a^2 + b^2$ 1
6

* Alternatively

(b) The pair of tangents through P(h, k) is

$(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1)(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1) = (\frac{hx}{a^2} + \frac{ky}{b^2} - 1)^2$ 2

$[\frac{1}{a^2}(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1) - \frac{h^2}{a^4}]x^2 + [\frac{1}{b^2}(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1) - \frac{k^2}{b^4}]y^2$
$- \frac{2hk}{a^2b^2}xy + \frac{2h}{a^2}x + \frac{2k}{b^2}y - (\frac{h^2}{a^2} + \frac{k^2}{b^2}) = 0$ 1

If the tangents are perpendicular, by (a) (i) and (ii)

$(\frac{k^2}{a^2b^2} - \frac{1}{a^2}) + (\frac{h^2}{a^2b^2} - \frac{1}{b^2}) = 0$. 2

$k^2 + h^2 = a^2 + b^2$

$\therefore$ P(h, k) lies on the circle $x^2 + y^2 = a^2 + b^2$. 1
6

*Note: There are a number of alternative methods used in common textbooks

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3. (a) $$f'(0) = \lim_{\Delta x \to 0} \frac{(\Delta x + 0)^{n+1}|\Delta x + 0| - 0}{\Delta x}$$ .... II 1

$$= \lim_{\Delta x \to 0} (\Delta x)^n |\Delta x|$$

$$= 0$$ 1

$$f'(x) = \begin{cases} \frac{d}{dx} x^{n+1} \cdot x & x > 0 \\ 0 & x = 0 \\ \frac{d}{dx} x^{n+1} (-x) & x < 0 \end{cases}$$ 1

$$= \begin{cases} (n+2)\ x^n \cdot x & x > 0 \\ 0 & x = 0 \\ -(n+2)\ x^n \cdot x, & x < 0 \end{cases}$$

$$= (n + 2)x^n \cdot |x|, \qquad \forall x \in \mathbb{R}$$ 1

$$\therefore \int x^n \cdot |x|\ dx = \frac{1}{n+2} x^{n+1} \cdot |x| + c$$ 1

5

(b) $$\int_a^b \left| \sum_{j=1}^{2n} c_j x^j \right| dx \le \int_a^b \sum_{j=1}^{2n} |c_j x^j|\ dx$$ 1

$$= \sum_{j=1}^{2n} \int_a^b |c_j| |x^j|\ dx$$

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \int_a^b |x^{2j-1}| dx + |c_{2j}| \int_a^b |x^{2j}| dx \right\}$$ 1

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \int_a^b x^{2j-2} |x|\ dx + |c_{2j}| \int_a^b x^{2j}\ dx \right\}$$ 1

$$= \sum_{j=1}^{n} \left\{ \left[|c_{2j-1}| \frac{1}{2j} x^{2j-1} |x| + |c_{2j}| \frac{1}{2j+1} x^{2j+1} \right]_a^b \right\}$$ 1

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \frac{1}{2j} [b^{2j-1}|b| - a^{2j-1}|a|] + |c_{2j}| \frac{1}{2j+1} [b^{2j+1} - a^{2j+1}] \right.$$

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \frac{1}{2j} [b^{2j} + |a|^{2j}] + |c_{2j}| \frac{1}{2j+1} [b^{2j+1} + |a|^{2j+1}] \right\}$$

$$= \sum_{j=1}^{2n} |c_j| \left(\frac{b^{j+1} + |a|^{j+1}}{j+1} \right)$$ 1

5

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Alternatively

3. (b) $$\int_a^b \left| \sum_{j=1}^{2n} c_j x^j \right| dx$$ II

$$= \int_a^0 \left| \sum_{j=1}^{2n} c_j x^j \right| dx + \int_0^b \left| \sum_{j=1}^{2n} c_j x^j \right| dx$$ 1

$$\le \int_a^0 \sum_{j=1}^{2n} |c_j x^j| dx + \int_0^b \sum_{j=1}^{2n} |c_j x^j|\ dx$$ 1

$$= \int_a^0 \sum_{j=1}^{2n} |c_j| (-1)^j x^j\ dx + \int_0^b \sum_{j=1}^{2n} |c_j| x^j\ dx$$ 1

$$= \sum_{j=1}^{2n} [\,|c_j| (-1)^j \frac{x^{j+1}}{j+1}]_a^0 + \sum_{j=1}^{2n} [\,|c_j| \frac{x^{j+1}}{j+1}]_0^b$$ 1

$$= \sum_{j=1}^{2n} |c_j| (-1)^{j+1} \frac{(a)^{j+1}}{j+1} + \sum_{j=1}^{2n} |c_j| \frac{b^{j+1}}{j+1}$$

$$= \sum_{j=1}^{2n} |c_j| \frac{|a|^{j+1}}{j+1} + \sum_{j=1}^{2n} |c_j| \frac{b^{j+1}}{j+1}$$

$$= \sum_{j=1}^{2n} |c_j| \frac{b^{j+1} + |a|^{j+1}}{j+1}$$ 1

5

3. (c) $$\int_{-1}^1 \left| \sum_{j=1}^{2n} \frac{(-1)^j (j+1) x^j}{2^j} \right| dx$$

$$= 2 \int_0^1 \left| \sum_{j=1}^{2n} \frac{(-1)^j (j+1) x^j}{2^j} \right| dx$$ 1

$$\le 2 \int_0^1 \sum_{j=1}^{2n} \frac{(j+1) x^j}{2^j}\ dx$$ 1

$$= 2 \sum_{j=1}^{2n} \left[\frac{x^{j+1}}{2^j} \right]_0^1$$ 1

$$= 2 \sum_{j=1}^{2n} \frac{1}{2^j}$$

$$= 2 \left(\frac{\frac{1}{2} (1 - \frac{1}{2^{2n}})}{1 - \frac{1}{2}} \right)$$

$$< 2$$ ....

4. (a) (i) $f(0) = f(0 + 0)$ II

$= [f(0)]^2$ ... 1

$\Rightarrow f(0) = 1$ as $f(0) \neq 0$ 1

(ii) We shall prove by induction:

$\forall x \geqslant 0,\ f(x) \geqslant f(0) = 1$ as f is increasing)

For $k = 0$, R.S. $= [f(x)]^0 = 1 = f(0 \cdot x) =$ L.S.) ... 1

Assume that $f(kx) = [f(x)]^k$ for some $k \geqslant 0$.

$$f((k + 1)x) = f(kx + x)$$
$$= f(kx)f(x)$$
$$= [f(x)]^{k+1}$$

$\therefore f(kx) = [f(x)]^k \quad \forall$ integers $k \geqslant 0$ 1

(iii) Let $a = f(1) \geqslant f(0) = 1$. ← Indispensable because of, otherwise, a may be ≤ 1, By given, f is increasing ... 1

For any non-negative integer n,

$$f(n) = f(n \cdot 1)$$
$$[f(1)]^n \quad \text{by (ii)}$$
$$= a^n$$ .... $\frac{1}{6}$

(b) (i) For any $x \geqslant 0$, let n be the non-negative integer such that $n \leqslant x < n + 1$.

Then $a^n \leqslant a^x \leqslant a^{n+1}$ and $\frac{1}{a^{n+1}} \leqslant \frac{1}{a^x} \leqslant \frac{1}{a^n}$ ....(i) 1

Since $f(x)$ is increasing, $f(n) \leqslant f(x) \leqslant f(n+1)$

$\therefore a^n \leqslant f(x) \leqslant a^{n+1}$ ....(ii) 1

Since all quantities involved are positive, (i) and (ii) give $\frac{a^n}{a^{n+1}} \leqslant \frac{f(x)}{a^x} \leqslant \frac{a^{n+1}}{a^n}$

i.e. $\frac{1}{a} \leqslant \frac{f(x)}{a^x} \leqslant a$ ....(*) 1

(ii) As (*) is independent of x, replacing x by kx, we have

$\frac{1}{a} \leqslant \frac{f(kx)}{a^{kx}} \leqslant a \quad \forall k > 0$ 1

$\frac{1}{a} \leqslant \left(\frac{f(x)}{a^x}\right)^k \leqslant a$ 1

$\frac{1}{a^{\frac{1}{k}}} \leqslant \frac{f(x)}{a^x} \leqslant a^{\frac{1}{k}}$ 1

As both $\lim\limits_{k\to\infty} a^{\frac{1}{k}}$ and $\lim\limits_{k\to\infty} \frac{1}{a^{\frac{1}{k}}}$ equal 1

$\lim\limits_{k\to\infty} \frac{f(x)}{a^x} = 1$ 1

i.e. $f(x) = a^x \quad \forall x \geqslant 0$. 1

---

5. (a) Let x, y be the cartesian coordinates of P, $\tan\phi$ be the gradient of the curve at P II

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$\frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta$

$\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta$ 1

$\therefore \tan\phi = \frac{dy}{dx}$

$$= \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta}$$

$$= \frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}$$ 1

$\tan\psi = \tan(\phi - \theta)$ 1

$$= \frac{\dfrac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta} - \tan\theta}{1 + \dfrac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}\tan\theta}$$

$$= \frac{r(1 + \tan^2\theta)}{\frac{dr}{d\theta}(1 + \tan^2\theta)}$$

$$= \frac{r}{\left(\frac{dr}{d\theta}\right)}$$ .... $\frac{1}{4}$

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5. (b) (i) Substituting $r = 2$ in C_1 , $2 = 2(1 - \cos\theta)$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$ 1

∴ the points of intersection are $(2, \frac{\pi}{2})$ and $(2, \frac{3\pi}{2})$.

At $\theta = \frac{\pi}{2}$, the tangent to C_2 is parallel to the x-axis.

For C_1 , $\frac{dr}{d\theta} = 2 \sin\theta$

$= 2$ ……………………………………

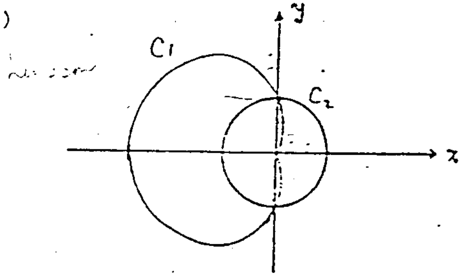
∴ $\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{2}{2} = 1$

The angle between C_1 and C_2 is $= \frac{\pi}{2} - \tan^{-1} 1$

$= \frac{\pi}{4}$

By symmetry, the angle between C_1 and C_2 at $\theta = \frac{3\pi}{2}$ is also $\frac{\pi}{4}$ …………………………………… 1

(ii)



1+1

For any point (r, θ) of C_1 lying inside C_2

$r = 2(1 - \cos\theta)$, $r < 2$

∴ $2(1 - \cos\theta) < 2$

$0 < \cos\theta$

$0 < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta < 2\pi$ …………………… 1 — or for lim of integra belo

Length of C_1 inside C_2

$= \int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta$ 1

$= 2 \int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta$ by symmetry

$= 2\int_0^{\frac{\pi}{2}} \sqrt{(\cos\theta\frac{dr}{d\theta} - r \sin\theta)^2 + (\sin\theta \frac{dr}{d\theta} + r \cos\theta)^2}\, d\theta$

$= 2 \int_0^{\frac{\pi}{2}} \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$ …………………… 1

$= 2\int_0^{\frac{\pi}{2}} \sqrt{4 \sin^2\theta + 4(1 - \cos\theta)^2}\, d\theta$

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5. (b) (ii) $= 4\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1 - \cos\theta}\, d\theta$ …………………… 1

$= 8 \int_0^{\frac{\pi}{2}} \sin \frac{\theta}{2}\, d\theta$

$= 16 [- \cos \frac{\theta}{2}]_0^{\frac{\pi}{2}}$

$= 8(2 - \sqrt{2})$ (≑ 4.69) …………………… 1

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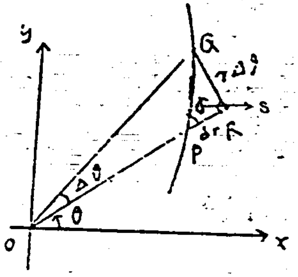
Alternatively

(a) From the diagram



PR ≑ Δr 1

QR ≑ r Δθ

tan ø ≑ $\frac{r \Delta\theta}{\Delta r}$ 1

∴ $\tan\psi = \lim_{\Delta s \to 0} r \frac{\Delta\theta}{\Delta r}$ 1

$= \frac{r}{\frac{dr}{d\theta}}$ …………………… 1

4

(b) (ii) $ds^2 = (r d\theta)^2 + (dr)^2$ …………………… 1

Length of C_1 inside $C_2 = \int ds$

$= \int \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$ 1

= etc

etc.

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6\. (a) $f(x) = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

$\therefore$ the stationary points are at $x = 0$ and $x = 2$. 1

x	-1	0	1	2	3
f(x)	0	4	2	0	4
f'(x)	+	0	−	0	+
f''(x)	−		0	+	

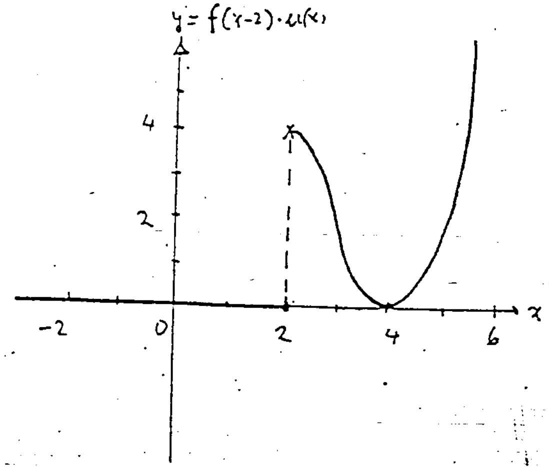
$\therefore$ (0, 4) is a maximum point, 1

(2, 0) is a minimum point,

and (1, 2) is the only point of inflexion. $\frac{1}{3}$

(b) $h(x) = f(x-2) \cdot u(x) = \begin{cases} 0 & \text{when } x < 2 \\ f(x-2) & \text{when } x \geq 2. \end{cases}$ 1

Translating the graph of $f(x)$ $(x \geq 0)$ horizontally to the right by 2 units, one obtains the graph of $h(x)$ for $x \geq 2$.



$\frac{2}{3}$

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6\. (c) $I_n = \int_0^n e^{-x} h(x)\, dx$

$= \int_2^n e^{-x} f(x-2)\, dx \quad (n \geqslant 2)$ 1

$= \int_0^{n-2} e^{-(t+2)} f(t)\, dt$ t=x-2

$= e^{-2} \int_0^{n-2} e^{-t} (t^3 - 3t^2 + 4)\, dt$ 1

Now $\int te^{-t}dt = -te^{-t} + \int e^{-t}dt$ 1

$= -(t + 1)e^{-t} + c$

$\int t^2e^{-t}dt = -t^2e^{-t} + 2\int te^{-t}dt$

$= -(t^2 + 2t + 2)e^{-t} + c$

$\int t^3e^{-t}dt = -t^3e^{-t} + 3\int t^2e^{-t}dt$

$= -(t^3 + 3t^2 + 6t + 6)e^{-t} + c$ 1

$\therefore I_n = -e^{-2}e^{-t}[(t^3 + 3t^2 + 6t + 6) - 3(t^2 + 2t + 2) + 4] \Big|_0^{n-2}$ 1

$= -e^{-2}e^{-t}(t^3 + 4) \Big|_0^{n-2}$

$= -e^{-2}[e^{-(n-2)}((n-2)^3 + 4) - 4]$ 1

By L'Hospital's rule,

$\lim_{t\to\infty} \frac{t^3}{e^t} = \lim_{t\to\infty} \frac{3t^2}{e^t} = \lim_{t\to\infty} \frac{6t}{e^t} = \lim_{t\to\infty} \frac{6}{e^t} = 0$ 1

$\therefore \lim_{n\to\infty} I_n = -e^{-2}[\lim_{n\to\infty} e^{-(n-2)}(n - 2)^3 + \lim_{n\to\infty} e^{-(n-2)}4 - \lim_{n\to\infty} 4]$

$= 4e^{-2}$ .... $\frac{1}{8}$

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7. (a) (i) $\pi : \vec{r}\cdot\vec{n} = \rho$ passing through the pt position vector ⊥ π ....(1)

$l : \vec{r} = \vec{a} + t\vec{b}$ ....(2)

Putting (2) in (1) $(\vec{a} + t\vec{b})\cdot\vec{n} = \rho$ — 1

$t(\vec{b}\cdot\vec{n}) = \rho - \vec{a}\cdot\vec{n}$

$t = \dfrac{\rho - \vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}$ $(\vec{b}\cdot\vec{n} \neq 0)$ Solving for t

∴ l intersects π at the point with position vect

$\vec{a} + \left(\dfrac{\rho - \vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{b}$ 1

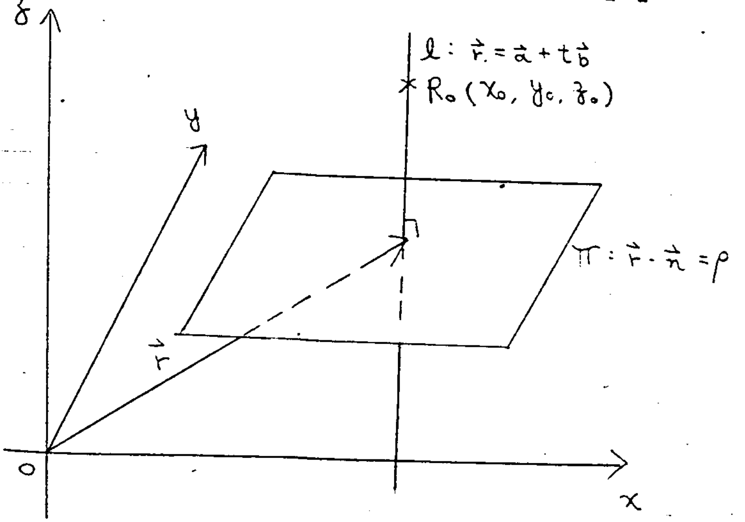
(ii) Let the position vector of R_0 be $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$.

The equation of the line through R_0 and perpendicular to π is $\vec{r} = \vec{r}_0 + t\vec{n}$, $t \in \mathbb{R}$ passing through r_0 2

By (i), the position vector of the foot of the perpendicular from R_0 to π is $r_0 + \dfrac{\rho - \vec{r}_0\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$. — 1

$\overline{5}$

Candidate: take $\vec{n}\cdot$



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7. (b) $\pi : x + y + z - 1 = 0$ can be written as

$(x, y, z)\cdot(1, 1, 1) = 1$????? 1

By (a)(ii), PP' intersects π ~~at the point M~~ with position vector $\vec{m} = \vec{p} + \dfrac{1 - \vec{p}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$, where $\vec{n} = (1, 1, 1)$ 1

Since M is the mid-point of PP',

P' is given by $\dfrac{\vec{p'} + \vec{p}}{2} = m$

$\vec{p'} = 2\vec{m} - \vec{p}$ 1

$= \vec{p} + \dfrac{2(1 - \vec{p}\cdot\vec{n})}{\vec{n}\cdot\vec{n}}\vec{n}$

$= (\alpha, \beta, \gamma) + \left[\dfrac{2(1 - (\alpha + \beta + \gamma))}{(1, 1, 1)\cdot(1, 1, 1)}\right](1, 1, 1)$ 1

i.e., the coordinates of P' are given by

$$\begin{cases} x = \alpha + \frac{2}{3}(1 - \alpha - \beta - \gamma) \\ y = \beta + \frac{2}{3}(1 - \alpha - \beta - \gamma) \\ z = \gamma + \frac{2}{3}(1 - \alpha - \beta - \gamma) \end{cases} \quad \text{.................(3)}$$ 1

Now $l : \dfrac{x - 1}{1} = \dfrac{y - 2}{2} = \dfrac{z - 3}{3}$ can be written as

$\vec{r} = (1, 2, 3) + t(1, 2, 3)$ 1

If $P(\alpha, \beta, \gamma)$ lies on l, $\alpha = 1 + t$

$\beta = 2 + 2t$

$\gamma = 3 + 3t$ 1

Substituting in (3)

$x = (1 + t) + \frac{2}{3}(1 - 1 - t - 2 - 2t - 3 - 3t) = -\frac{7}{3} - 3t$

$y = (2 + 2t) + \frac{2}{3}(-5 - 6t) = -\frac{4}{3} - 2t$

$z = (3 + 3t) + \frac{2}{3}(-5 - 6t) = -\frac{1}{3} - t$

∴ the locus of P' is $l' : \dfrac{x + \frac{7}{3}}{3} = \dfrac{y + \frac{4}{3}}{2} = \dfrac{z + \frac{1}{3}}{1}$ — 1

[or $\vec{r} = (-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}) + t(3, 2, 1)$, $t \in \mathbb{R}$]

$\overline{9}$

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8\. (a) $F'(x) = \lambda g'(\lambda x + (1-\lambda)a) - \lambda g'(x)$ 1

$\forall$ x for $x<a$

If $x = a$,

$\lambda x + (1-\lambda)a = x$

$\therefore F'(a) = 0$ 1

$\lambda x+(1-\lambda)a > \lambda x+(1-\lambda)x = x$

$\forall\ x < a,\ \lambda x + (1-\lambda)a > x$ (as $0 < \lambda < 1$)

$\Rightarrow F'(x) \geqslant 0$ as g' is increasing 1

Similarly, $\forall\ x > a,\ F'(x) \leqslant 0.$

$\therefore F(x)$ attains its greatest value at $x = a$. $\frac{1}{4}$

(b) (i) By (a), $F(x) \leqslant F(a)$

Introduction

$= g(\lambda a + (1-\lambda)a) - \lambda g(a) - (1-\lambda)g(a)$

$= 0$ .... $\lambda_1+\lambda_2=1$ .... 2 ✓

$1-\lambda=\lambda_2$

For m = 2, let $\lambda = \lambda_1$, $1 - \lambda = \lambda_2$, we have from (a)

$g(\lambda_1 x_1 + \lambda_2 x_2) - [\lambda_1 g(x_1) + \lambda_2 g(x_2)] \leq 0$ $\Rightarrow$ see introduction

i.e. $g(\lambda x_1 + \lambda_2 x_2) \leq \lambda_1 g(x_1) + \lambda_2 g(x_2)$ 1

Suppose the given statement is true for $2 \leqslant m < k$.

Let $\lambda_1 + \lambda_2 + \dots + \lambda_{k-1} = \lambda$, $1-\lambda=\lambda_k$

$x = \frac{1}{\lambda}(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{k-1}x_{k-1})$ 2

Then $g(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k)$

$= g(\lambda x + (1-\lambda)x_k)$

✓ $\leq \lambda g(x) + (1-\lambda)g(x_k)$

$= \lambda g(\frac{\lambda_1}{\lambda}x_1 + \frac{\lambda_2}{\lambda}x_2 + \dots + \frac{\lambda_{k-1}}{\lambda}x_{k-1}) + (1-\lambda)g(x_k)$

$\leqslant \lambda_1 g(x_1) + \lambda_2 g(x_2) + \dots + \lambda_{k-1}g(x_{k-1}) + \lambda_k g(x_k)$

$\therefore$ the statement is true for m=k and hence $\forall\ m \geqslant 2$ 2

(ii) Let $g(x) = e^x$, which is differentiable and

$g'(x) = e^x$ is increasing. 1

For any positive numbers $a_1, a_2, \dots, a_m$,

let $a_i = e^{x_i}$ $(1 \leqslant i \leqslant m)$ 1

Then by (i),

$e^{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_m x_m} \leqslant \lambda_1 e^{x_1} + \lambda_2 e^{x_2} + \dots + \lambda_m e^{x_m}$

i.e. $a_1^{\lambda_1} \cdot a_2^{\lambda_2} \dots a_m^{\lambda_m} \leq \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_m a_m$ 1

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II

9\. (a) For any $x \in I$

$f(x) = f(x) - f(0)$

$= \int_0^x f'(t)\,dt$ 1

$\leq \int_0^x f'(x)\,dt$ as f' is increasing $\because f'(x) > f'(t)\ \forall t \in [0,x]$ 1

$= x f'(x)$ $\because [f'(x)t]_0^x$ .... $\frac{1}{3}$

f' independent of t.

(b) (i) $G(x) = 2f(x)\sqrt{1 + f'(x)^2}$

$\geqslant 2f(x)\,|f'(x)|$

$\geqslant 2f(x)f'(x)$ as $f'(x) \geq 0.$ 1

(ii) $F(x) = f(x)\sqrt{x^2 + f(x)^2}$

$$F'(x) = f'(x)\sqrt{x^2 + f(x)^2} + \frac{f(x)}{2}\,\frac{(2x + 2f(x)f'(x))}{\sqrt{x^2 + f(x)^2}}$$

$$= \frac{xf(x) + x^2 f'(x) + 2f(x)^2 f'(x)}{\sqrt{x^2 + f(x)^2}}$$ 1

$[F'(x)]^2 - [G(x)]^2$

$$= \frac{x^2 f(x)^2 + x^4 f'(x)^2 + 4f(x)^4 f'(x)^2 + 2x^3 f(x)f'(x) + 4xf(x)^3 f'(x) + 4x^2 f(x)^2 f'(x)^2}{x^2 + f(x)^2}$$

$- 4f^2(x)(1 + f'(x)^2)$

$$= \frac{1}{x^2+f(x)^2}[-3x^2 f(x)^2 + x^4 f'(x)^2 + 2x^3 f(x)f'(x) + 4xf(x)^3 f'(x) - 4f(x)^4]$$

$$\geq \frac{1}{x^2+f(x)^2}[-3x^2 f(x)^2 + x^2 f(x)^2 + 2x^2 f(x)^2 + 4f(x)^3 f(x) - 4f(x)^4]$$ 1

$f(x) \leq xf'(x)$

$= 0$ (as $0 \leqslant f(x) \leqslant xf'(x)$) 1

From (a) $f'(x) \geq 0$, $\therefore F'(x) \geq 0$ as all quantities involved are non-negative. 1

$\therefore F'(x) \geq G(x)$ .... $\frac{1}{6}$

(c) $S = 2\pi \int_0^a f(x)\sqrt{1 + [f'(x)]^2}\,dx$ 2

$= \pi\int_0^a G(x)\,dx$ $(G(x) \geq 2f(x)f'(x))$ $(F'(x) \geq G(x))$

$\therefore 2\pi\int_0^a f(x)f'(x)\,dx \leqslant S \leq \pi\int_0^a F'(x)\,dx$ 1

$\pi[f(x)^2]_0^a \leqslant S \leqslant \pi[F(x)]_0^a$

$\pi[f(a)]^2 \leqslant S \leqslant \pi f(a)\sqrt{a^2 + [f(a)]^2}$ .... $\frac{1+1}{5}$