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HONG KONG ADVANCED LEVEL EXAMINATION, 1985

PURE MATHEMATICS (I)

MARKING SCHEME

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SOLUTIONS	MARKS	REMARKS
85 I		
(a) (i) Let $\lim_{n\to\infty} a_n = \ell$.		
Then $\lim_{n\to\infty} a_{n+p} = \ell$	1	
$\therefore \lim_{n\to\infty}(a_n + a_{n+p})$ exists and equals		
$\lim_{n\to\infty} a_n + \lim_{n\to\infty} a_{n+p}$	2	
$= 2\ell$		
(ii) Consider the sequence $\{c_n\}$ defined by $c_n = (-1)^n$. The sequence whose nth term is $c_n + c_{n+1}$ is the convergent	2	
sequence 0, 0, 0, ... but $\{c_n\}$ itself is divergent.	2	
	7	
(b) Since $\lim_{n\to\infty}(a_n + a_{n+1}) = A$ and $\lim_{n\to\infty}(a_n + a_{n+2}) = B$,		
$\lim_{n\to\infty}[(a_n + a_{n+1}) + (a_n + a_{n+2})] = A + B$.	2	
But $\lim_{n\to\infty}(a_{n+1} + a_{n+2}) = \lim_{n\to\infty}(a_n + a_{n+1})$		
$= A$	1	
$\therefore \lim_{n\to\infty}(2a_n)$ exists and equals B	1	
$\lim_{n\to\infty} a_n = \frac{B}{2}$	1	
Further, $A = \lim_{n\to\infty}(a_n + a_{n+1})$		
$= \lim_{n\to\infty} a_n + \lim_{n\to\infty} a_n$		
$= B$	2	
	7	

SOLUTIONS	MARKS	REMARKS
85 I		
2(a) (i) $\Delta = \begin{vmatrix} 0 & \lvert a_1 - a_2\rvert & \lvert a_1 - a_3\rvert \\ \lvert a_2 - a_1\rvert & 0 & \lvert a_2 - a_3\rvert \\ \lvert a_3 - a_1\rvert & \lvert a_3 - a_2\rvert & 0 \end{vmatrix}$		
$= \lvert(a_1-a_2)(a_2-a_3)(a_3-a_1)\rvert + \lvert(a_2-a_1)(a_3-a_2)(a_1-a_3)\rvert$	1	
> 0 as a_1, a_2, a_3 are distinct.	1	
$\therefore$ (E) has a unique solution.		
(ii) Assume that $a_1 > a_2 > a_3$ 這個 case		
If $b_1 = b_2 = b_3\ (= b) \neq 0$,		
$x_1 = \frac{b}{\Delta}\begin{vmatrix} 1 & a_1 - a_2 & a_1 - a_3 \\ 1 & 0 & a_2 - a_3 \\ 1 & a_2 - a_3 & 0 \end{vmatrix}$		
$= \frac{b}{\Delta}[(a_1-a_2)(a_2-a_3) + (a_2-a_3)(a_1-a_3) - (a_2-a_3)(a_2-a_3)]$		
$= \frac{2b}{\Delta}(a_1 - a_2)(a_2 - a_3)$ $\neq 0$ ($\because$ all distinct)	1	
$x_2 = \frac{b}{\Delta}\begin{vmatrix} 0 & 1 & a_1 - a_3 \\ a_1 - a_2 & 1 & a_2 - a_3 \\ a_1 - a_3 & 1 & 0 \end{vmatrix}$		
$= \frac{b}{\Delta}[(a_1-a_2)(a_2-a_3) + (a_1-a_2)(a_1-a_3) - (a_1-a_3)(a_1-a_3)]$		
$= 0$	1	
$x_3 = \frac{b}{\Delta}\begin{vmatrix} 0 & a_1 - a_2 & 1 \\ a_1 - a_2 & 0 & 1 \\ a_1 - a_3 & a_2 - a_3 & 1 \end{vmatrix}$		
$= \frac{b}{\Delta}[(a_1-a_2)(a_2-a_3) + (a_1-a_2)(a_1-a_3) - (a_1-a_2)(a_1-a_2)]$		
$= \frac{2b}{\Delta}(a_1 - a_2)(a_2 - a_3)$ $\neq$ ($\because$ all distinct)	1	
$\therefore$ only $x_2 = 0$.		
Similarly, if $a_\ell > a_m > a_n$ where ℓ, m, n is any other permutation of 1, 2, 3, we have		
$x_m = 0$ but $x_\ell = x_n \neq 0$	1	
	6	

(b) Assume for contradiction that a_1, a_2, a_3 are not all distinct. 答一個case

Let $a_1 = a_2$. Then the system (E) becomes

$$\begin{cases} |a_1 - a_3|x_3 = b_1 \\ |a_1 - a_3|x_3 = b_2 \\ |a_3 - a_1|x_1 + |a_3 - a_1|x_2 = b_3 \end{cases}$$ 2

If this system is consistent, the first two equations imply $b_1 = b_2$, contradicting the fact that b_1, b_2, b_3 are all distinct.

$\therefore$ (E) is not consistent. 2

Similarly if $a_1 = a_3$ or $a_2 = a_3$, (E) cannot be consistent. 1

$\therefore$ (E) is consistent only if a_1, a_2, a_3 are all distinct. necessary but not sufficient conditions

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(c) If $a_1 = a_2 = a_3$ the coefficient matrix of (E) is the zero matrix.

(E) is consistent iff $b_1 = b_2 = b_3 = 0$, 2

in which case the whole space $\mathbb{R}^3$ is the solution set. 1

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3(a) $$\frac{f(x)}{(x+a_1)(x+a_2)\dots(x+a_n)} = \frac{c_1}{x+a_1} + \frac{c_2}{x+a_2} + \dots + \frac{c_n}{x+a_n}$$

Combining the partial fractions of the R.S., the numerator is

$$c_1(x+a_2)(x+a_3)\dots(x+a_n) + c_2(x+a_1)(x+a_3)\dots(x+a_n) + \dots + c_n(x+a_1)(x+a_2)\dots(x+a_{n-1})$$

$= (c_1 + c_2 + \dots + c_n)x^{n-1}$ + (terms of degree $< n-1$) 2

Since $f(x)$ is a polynomial of degree $< n-1$, ... 1

$c_1 + c_2 + \dots + c_n = 0$ .... Comparing coefficients 1

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(b) $$F(x) = \frac{px+q}{(x+a)(x+a+1)(x+a+2)} = \frac{b_1}{x+a} + \frac{b_2}{x+a+1} + \frac{b_3}{x+a+2}$$

For $N > 3$,

$$\sum_{k=1}^{N} F(k) = \sum_{k=1}^{N}\frac{b_1}{k+a} + \sum_{k=1}^{N}\frac{b_2}{k+a+1} + \sum_{k=1}^{N}\frac{b_3}{k+a+2}$$ 1

$$= \left[\frac{b_1}{1+a} + \frac{b_1}{2+a} + \sum_{k=3}^{N}\frac{b_1}{k+a}\right] + \left[\frac{b_2}{2+a} + \frac{b_2}{N+a+1} + \sum_{k=3}^{N}\frac{b_2}{k+a}\right]$$

$$+ \left[\frac{b_3}{N+a+1} + \frac{b_3}{N+a+2} + \sum_{k=3}^{N}\frac{b_3}{k+a}\right]$$

$$= \frac{b_1}{1+a} + \frac{b_1+b_2}{2+a} + \frac{b_2+b_3}{N+a+1} + \frac{b_3}{N+a+2} + \sum_{k=3}^{N}\frac{b_1+b_2+b_3}{k+a}$$ 2

The last term vanishes since $px+q$ is of degree 1 and therefore $b_1 + b_2 + b_3 = 0$ by (a). 2

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(c) Let $$\frac{1}{(2k+1)(2k+3)(2k+5)} = \frac{b_1}{2k+1} + \frac{b_2}{2k+3} + \frac{b_3}{2k+5}$$

$1 = b_1(2k+3)(2k+5) + b_2(2k+1)(2k+5) + b_3(2k+1)(2k+3)$ 1

Put $k = -\frac{1}{2}$, $1 = b_1(2)(4) \Rightarrow b_1 = \frac{1}{8}$

$k = -\frac{3}{2}, -\frac{5}{2} \Rightarrow b_2 = -\frac{1}{4}$, $b_3 = \frac{1}{8}$ 1

$$\therefore \frac{1}{(2k+1)(2k+3)(2k+5)} = \frac{1}{8}\left(\frac{1}{2k+1}\right) - \frac{1}{4}\left(\frac{1}{2k+3}\right) + \frac{1}{8}\left(\frac{1}{2k+5}\right)$$

$$\frac{1}{8}\cdot\frac{1}{(k+\frac{1}{2})(k+\frac{3}{2})(k+\frac{5}{2})} = \frac{1}{8}\left(\frac{\frac{1}{2}}{k+\frac{1}{2}} - \frac{1}{k+\frac{3}{2}} + \frac{\frac{1}{2}}{k+\frac{5}{2}}\right)$$ 1

Let $a = \frac{1}{2}$, by (b),

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{(2k+1)(2k+3)(2k+5)}$$

$$= \lim_{N\to\infty}\frac{1}{8}\left[\frac{\frac{1}{2}}{1+\frac{1}{2}} + \frac{-\frac{1}{2}}{2+\frac{1}{2}} + \frac{-\frac{1}{2}}{N+\frac{3}{2}} + \frac{\frac{1}{2}}{N+\frac{5}{2}}\right]$$

$= \frac{1}{60}$ 2

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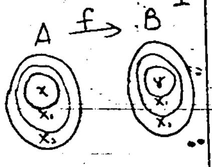
SOLUTIONS	MARKS	REMARKS

4(a) (i) Let $X_1 \subset X_2$. For any y,



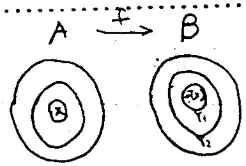
$y \in f[X_1] \Rightarrow y = f(x)$ for some $x \in X_1$

$\Rightarrow y = f(x)$ for some $x \in X_2$

$\Rightarrow y \in f[X_2]$

$\therefore f[X_1] \subset f[X_2]$ 2

(ii) Let $Y_1 \subset Y_2$. For any x,



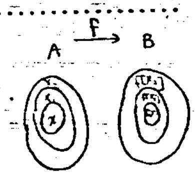
$x \in f^{-1}[Y_1] \Rightarrow f(x) \in Y_1$

$\Rightarrow f(x) \in Y_2$

$\Rightarrow x \in f^{-1}[Y_2]$

$\therefore f^{-1}[Y_1] \subset f^{-1}[Y_2]$ .... 2 / 4

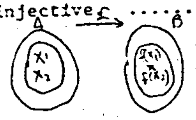
(b) 'Only if' part.



Let f be injective and let $X_1, X_2 \subset A$ s.t. $f[X_1] \subset f[X_2]$. For any $x \in X_1$,

$f(x) \in f[X_1] \Rightarrow f(x) \in f[X_2]$

$\Rightarrow x \in X_2$ since f is injective ..... 2 — Explanation necessary.



$\therefore X_1 \subset X_2$

'If' part.

For any $x_1, x_2 \in A$, suppose $f(x_1) = f(x_2)$. 1

Let $X_1 = \{x_1\}$, $X_2 = \{x_2\}$. Then $f[X_1] \subset f[X_2]$ and $f[X_2] \subset f[X_1]$.

$\therefore X_1 = X_2$ i.e. $x_1 = x_2$

Hence f is injective. .... 2 / 5

Alt. Solution:

'If' part. We shall show by contradiction that the given statement is not true. Suppose f is not injective. $\exists\, x, y \in A$ s.t.

$f(x) = f(y)$ but $x \neq y$. 1

Take $X_1 = \{x, y\}$, $X_2 = \{x\}$. Then $f[X_1] \subset f[X_2]$ but $X_1 \not\subset X_2$. 2

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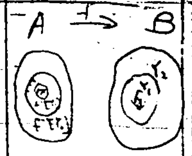


4(c) 'Only if' part.

Let f be surjective and let $Y_1, Y_2 \subset B$ s.t. $f^{-1}[Y_1] \subset f^{-1}[Y_2]$. For any $y \in Y_1$, let $y = f(x)$ for some $x \in A$.

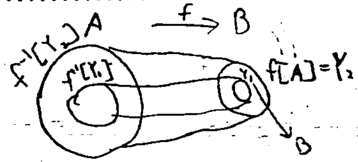
Then $x \in f^{-1}[Y_1] \Rightarrow x \in f^{-1}[Y_2]$

$\Rightarrow y \in Y_2$. $\therefore Y_1 \subset Y_2$ 2

'If' part.

Let $Y_1 = B$, $Y_2 = f[A]$ 1

Then $f^{-1}[Y_2] = A$.



$\therefore f^{-1}[Y_1] \subset f^{-1}[Y_2]$

$\Rightarrow Y_1 \subset Y_2$

$\Rightarrow B \subset f[A]$.

$\therefore f$ is surjective. .... 2 / 5

Alternative Solution:

"If" part.

Suppose f is not surjective.

$\exists\, y \in B$ s.t. $y \notin f[A]$. 1

Let $Y_1 = B$, $Y_2 = B \setminus \{y\}$

Then $f^{-1}[Y_1] \subset f^{-1}[Y_2]$ but $Y_1 \not\subset Y_2$. 2

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5(a) Let $f(x) = x^k - kx + k - 1$

$f'(x) = k(x^{k-1} - 1)$

$$\begin{cases} < 0 & \text{if } 0 \leqslant x < 1 \\ = 0 & x = 1 \\ > 0 & x > 1 \end{cases}$$

$\therefore$ $f(x)$ has an absolute minimum at $x = 1$ 2

But $f(1) = 0$

$\therefore$ $x^k - kx + k - 1 \geqslant 0$ 1

i.e. $x^k + k - 1 \geqslant kx$ ...... (*)

The equality holds iff $x = 1$. .... $\frac{1}{4}$

(b) (i) $\left(\frac{G_m}{G_{m-1}}\right)^m = \frac{(G_m)^m}{(G_{m-1})^{m-1}\, G_{m-1}}$

$= \frac{\prod_{i=1}^{m} a_i}{\left(\prod_{i=1}^{m-1} a_i\right) G_{m-1}}$

$= \frac{a_m}{G_{m-1}}$

$= \frac{mA_m - (m-1)A_{m-1}}{G_{m-1}}$ ....(**) 2

$A_m = \frac{1}{m}\sum_{i=1}^{m} a_i$

$mA_m = a_1 + a_2 + \cdots + a_m$ — ①

$A_{m-1} = \frac{1}{m-1}\sum_{i=1}^{m-1} a_i$

$(m-1)A_{m-1} = a_1 + a_2 + \cdots + a_{m-1}$ — ②

① − ②

(ii) Putting $x = \frac{G_m}{G_{m-1}}$ in (*), for $m = 2, 3, \ldots, n$,

$\left(\frac{G_m}{G_{m-1}}\right)^m + m - 1 \geqslant m\left(\frac{G_m}{G_{m-1}}\right)$ 1

By (**), $\frac{mA_m - (m-1)A_{m-1}}{G_{m-1}} + (m-1) \geqslant m\left(\frac{G_m}{G_{m-1}}\right)$ 1

$\therefore$ $m(A_m - G_m) + (m-1)(G_{m-1} - A_{m-1}) \geqslant 0$

$A_m - G_m \geqslant \frac{m-1}{m}(A_{m-1} - G_{m-1})$ 1

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5(b) (iii) By (ii), $A_n - G_n \geqslant \frac{n-1}{n}(A_{n-1} - G_{n-1})$ 1

$\geqslant$ ....

$\geqslant \frac{(n-1)(n-2)\ldots(2)(1)}{n(n-1)\ldots(3)(2)}(A_1 - G_1)$

$= 0$ ....(***) 1

$\therefore$ $A_n \geqslant G_n$

Next, if $a_1 = a_2 = \ldots = a_n = a$, then both A_n, G_n equal a. 1

Conversely, if $A_n = G_n$, then all equalities in (***) hold.

By (a), this is true iff $x = \frac{G_m}{G_{m-1}} = 1$, ... 1

i.e. $(a_1a_2\ldots a_m)^{\frac{1}{m}} = (a_1a_2\ldots a_{m-1})^{\frac{1}{m-1}}$

$a_m = (a_1a_2\ldots a_{m-1})^{\frac{1}{m-1}}$ $(1 < m \leqslant n)$.

$\therefore$ $a_2 = a_1$

$a_3 = (a_1a_2)^{\frac{1}{2}} = a_1$

....

Hence $a_n = a_{n-1} = \ldots = a_1$ .... $\frac{1}{10}$

(c)

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6(a) For $r = 0$ or 1, the statement is trivially true. 1

Consider $r > 1$.

(i) When $k \geqslant r$, $P_r(k) = \dfrac{k(k-1)\ \dots\ k-r+1}{r!} = C^k_r$,

which is an integer. 1

(ii) When $0 \leqslant k < r$, $P_r(k) = 0$ since $(x - k)$

is a factor of $P_r(k)$. 1

(iii) When $k < 0$, putting $k = -\ell$

$$P_r(k) = \frac{(-1)^r \ell(\ell+1)\ \dots\ (\ell+r-1)}{r!} = (-1)^r C^{\ell+r-1}_r,$$

which is also an integer. .... 1 — 4

(b) Let $0 < m \leqslant n$.

$\underbrace{\qquad\qquad\qquad}_{}$ $P_r(m) = C^m_r$

$P(m) = a_0P_0(m) + a_1P_1(m) + \dots + a_mP_m(m) + \dots + a_nP_n(m)$. $\longrightarrow 0$

By (a) $P_r(m) = 0$ for $m < r$ and $P_r(m) = C^m_r$ for $m \geqslant r$

$\therefore P(m) = a_0C^m_0 + a_1C^m_1 + \dots + a_mC^m_m$ 2

If $a_0, a_1, \dots, a_{m-1}$ (given) are integers while a_m is not,

$P(m) = (a_0C^m_0 + a_1C^m_1 + \dots + a_{m-1}C^m_{m-1}) + a_m$ cannot be an integer. 1

(i) First $a_0 = P(0)$ is an integer. Assume for contradiction

that $a_0, a_1, \dots, a_n$ are not all integers. Let m

be the least suffix such that a_m is not an integer

$(0 < m \leqslant n)$. By the above result, $P(m)$ cannot be an 1

integer, which contradicts the given conditions. 1

(ii) $P(k) = \sum\limits_{r=0}^{n} a_rP_r(k)$.

Since $a_0, a_1, \dots, a_n$ are integers by b(i), and $P_r(k)$

are integers by (a), $P(k)$ is therefore an integer for all k. 2 — 7

(c) Let $Q(x) = P_2(x) = \dfrac{x(x-1)}{2}$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x.$$

$Q(k)$ is an integer $\forall k$ by (a), but the coefficients of $Q(x)$ are

not all integral. .... 3 — 3

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7(a) (i) If $X, Y \in M_1$, $AX = k_1X$ and $AY = k_1Y$

For any $a, b \in \mathbb{R}$, $(aX + bY) \in M$ and (3×1 matrics)

$A(aX + bY) = A(aX) + A(bY)$

$= aAX + bAY$

$= ak_1X + bk_1Y$ 1

$= k_1(aX + bY)$ 1

$\therefore aX + bY \in M_1$.

Similarly for $i = 2$.

(ii) Obviously $0 \in M_1 \cap M_2$ 1

If $X \in M_1 \cap M_2$, $AX = k_1X$ and $AX = k_2X$

$\therefore k_1X = k_2X \Rightarrow (k_1 - k_2)X = 0$

$\Rightarrow X = 0$ as $k_1 \neq k_2$ and 1

$\therefore M_1 \cap M_2 = \{0\}$. — 4

(b) For any $X \in M$, $(A - k_2I)X \in M$

and $A[(A - k_2I)X] = (A^2 - k_2A)X$ 1

$= (k_1A - k_1k_2I)X$ 1

$= k_1[(A - k_2I)X]$ 1

$\therefore (A - k_2I)X \in M_1$.

Similarly $(A - k_1I)X \in M_2$. 1 — 4

$A^2 - (k_1+k_2)A + k_1k_2I = 0$

$A^2 - k_2A = k_1A - k_1k_2$

(c) For any $X \in M$, by (b), (2)

$(A - k_2I)X \in M_1$ and $(A - k_1I)X \in M_2$ 要作

By (a) $X_1 = \left(\dfrac{1}{k_1 - k_2}\right)(A - k_2I)X \in M_1$ and 1

$X_2 = \left(\dfrac{-1}{k_1 - k_2}\right)(A - k_1I)X \in M_2$ 1

and $X_1 + X_2 = \dfrac{1}{k_1 - k_2}(A - k_2I)X + \dfrac{-1}{k_1 - k_2}(A - k_1I)X$

$= X$ 1

uniqueness (1)

If $\exists X'_1 \in M_1$ and $X'_2 \in M_2$ such that $X = X'_1 + X'_2$ 1

then $X_1 + X_2 = X'_1 + X'_2$

$X_1 - X'_1 = X'_2 - X_2 \in M_1 \cap M_2$ by (a)(i). 1

$\therefore X_1 - X'_1 = X'_2 - X_2 = 0$ by (a)(ii) 1

i.e. $X_1 = X'_1$, $X_2 = X'_2$ — 6

$x = M_1$

$y = M_2$

if $x = y$,

$x = y \in M_1 \cap$

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8(a) We shall prove the first part by induction. The case where $n = 1$ is trivial.

Assume that $g(kx) = kg(x)$ for some integer $k \geq 1$.
Then
$$g((k+1)x) = g(kx + x) = g(kx) + g(x) = kg(x) + g(x) = (k+1)g(x)$$

Hence $g(nx) = ng(x) \quad \forall n \geq 1$. 2

Next assume for contradiction that $g(x) \not\equiv 0$.
Let $|g(x_0)| = b > 0$ for some $x_0 \in \mathbb{R}$. 1
Since g is bounded on $\mathbb{R}$, let $|g(x)| \leq M \ \forall x \in \mathbb{R}$ and let n be an integer greater than $\frac{M}{b}$. 1
Then
$$|g(nx_0)| = |ng(x_0)| = n|g(x_0)| > \frac{M}{b}|g(x_0)| = M,$$
which contradicts the boundedness of g. 1

$\therefore g(x) \equiv 0$ — 5

(b) (i) For any $x, y \in \mathbb{R}$,
$$h(x+y) = f(x+y) - \frac{f(a)}{a}(x+y) = f(x) + f(y) - \frac{f(a)}{a}x - \frac{f(a)}{a}y = h(x) + h(y)$$ 2

$\therefore h$ is additive.

Let f be bounded by M on $[0, a]$.

For any $x \in [0, a]$,
$$|h(x)| = \left|f(x) - \frac{f(a)}{a}x\right| \leq |f(x)| + \left|\frac{f(a)}{a}\right||x| \leq |f(x)| + |f(a)| \leq 2M$$

$\therefore h$ is bounded on $[0, a]$. 2

(ii) For any $x \in \mathbb{R}$,
$$h(x+a) = h(x) + h(a) = h(x) + f(a) - \frac{f(a)}{a}a = h(x)$$ 1

Next for any $x \in \mathbb{R}$, since h is periodic of period a, let y be in $[0, a]$ such that $h(x) = h(y)$. 1

$\therefore |h(x)| = |h(y)| \leq 2M$ by (i)
$\therefore h$ is bounded on $\mathbb{R}$. 1

(iii) As h is additive and bounded on $\mathbb{R}$, by (a)

$h(x) \equiv 0$, i.e. $f(x) = \frac{f(a)}{a}x \quad \forall x \in \mathbb{R}$ 2

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9(a) (i)
$$(1+x)^{m-p}(1+x)^m = \left(\sum_{r=0}^{m-p} C_r^{m-p} x^r\right)\left(\sum_{r=0}^{m} C_r^m x^r\right)$$

The coefficient of $x^m = \sum_{s=0}^{m-p} C_s^{m-p} C_{m-s}^m$ 1

$$= \sum_{r=p}^{m} C_{m-r}^{m-p} C_r^m$$

$$= \sum_{r=p}^{m} C_{r-p}^{m-p} C_r^m$$ 1

But the coefficient of x^m in $(1+x)^{2m-p}$ is C_m^{2m-p} 1

Hence the result.

(ii)
$$\sum_{r=0}^{m} r(C_r^m)^2 = \sum_{r=0}^{m}(r)(C_r^m)(C_r^m)$$

$$= m\sum_{r=1}^{m} C_{r-1}^{m-1} C_r^m$$ 1

$$= m \cdot C_m^{2m-1}, \text{ from (i) } (p = 1)$$ 1

$$\sum_{r=0}^{m} r^2(C_r^m)^2 = m^2 \sum_{r=1}^{m}(C_{r-1}^{m-1})^2$$

$$= m^2 \sum_{r=0}^{m-1}(C_r^{m-1})^2$$

$$= m^2 \cdot C_{m-1}^{2m-2}, \text{ from (i) } (p = 0)$$ .... 1 — 6

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SOLUTIONS	MARKS	REMARKS
(b) Suppose bag A contains r ($0 \le r \le m$) white balls. If a ball is drawn from each bag, the probability that they have the same colour is		
$\frac{r}{m}\cdot\frac{m-r}{m}+\frac{m-r}{m}\cdot\frac{r}{m}$		
$=\frac{2r(m-r)}{m^2}$	1	
If m balls are selected at random and put into each bag, the probability that bag A contains r white balls is		
$\frac{C^m_r\cdot C^m_{m-r}}{C^{2m}_m}$	2	
$=\frac{(C^m_r)^2}{C^{2m}_m}$		
$\therefore$ the probability that the two balls subsequently drawn have the same colour is		
$\frac{(C^m_r)^2}{C^{2m}_m}\cdot\frac{2r(m-r)}{m^2}$	1	
The required probability is		
$\sum_{r=0}^{m}\frac{(C^m_r)^2}{C^{2m}_m}\cdot\frac{2r(m-r)}{m^2}$	2	
$=\frac{2}{C^{2m}_m}\left[\frac{1}{m}\sum_{r=0}^{m} r(C^m_r)^2-\frac{1}{m^2}\sum_{r=0}^{m} r^2(C^m_r)^2\right]$		
$=\frac{2}{C^{2m}_m}\left[C^{2m-1}_m - C^{2m-2}_{m-1}\right]$	1	
$=\frac{2(m!)^2}{(2m)!}\left[\frac{(2m-1)!}{m!\,(m-1)!}-\frac{(2m-2)!}{[(m-1)!]^2}\right]$		
$=\frac{m-1}{2m-1}$	1	
	8	

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香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八五年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1985

PURE MATHEMATICS (II)
MARKING SCHEME

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1 (a) The tangents to (E) at P and Q are given by

$$\frac{\cos\theta}{a}x + \frac{\sin\theta}{b}y = 1$$

$$\frac{\cos\phi}{a}x + \frac{\sin\phi}{b}y = 1 \quad \ldots\ldots \quad 1$$

Solving these equations, the coordinates of T are

by Cramer's rule

$$x = \frac{a(\sin\phi - \sin\theta)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad 1$$

$$= \frac{a(\sin\phi - \sin\theta)}{\sin(\phi - \theta)}$$

$$y = \frac{b(\cos\theta - \cos\phi)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad 1$$

$$= \frac{b(\cos\theta - \cos\phi)}{\sin(\phi - \theta)}$$

The tangents to (K) at M and N are

$$x\cos\theta + y\sin\theta = a$$

$$x\cos\phi + y\sin\phi = a \quad \ldots\ldots \quad 1$$

The coordinates of R are

$$x = \frac{a(\sin\phi - \sin\theta)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad 1$$

$$y = \frac{a(\cos\theta - \cos\phi)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad 1$$

6

(which could also be obtained by putting b = a in the first part)

(b) (i) Substituting the coordinates of T in (F),

$$\frac{a^2(\sin\phi - \sin\theta)^2}{a^2\sin^2(\phi - \theta)} + \frac{b^2(\cos\theta - \cos\phi)^2}{b^2\sin^2(\phi - \theta)} = 2 \quad \ldots\ldots \quad 2$$

$$\sin^2\phi + \sin^2\theta - 2\sin\phi\sin\theta + \cos^2\phi + \cos^2\theta - 2\cos\phi\cos\theta$$

$$= 2\sin^2(\phi - \theta)$$

$$\cos(\phi - \theta) = \sin^2(\phi - \theta) \quad \ldots\ldots \quad 1$$

$$\cos(\phi - \theta)\,[\cos(\phi - \theta) - 1] = 0$$

$\therefore \cos(\phi - \theta) = 0$ as θ and ϕ are distinct ……… 1

(ii) Squaring and adding the coordinates of R,

$$x^2 + y^2 = \frac{a^2(\sin\phi - \sin\theta)^2}{\sin^2(\phi - \theta)} + \frac{a^2(\cos\theta - \cos\phi)^2}{\sin^2(\phi - \theta)} \quad \ldots\ldots \quad 2$$

$$= a^2\,\frac{\sin^2\phi + \sin^2\theta - 2\sin\theta\sin\theta + \cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi}{\sin^2(\phi - \theta)}$$

$$= 2a^2\,\frac{[1 - \cos(\theta - \phi)]}{\sin^2(\phi - \theta)}$$

$= 2a^2$ since $\cos(\theta - \phi) = 0$ ……… 1

$\therefore$ R moves on a circle with radius $a\sqrt{2}$ ……… 1

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(Note: This part can also be deduced by geometric method.)

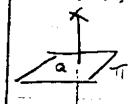
2 (a) The direction numbers of a line normal to π are A, B, C. ……… 1

If this line passes through P, its equations are

$$\begin{cases} x = x_0 + At \\ y = y_0 + Bt \quad t \in \mathbb{R} \quad \ldots (*) \\ z = z_0 + Ct \end{cases} \quad \ldots\ldots \quad 1$$

Substituting in π,

$P(x_0, y_0, z_0)$



$$A(x_0 + At) + B(y_0 + Bt) + C(z_0 + Ct) + D = 0 \quad \ldots\ldots \quad 1$$

$$Ax_0 + By_0 + Cz_0 + D + (A^2 + B^2 + C^2)t = 0$$

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2} \quad \ldots\ldots \quad 1$$

Putting this value of t in (*), we obtain the coordinates of Q. 1

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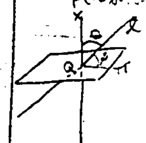
(b) The direction numbers of ℓ are p, q, r. ……… 1

$P(x_0, y_0, z_0)$



The acute angle θ between ℓ and the normal to π is given by

$$\cos\theta = \frac{|Ap + Bq + Cr|}{\sqrt{A^2 + B^2 + C^2}\sqrt{p^2 + q^2 + r^2}} \quad \ldots\ldots \quad 2$$

$\therefore$ the angle ϕ between ℓ and π is given by

$$\phi = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \cos^{-1}\frac{|Ap + Bq + Cr|}{\sqrt{A^2 + B^2 + C^2}\sqrt{p^2 + q^2 + r^2}} \quad \ldots\ldots \quad 1$$

4

$\vec{a}\cdot\vec{n} = |\vec{a}||\vec{n}|\cos$

$\cos\theta = \frac{\vec{a}\cdot\vec{n}}{|\vec{a}||\vec{n}|}$

(c) Substituting the coordinates of a point on ℓ into π, ……… 1

$$A(a + tp) + B(b + tq) + C(c + tr) + D = 0$$

$$(Aa + Bb + Cc + D) + (Ap + Bq + Cr)t = 0 \quad \ldots\ldots \quad 1$$

ℓ lies on π if and only if the above equation holds for all real t. ……… 1

i.e. $Aa + Bb + Cc + D = 0$

$Ap + Bq + Cr = 0$ ……… 2

5

[Note: Candidates may use vectors in their solution.]

3 (a) For $t \in (-1, 1)$, the sum of the GP

$$1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} = \frac{1-(-t)^n}{1+t} \Leftarrow n \text{ terms}$$

$$\therefore \frac{1}{1+t} = 1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t} \qquad 1$$

Putting $u = 1 + t$, for $x \in (-1, 1)$,

$$\ln(1+x) = \int_1^{1+x} \frac{1}{u}\,du \qquad \begin{matrix} u = 1+t \\ du = dt \\ 1+x = 1+t \;\; 1 = 1+t \\ \Rightarrow t = x \;\; \Rightarrow t = 0 \end{matrix}$$

$$= \int_0^x \frac{1}{1+t}\,dt \quad \dots\dots \quad 1$$

$$= \int_0^x \left[1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t}\right] dt$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t}\,dt \qquad 1$$

Similarly, $\frac{1}{1-t} = 1 + t + t^2 + \dots + t^{n-1} + \frac{t^n}{1-t}$ …… 1

$$\ln(1-x) = \int_1^{1-x} \frac{1}{u}\,du \qquad \begin{matrix} u = 1-t \\ du = -dt \\ 1-x = 1-t \;\; 1 = 1-t \\ \Rightarrow t = x \;\; \Rightarrow t = 0 \end{matrix}$$

$$= -\int_0^x \frac{1}{1-t}\,dt$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} - \int_0^x \frac{t^n}{1-t}\,dt \quad \dots\dots \quad \frac{1}{5}$$

(or putting $x = -u$ in the first result)

3 (b) Putting $n = 2k + 1$,

$$\frac{t^{2k+1} + t^{2k+2} - t^{2k+1} + t^{2k+2}}{1-t^2}$$

$$\ln\left(\frac{1+x}{1-x}\right)$$

$$= \ln(1+x) - \ln(1-x) \quad \dots\dots \quad 1$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2k+1}}{2k+1}\right] + \int_0^x \left[\frac{t^{2k+1}}{1-t} - \frac{t^{2k+1}}{1+t}\right] dt$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2k+1}}{2k+1}\right] + \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt \quad \dots\dots \quad 1$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) - 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2k+1}}{2k+1}\right]$$

$$= \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt$$

from the $\int_0^x$

For $0 < t < x < 1$, $0 < \frac{t^{2k+2}}{1-t^2} < \frac{t^{2k+2}}{1-x^2}$

$$\therefore \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt \geq 0 \quad \dots\dots \quad 1$$

$$\text{and } \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt \leq \frac{1}{1-x^2}\int_0^x 2t^{2k+2}\,dt \quad \dots\dots \quad 1$$

$$= \frac{2}{1-x^2}\left(\frac{x^{2k+3}}{2k+3}\right) \quad \dots\dots \quad \frac{1}{5}$$

The result follows.

(c) Putting $x = \frac{1}{2}$,

$$0 \leq \ln 3 - 2\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \dots + \frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\right] \leq \frac{8}{3} \cdot \frac{1}{2k+3}\left(\frac{1}{2}\right)^{2k+3} \qquad 1$$

As $\lim_{k \to \infty} \frac{8}{3} \cdot \frac{1}{2k+3}\left(\frac{1}{2}\right)^{2k+3} = 0$, …… 1

By Sandwich theorem,

$$\lim_{k \to \infty}\left[\ln 3 - 2\left(\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \dots + \frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\right)\right] = 0 \qquad 1$$

$\therefore \lim_{k \to \infty}\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \dots + \frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\right]$ exists and

equals $\frac{1}{2}\ln 3$ …… $\frac{1}{4}$

SOLUTIONS | 85 II | MARKS | REMARKS

4.(a) The coordinates of the centre of the circle are $(r\not{0}, r)$. arc length 1

The coordinates of Q with respect to the centre are $(-r\sin\emptyset, -r\cos\emptyset)$ 1

$\therefore$ the locus of Q is given by

$$\begin{cases} x = r\emptyset - r\sin\emptyset \\ y = r - r\cos\emptyset, \quad 0 \le \emptyset \le 2\pi \end{cases}$$ 1, 1

i.e. the locus of P is given by

$$\begin{cases} x = r(\emptyset - \sin\emptyset) \\ y = r(\cos\emptyset - 1), \quad 0 \le \emptyset \le 2\pi . \end{cases}$$ 1

greatest when P reaches P_L

When P reaches P_L, the value of $\emptyset = \pi$. 1

Total: 6

(b) $dx = r(1 - \cos\emptyset)\,d\emptyset$, $\quad dy = -r\sin\emptyset\,d\emptyset$

$$\frac{dy}{dx} = \frac{-\sin\emptyset}{1-\cos\emptyset}$$ 1

$$T = \int_{\emptyset_s}^{\pi} \sqrt{\frac{1 + \left(\frac{\sin\emptyset}{1-\cos\emptyset}\right)^2}{2gr(\cos\emptyset_s - \cos\emptyset)}}\; r(1-\cos\emptyset)\,d\emptyset$$

$$= \sqrt{\frac{r}{g}}\int_{\emptyset_s}^{\pi}\sqrt{\frac{1-\cos\emptyset}{\cos\emptyset_s - \cos\emptyset}}\,d\emptyset$$ 2

$$= \sqrt{\frac{r}{g}}\int_{\emptyset_s}^{\pi}\sqrt{\frac{2\sin^2\frac{\emptyset}{2}}{(2\cos^2\frac{\emptyset_s}{2} - 1) - (2\cos^2\frac{\emptyset}{2} - 1)}}\,d\emptyset$$ 2

$$= \sqrt{\frac{r}{g}}\int_{\emptyset_s}^{\pi}\frac{-2\,d(\cos\frac{\emptyset}{2})}{\sqrt{\cos^2\frac{\emptyset_s}{2} - \cos^2\frac{\emptyset}{2}}}$$

$$= 2\sqrt{\frac{r}{g}}\left[-\sin^{-1}\left(\frac{\cos\frac{\emptyset}{2}}{\cos\frac{\emptyset_s}{2}}\right)\right]_{\emptyset_s}^{\pi}$$ 2

$$= 2\sqrt{\frac{r}{g}}\,(-\sin^{-1}0 + \sin^{-1}1)$$

$$= \pi\sqrt{\frac{r}{g}},$$ 1

Total: 8

which is independent of $\emptyset_s$.

SOLUTIONS | 85 II | MARKS | REMARKS

5.(a) Since $x_1 + x_2 = 2t$, $x_1x_2 = 1$, for $n \ge 1$, ... 1

sum of roots, product of root

$$x_1^{n+1} + x_2^{n+1} = (x_1^n + x_2^n)(x_1 + x_2) - x_1x_2(x_1^{n-1} + x_2^{n-1})$$ 1

$$= 2F_n(t)(2t) - 2F_{n-1}(t)$$

$$\therefore F_{n+1}(t) = 2tF_n(t) - F_{n-1}(t)$$ 1

Now $F_1(t) = \frac{1}{2}(x_1 + x_2) = t$ is a polynomial in t of degree 1 and with leading coefficient 1. 1

For all positive integers k less than or equal to some positive n, ($k \le n$) assume that $F_k(t)$ is a polynomial in t of degree k and with leading coefficient 2^{k-1}.

Then $F_{n+1}(t) = 2tF_n(t) - F_{n-1}(t)$ is also a polynomial in t of degree $(n + 1)$. Further the leading coefficient of $F_{n+1}(t) = 2 \times 2^{n-1} = 2^n$. The result follows by induction. second principle 2

Total: 6

(b) As $-1 \le t \le 1$, we may let $\cos\theta = t$ 1

$F_0(t) = 1 = \cos 0$, $F_1(t) = t = \cos[\cos^{-1}t]$. 1

Assume that $F_k(t) = \cos[k\cos^{-1}t]$, where $0 \le k \le n$, $n \ge 1$.

$$F_{n+1}(t) = 2t\,F_n(t) - F_{n-1}(t)$$

$$= 2\cos\theta\cos n\theta - \cos(n-1)\theta$$

$$= 2\cos\theta\cos n\theta - \cos n\theta\cos\theta - \sin n\theta\sin\theta$$

$$= \cos(n+1)\theta$$

$$= \cos[(n+1)\cos^{-1}t]$$

$$\therefore F_n(t) = \cos[n\cos^{-1}t] \quad \forall\ n \ge 0.$$ 2

$$\int_0^{\pi} F_m(\cos\theta)\,F_n(\cos\theta)\,d\theta = \int_0^{\pi}\cos m\theta\cos n\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}[\cos(m+n)\theta + \cos(m-n)\theta]\,d\theta$$ 1

$$= \begin{cases} \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right]_0^{\pi} & \text{if } m \ne n \\ [\theta]_0^{\pi} & \text{if } m = n = 0 \\ \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n} + \theta\right]_0^{\pi} & \text{if } m = n > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n = 0 \\ \frac{\pi}{2} & \text{if } m = n > 0 \end{cases}$$ 1, 1, 1

Total: 8

SOLUTIONS | 85 II | MARKS | REMARKS

Alternative Solution :

5(a) Solving $x^2 - 2tx + 1 = 0$, $\quad x = \dfrac{2t \pm \sqrt{4t^2 - 4}}{2}$

$= t \pm \alpha$, where $\alpha = \sqrt{t^2 - 1}$ — 1

Let $x_1 = t + \alpha$, $x_2 = t - \alpha$.

$F_{n+1}(t) - 2t\,F_n(t) + F_{n-1}(t)$

$= \frac{1}{2}\,[\,(t+\alpha)^{n+1} + (t-\alpha)^{n+1}\,] - \frac{2t}{2}\,[(t+\alpha)^n + (t-\alpha)^n]$

$+ \frac{1}{2}\,[(t+\alpha)^{n-1} + (t-\alpha)^{n-1}\,]$ — 1

$= \frac{1}{2}\,(t+\alpha)^{n-1}\,[(t+\alpha)^2 - 2t(t+\alpha) + 1]$

$+ \frac{1}{2}(t-\alpha)^{n-1}\,[(t-\alpha)^2 - 2t(t-\alpha) + 1]$

$= \frac{1}{2}\,(t+\alpha)^{n-1}\,[t^2 + (t^2-1) + 2\alpha t - 2t^2 - 2\alpha t + 1]$

$+ \frac{1}{2}\,(t-\alpha)^{n-1}\,[t^2 + (t^2-1) - 2\alpha t - 2t^2 + 2\alpha t + 1]$

$= 0$ — 1

Next, for each given n,

$F_n(t) = \frac{1}{2}\,[(t+\alpha)^n + (t-\alpha)^n]$ — 1

$= \frac{1}{2}\,[\,\sum_{r=0}^{n} C^n_r\, t^{n-r}\alpha^r + \sum_{r=0}^{n} (-1)^r\, t^{n-r}\,\alpha^r\,]$

$= C^n_0\, t^n + C^n_2\, t^{n-2}\,\alpha^2 + C^n_4\, t^{n-4}\,\alpha^4 + \ldots$

We see that each term is in the form $C^n_{2k}\, t^{n-2k}(t^2 - 1)^k$.

$\therefore$ $F_n(t)$ is a polynomial in t, of degree n and with leading coefficient $(C^n_0 + C^n_2 + C^n_4 + \ldots) = 2^{n-1}$. — $\frac{2}{6}$

SOLUTIONS | 85 II | MARKS | REMARKS

6. (a) Put $u = f_n(t)$, $dv = (x - t)^{m-1}\,dt$

$du = f_{n-1}(t)\,dt$, $v = -\dfrac{(x-t)^m}{m}$

Integrating by parts, we have

$$\int_0^x (x-t)^{m-1} f_n(t)dt = \left[-\frac{(x-t)^m f_n(t)}{m}\right]_0^x + \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)dt$$ — 2

$$= \frac{x^m f_n(0)}{m} + \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)\,dt$$

$$= \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)\,dt \quad \ldots\ldots\ldots\ldots$$ — $\frac{2}{4}$

(b) Applying (a) repeatedly, for n = 1, 2, 3, ...,

$$f_n(x) = \int_0^x f_{n-1}(t)\,dt$$

$$= \int_0^x (x-t)\,f_{n-2}(t)\,dt \quad \ldots\ldots\ldots\ldots$$ — 2

$$= \frac{1}{2}\int_0^x (x-t)^2 f_{n-3}(t)\,dt$$

$= \ldots\ldots\ldots$

$$= \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f_0(t)\,dt \quad \ldots\ldots\ldots\ldots$$ — $\frac{2}{4}$

(c) For $0 \le t \le x \le 1$, $0 \le (x-t)^{n-1} \le 1$. .... — 1

$$0 \le |f_n(x)| = \frac{1}{(n-1)!}\left|\int_0^x (x-t)^{n-1} f_0(t)\,dt\right| \quad \ldots\ldots\ldots\ldots$$ — 2

← NB

$$\le \frac{M}{(n-1)!}\int_0^x (x-t)^{n-1}\,dt$$

$$= \frac{M}{(n-1)!}\left[-\frac{(x-t)^n}{n}\right]_0^x$$

$$= \frac{Mx^n}{n!} \quad \ldots\ldots\ldots\ldots$$ — 1

$$\le \frac{M}{n!} \quad \text{as } 0 \le x \le 1 \quad \ldots\ldots\ldots\ldots$$ — 1

But $\lim_{n\to\infty} \frac{M}{n!} = 0$,

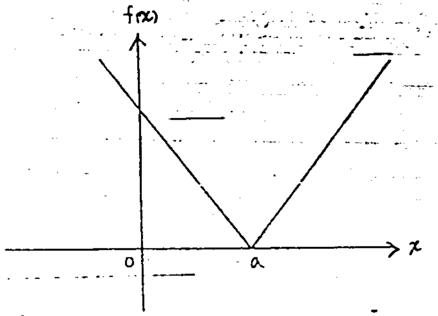
$\therefore$ $\lim_{n\to\infty} |f_n(x)| = 0$ — by Sandwich theorem

i.e. $\lim_{n\to\infty} f_n(x) = 0$ .... — 1

SOLUTIONS 85 II | MARKS | REMARKS

(a) $f(x) = |x - a|$

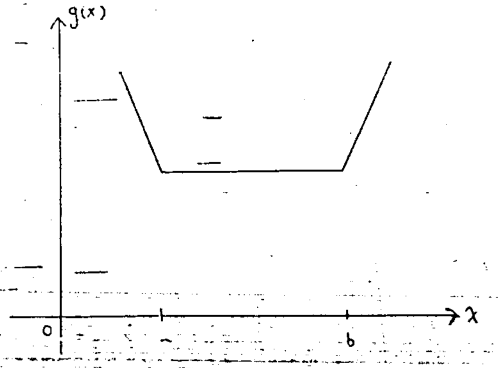
Value of x	x < a	x = a	x > a	Marks
Value of f	a − x	0	x − a	1
Value of f'	−1	f' doesn't exist ($f'_+ \neq f'_-$)	1	1
Behaviour of f	↘	minimum	↗	1



$\frac{1}{4}$

(b) $g(x) = |x - a| + |x - b|$.

Value of x	x < a	x = a	a<x<b	x = b	x>b	Marks
Value of g	(a+b)−2x	b − a	b − a	b − a	2x−(a+b)	1
Value of g'	−2	Doesn't exist	0	Doesn't exist	2	1
Behaviour of g	↘	minimum			↗	1

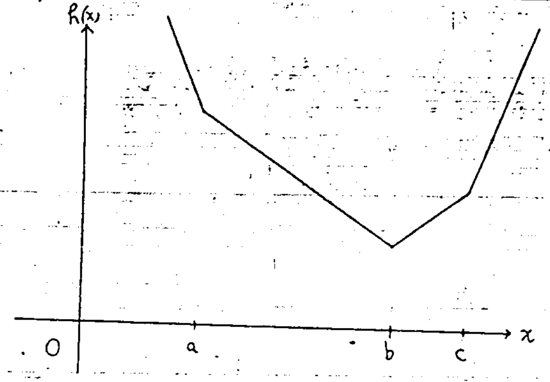


$\frac{2}{5}$

SOLUTIONS 85 II | MARKS | REMARKS

(c) $h(x) = |x - a| + |x - b| + |x - c|$

Value of x	x< a	x = a	a<x<b	x = b	b<x<c	x = c	x > c	Marks
Value of h	(a+b+c)−3x	b+c−2a	(b+c−a)−x	c − a	x−(a+b−c)	2c−(a+b)	3x−(a+b+c)	1
Value of h'	−3	Doesn't exist	−1	Doesn't exist	1	Doesn't exist	3	1
Behaviour of h	↘	↘	↘	minimum	↗	↗	↗	1



$\frac{2}{5}$

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Differentiable iff $f'_+(a) = f'_-(a) = \ell$

$f'(a) = \ell$

SOLUTIONS	85 II MARKS	REMARKS

8 (a) (i) Put $u = \pi - x$ 1

$$\int_{\frac{\pi}{2}}^{\pi-2r} \ln \sin x \, dx = -\int_{2r}^{\frac{\pi}{2}} \ln \sin(\pi - x) \, dx \qquad \text{NB } = \int_{\frac{\pi}{2}}^{2r} \ln\sin(\pi-u)\,d(\pi-u)$$

$$= \int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx \text{ .... } 1$$

$x = \pi - 2r \Rightarrow u = \pi - \pi + 2r = 2r$

$x = \frac{\pi}{2} \Rightarrow u = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

(ii) Put $u = 2x$　　$x = \frac{u}{2}$, $dx = \frac{1}{2}du$

$$\int_{r}^{\frac{\pi}{2}-r} \ln \sin 2x \, dx = \frac{1}{2}\int_{2r}^{\pi-2r} \ln \sin u \, du \text{ .... } 1$$

$$= \frac{1}{2}\left[\int_{2r}^{\frac{\pi}{2}} \ln \sin u \, du + \int_{\frac{\pi}{2}}^{\pi-2r} \ln \sin u \, du\right] \quad \text{by (i)}$$

$$= \int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx \text{ .... } 1$$

$x = \frac{\pi}{2} - r \Rightarrow u = \pi - 2r$

$x = r \Rightarrow u = 2r$

$$\int_{r}^{\frac{\pi}{2}-r} (\ln \sin x + \ln \cos x)dx = \int_{r}^{\frac{\pi}{2}-r} \ln \sin x \cos x \, dx \text{ ...... } 1$$

$$= \int_{r}^{\frac{\pi}{2}-r} \ln \frac{\sin 2x}{2} \, dx \text{ .... } 1$$

$$= \int_{r}^{\frac{\pi}{2}-r} \ln \sin 2x \, dx - \int_{r}^{\frac{\pi}{2}-r} \ln 2 \, dx \text{ .... } 1$$

$$= \int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx - \left(\frac{\pi}{2} - 2r\right) \ln 2 \text{ .... } \frac{1}{8}$$

SOLUTIONS	85 II MARKS	REMARKS

8. (b) Put $u = \frac{\pi}{2} - x$ 1

$$\int_{r}^{\frac{\pi}{2}-r} \ln \sin x \, dx = -\int_{\frac{\pi}{2}-r}^{r} \ln \sin\left(\frac{\pi}{2} - u\right) du$$

$$= \int_{r}^{\frac{\pi}{2}-r} \ln \cos x \, dx \text{ .... } 1$$

By (*),

$$\lim_{r\to 0}\int_{r}^{\frac{\pi}{2}-r} (\ln \sin x + \ln \cos x)dx$$

$$= \lim_{r\to 0}\left[\int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx - \left(\frac{\pi}{2} - 2r\right) \ln 2\right] \text{ .... } 1$$

$$\left(\overbrace{\lim_{r\to 0}\int_{r}^{\frac{\pi}{2}-r} \ln \sin x \, dx}^{A}\right) + \left(\overbrace{\lim_{r\to 0}\int_{r}^{\frac{\pi}{2}-r} \ln \cos x \, dx}^{A}\right)$$

$$= \left(\overbrace{\lim_{r\to 0}\int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx}^{A}\right) - \lim_{r\to 0}\left[\left(\frac{\pi}{2} - 2r\right) \ln 2\right] \text{ .... } 1$$

(Note that all limits involved exist)

$$\therefore\ A = \lim_{r\to 0}\int_{r}^{\frac{\pi}{2}-r} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2 \text{ .... } \frac{2}{6}$$

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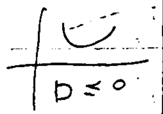
9.(a) Since $[\lambda f(x) + g(x)]^2 \geq 0$ for any λ and x,

$\int_a^b [\lambda f(x) + g(x)]^2\,dx \geq 0$ 1

$\int_a^b [\lambda^2[f(x)]^2 + 2\lambda f(x)\,g(x) + [g(x)]^2]\,dx \geq 0$ 1

$\therefore\ \lambda^2\int_a^b [f(x)]^2 dx + 2\lambda\int_a^b f(x)g(x)dx + \int_a^b [g(x)]^2 dx \geq 0$ for any λ. 1

Since the inequality holds for any λ, the discriminant ≤ 0. 1



$\therefore\ [2\int_a^b f(x)g(x)dx]^2 - 4\left(\int_a^b [f(x)]^2 dx\right)\left(\int_a^b [g(x)]^2 dx\right) \leq 0$

i.e. $[\int_a^b f(x)g(x)dx]^2 \leq \left(\int_a^b [f(x)]^2 dx\right)\left(\int_a^b [g(x)]^2 dx\right)$ .... $\frac{1}{4}$

(b) (i) As $f(0) = f(1) = 0$,

$\int_0^x f'(t)\,dt = f(x) - f(0)$

$= f(x)$ 1

and $-\int_x^1 f'(t)\,dt = -f(1) + f(x)$

$= f(x)$ 1

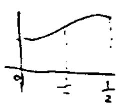
(ii) If $x \in [0, \frac{1}{2}]$, from (i),

$[f(x)]^2 = [\int_0^x f'(t)\,dt]^2$

$\leq \left(\int_0^x dt\right)\left(\int_0^x [f'(t)]^2\,dt\right)$ (by (a)) 1

$= x\int_0^x [f'(t)]^2\,dt$

$\leq x\int_0^{\frac{1}{2}} [f'(t)]^2\,dt$ ($\because [f'(t)]^2 \geq 0$) 1



If $x \in [\frac{1}{2}, 1]$, $[f(x)]^2 = [-\int_x^1 f'(t)\,dt]^2$

$\leq \left(\int_x^1 dt\right)\left(\int_x^1 [f'(t)]^2\,dt\right)$ 1

$= (1 - x)\int_x^1 [f'(t)]^2\,dt$

$\leq (1 - x)\int_{\frac{1}{2}}^1 [f'(t)]^2\,dt$ 1

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(b) (iii)

By (ii), $\int_0^{\frac{1}{2}} [f(x)]^2\,dx \leq \left(\int_0^{\frac{1}{2}} [f'(t)]^2\,dt\right)\left(\int_0^{\frac{1}{2}} x\,dx\right)$ 1

$= \frac{1}{8}\int_0^{\frac{1}{2}} [f'(t)]^2\,dt$ 1

and $\int_{\frac{1}{2}}^1 [f(x)]^2\,dx \leq \left(\int_{\frac{1}{2}}^1 [f'(t)]^2\,dt\right)\left(\int_{\frac{1}{2}}^1 (1 - x)\,dx\right)$

$= \frac{1}{8}\int_{\frac{1}{2}}^1 [f'(t)]^2\,dt$ 1

Adding these two results

$\int_0^{\frac{1}{2}} [f(x)]^2 dx + \int_{\frac{1}{2}}^1 [f(x)]^2 dx \leq \frac{1}{8}\left[\int_0^{\frac{1}{2}} [f'(t)]^2 dt + \int_{\frac{1}{2}}^1 [f'(t)]^2 dt\right]$

$\int_0^1 [f(x)]^2\,dx \leq \frac{1}{8}\int_0^1 [f'(t)]^2\,dt$ .... $\frac{1}{10}$