8. Let f be a real-valued function which is continuously differentiable and strictly increasing on the interval $I = [0, \infty)$. Suppose $f(0) = 0$. Let $a \in I$ and $b \in f[I]$.

(a) For any $t \in I$, define $g(t) = bt - \int_0^t f(x)\,dx$.

Prove that g attains its greatest value at $f^{-1}(b)$.

(b) (i) Show that $\int_0^{f^{-1}(b)} x f'(x)\,dx = g(f^{-1}(b))$.

(ii) By a change of variable, show that

$$\int_0^{f^{-1}(b)} x f'(x)\,dx = \int_0^b f^{-1}(x)\,dx .$$

(c) Use (a) and (b) to prove that $\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geqslant ab$.

Referring to Figure 3, what is the geometric meaning of the above inequality if the integrals are interpreted as areas ?

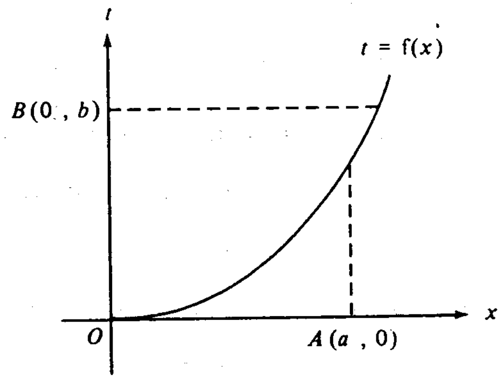


Figure 3

(d) Using (c), show that

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geqslant ab ,$$

where $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1985

純數學 試卷一

PURE MATHEMATICS PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. (a) (i) Let p be a given positive integer. Show that if a sequence $\{a_n\}$ converges, then $\lim_{n\to\infty}(a_n + a_{n+p})$ exists.

(ii) Show that the converse of (i) does not hold for $p = 1$ by constructing a sequence $\{c_n\}$ such that $c_n + c_{n+1} = 0$ for all positive integers n, but $\lim_{n\to\infty} c_n$ does not exist.

(b) $\{a_n\}$ is a sequence such that

$$\lim_{n\to\infty}(a_n + a_{n+1}) = A \quad \text{and} \quad \lim_{n\to\infty}(a_n + a_{n+2}) = B.$$

Show that $\lim_{n\to\infty} a_n$ exists; find its value and show that $A = B$.

2. Consider the system of linear equations in the unknowns x_i $(i = 1, 2, 3)$:

$$(E)\begin{cases} |a_1 - a_2|x_2 + |a_1 - a_3|x_3 = b_1 \\ |a_2 - a_1|x_1 + |a_2 - a_3|x_3 = b_2 \\ |a_3 - a_1|x_1 + |a_3 - a_2|x_2 = b_3 \end{cases}$$

(a) Suppose a_1, a_2 and a_3 are all distinct.

(i) Show that (E) has a unique solution.

(ii) If, furthermore, $b_1 = b_2 = b_3 \neq 0$, show that exactly one x_i in the solution (x_1, x_2, x_3) is zero.

(b) Suppose b_1, b_2 and b_3 are all distinct. Show that (E) is consistent only if a_1, a_2 and a_3 are all distinct.

(c) If $a_1 = a_2 = a_3$, are there any b_1, b_2 and b_3 such that (E) is consistent? Prove your assertion.

3. (a) Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. Suppose $f(x)$ is a polynomial of degree less than $n - 1$ and the expression $\dfrac{f(x)}{(x + a_1)(x + a_2)\ldots(x + a_n)}$ is resolved into partial fractions as

$$\frac{c_1}{x + a_1} + \frac{c_2}{x + a_2} + \ldots + \frac{c_n}{x + a_n}.$$

Show that $c_1 + c_2 + \ldots + c_n = 0$.

(b) Let $F(x) = \dfrac{px + q}{(x + a)(x + a + 1)(x + a + 2)}$ be resolved into partial fractions as $\dfrac{b_1}{x + a} + \dfrac{b_2}{x + a + 1} + \dfrac{b_3}{x + a + 2}$.

Show that for $N > 3$,

$$\sum_{k=1}^{N} F(k) = \frac{b_1}{1 + a} + \frac{b_1 + b_2}{2 + a} + \frac{b_2 + b_3}{N + a + 1} + \frac{b_3}{N + a + 2}.$$

(c) Using (b), or otherwise, evaluate

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{(2k + 1)(2k + 3)(2k + 5)}.$$

4. Consider a mapping $f: A \longrightarrow B$.

(a) (i) Show that for any subsets X_1, X_2 of A,

$$f[X_1] \subset f[X_2] \quad \text{if} \quad X_1 \subset X_2.$$

(ii) Show that for any subsets Y_1, Y_2 of B,

$$f^{-1}[Y_1] \subset f^{-1}[Y_2] \quad \text{if} \quad Y_1 \subset Y_2.$$

(b) Show that f is injective if and only if for any subsets X_1, X_2 of A, $f[X_1] \subset f[X_2] \Rightarrow X_1 \subset X_2$.

(c) Show that f is surjective if and only if for any subsets Y_1, Y_2 of B, $f^{-1}[Y_1] \subset f^{-1}[Y_2] \Rightarrow Y_1 \subset Y_2$.

5. (a) For any non-negative number x and for any integer $k > 1$, prove that

$$x^k + k - 1 \geqslant kx \quad \text{.........................}(*)\ .$$

When does the equality hold ?

(b) Let n be an integer greater than 1, $\{a_1, a_2, \ldots, a_n\}$ be a set of positive real numbers. For $m = 1, 2, \ldots, n$, let

$$A_m = \frac{1}{m}\sum_{i=1}^{m} a_i\ ,$$

$$G_m = \left(\prod_{i=1}^{m} a_i\right)^{\frac{1}{m}}\ .$$

(i) Show that, for $m = 2, 3, \ldots, n$,

$$\left(\frac{G_m}{G_{m-1}}\right)^m = \frac{mA_m - (m-1)A_{m-1}}{G_{m-1}} \quad \text{.........................}(**)\ .$$

(ii) Making use of (*) and (**), or otherwise, prove that

$$A_m - G_m \geqslant \frac{m-1}{m}(A_{m-1} - G_{m-1})$$

for $m = 2, 3, \ldots, n$.

(iii) Deduce that $A_n \geqslant G_n$, and show that the equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

6. The polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, ... are defined by

$$P_0(x) = 1\ ,\ P_1(x) = x\ ,$$

$$P_r(x) = \frac{x(x-1)\ldots(x-r+1)}{r!},\ \text{when } r = 2, 3, \ldots\ .$$

(a) Show that $P_r(k)$ is an integer for $r \geqslant 0$ and for any integer k.
[Hint: For $r \geqslant 2$, consider the cases $r \leqslant k$, $0 \leqslant k < r$ and $k < 0$ and use the fact that the binomial coefficients C^p_q are integers.]

(b) Let $P(x) = \sum_{r=0}^{n} a_r P_r(x)$, where $a_0, a_1, \ldots, a_n$ are constants.
If $a_0, a_1, \ldots, a_{m-1}$ are integers but $a_m (0 < m \leqslant n)$ is not, show that $P(m)$ is not an integer.
Deduce that if $P(0)$, $P(1)$, ..., $P(n)$ are integers, then

(i) $a_0, a_1, \ldots, a_n$ are integers,

(ii) $P(k)$ is an integer for any integer k.

(c) Find a polynomial $Q(x) = ax^2 + bx + c$ such that $Q(k)$ is an integer for any integer k, but not all of a, b and c are integers.

7. A is a 3×3 matrix such that

$$A^2 - (k_1 + k_2)A + k_1k_2I = 0 ,$$

where k_1 and k_2 are distinct real numbers and I is the identity matrix of order 3.

M is the set of all 3×1 matrices and

$$M_1 = \{X \in M: \ AX = k_1X\},$$
$$M_2 = \{X \in M: \ AX = k_2X\}.$$

(a) Show that

(i) if $X, Y \in M_i$, then $aX + bY \in M_i$ for any real numbers a and b, where $i = 1$ or 2,

(ii) $M_1 \cap M_2 = \{0\}$.

(b) For any $X \in M$, show that

$$(A - k_2I)X \in M_1 \quad \text{and} \quad (A - k_1I)X \in M_2 .$$

(c) Using the above results, or otherwise, show that for any $X \in M$, there exist unique matrices $X_1 \in M_1$ and $X_2 \in M_2$ such that $X = X_1 + X_2$.

8. A function $g: \mathbf{R} \longrightarrow \mathbf{R}$ is said to be *additive* if

$$g(x + y) = g(x) + g(y) \quad \text{for any } x, y \in \mathbf{R}.$$

g is said to be *bounded* on a subset S of $\mathbf{R}$ if there is a number M such that

$$|g(x)| < M \quad \text{for any } x \in S.$$

(a) Let g be additive. Show that

$$g(nx) = ng(x) \quad \text{for any positive integer } n.$$

Deduce that if g is also bounded on $\mathbf{R}$, then $g(x) \equiv 0$.

(b) Suppose f is an additive function and is bounded on the interval $[0, a]$, where $a > 0$. Let $h: \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$h(x) = f(x) - \frac{f(a)}{a}x$$

for $x \in \mathbf{R}$.

(i) Show that h is additive and bounded on $[0, a]$.

(ii) Show that h is a periodic function of period a, i.e. $h(x + a) = h(x)$ for any $x \in \mathbf{R}$.

Hence deduce that h is bounded on $\mathbf{R}$.

(iii) Prove that $f(x) = \frac{f(a)}{a}x$ for any $x \in \mathbf{R}$.

9. (a) Let m be an integer greater than 1.

(i) By considering the coefficients of x^m in the expansion of $(1+x)^{m-p}(1+x)^m$ and $(1+x)^{2m-p}$, or otherwise, show that

$$\sum_{r=p}^{m} C_{r-p}^{m-p} C_r^m = C_m^{2m-p} \quad (p = 0, 1, 2, \ldots, m).$$

(ii) Making use of the equality $r \cdot C_r^m = m \cdot C_{r-1}^{m-1}$, or otherwise, show that

$$\sum_{r=0}^{m} r(C_r^m)^2 = m \cdot C_m^{2m-1}$$

and $$\sum_{r=0}^{m} r^2(C_r^m)^2 = m^2 \cdot C_{m-1}^{2m-2}.$$

(b) From a total of m white balls and m black balls $(m > 1)$, m balls are selected at random and put into a bag A. The remaining m balls are put into another bag B. A ball is then drawn at random from each bag. Find the probability that the two balls have the same colour. Use (a) to show that this probability is $\dfrac{m-1}{2m-1}$.

[Hint: First show that in the case when bag A contains r white balls, the probability that the two balls drawn have the same colour is $\dfrac{2r(m-r)}{m^2}$.]

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1985

純數學 試卷二

PURE MATHEMATICS PAPER II

2.00 pm–5.00 pm (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. Consider the ellipse

$$(E):\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b > 0)$$

and the circle

$$(K):\ x^2 + y^2 = a^2 .$$

Let $M(a\cos\theta\ ,\ a\sin\theta)$ and $N(a\cos\phi\ ,\ a\sin\phi)$ be two points on (K) corresponding to two distinct points $P(a\cos\theta\ ,\ b\sin\theta)$ and $Q(a\cos\phi\ ,\ b\sin\phi)$ on (E). The tangents to (E) at P and Q intersect at T and the tangents to (K) at M and N intersect at R.

(a) Find the coordinates of T and R in terms of θ and ϕ.

(b) Suppose P and Q move on (E) in such a way that T lies on the ellipse

$$(F):\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 .$$

(i) Show that $\cos(\theta - \phi) = 0$.

(ii) Show that R moves on a fixed circle and find its radius.

2. Let $\pi:\ Ax + By + Cz + D = 0$ be a plane, $P = (x_0\ ,\ y_0\ ,\ z_0)$ be a point, and

$$\ell: \begin{cases} x = a + pt \\ y = b + qt \\ z = c + rt \end{cases} \qquad t \in \mathbf{R}$$

be a line.

(a) If P does not lie on π, find the foot Q of the perpendicular drawn from P to π.

(b) Find the angle between π and ℓ.

(c) Show that ℓ lies on π if and only if

$$\begin{cases} Ap + Bq + Cr = 0 \\ Aa + Bb + Cc + D = 0 . \end{cases}$$

3. (a) For any positive integer n and for $t \in (-1\ ,\ 1)$, show that

$$\frac{1}{1+t} = 1 - t + t^2 - \ldots + (-1)^{n-1}\, t^{n-1} + \frac{(-1)^n\, t^n}{1+t} .$$

Hence deduce that for any $x \in (-1\ ,\ 1)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\,\frac{x^n}{n} + \int_0^x \frac{(-1)^n\, t^n}{1+t}\,dt$$

and $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \ldots - \dfrac{x^n}{n} - \displaystyle\int_0^x \frac{t^n}{1-t}\,dt$.

(b) Using (a), or otherwise, show that for any $x \in (0\ ,\ 1)$ and for any positive integer k,

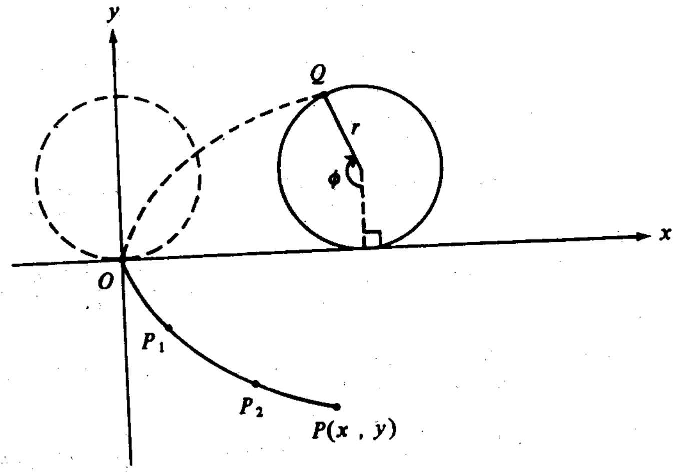
$$0 < \ln\left(\frac{1+x}{1-x}\right) - 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2k+1}}{2k+1}\right) < \frac{2}{1-x^2}\left(\frac{x^{2k+3}}{2k+3}\right).$$

(c) Using (b), or otherwise, show that

$$\lim_{k\to\infty}\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \ldots + \frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\right]$$

exists and find its value.

4.



(a) A circle of radius r rolls along the positive x-axis without slipping. A point Q on the circumference of the circle starts from the origin and reaches the position as shown in the diagram after the circle has rolled through an angle of ϕ, where $0 \leqslant \phi \leqslant 2\pi$. Let P be the mirror image of Q about the x-axis. Show that the locus of P is given by the parametric equations

$$\begin{cases} x = r(\phi - \sin\phi) \\ y = r(\cos\phi - 1), \quad 0 \leqslant \phi \leqslant 2\pi . \end{cases}$$

What is the value of ϕ when P reaches its lowest position P_L ?

(b) Let $y = f(x)$ be the function whose graph is the locus of P and let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the graph with parameters ϕ_1 and ϕ_2 respectively, where $0 \leqslant \phi_1 < \phi_2 \leqslant \pi$. Suppose a bead B slides smoothly along the curve $y = f(x)$ under gravity. It is known that the time T required for B to start at rest at P_1 and slide to P_2 is given by

$$T = \int_{x_1}^{x_2} \sqrt{\frac{1 + [f'(x)]^2}{2g[f(x_1) - f(x)]}}\, dx ,$$

where g is a positive constant. Show that the time for B to start at rest at any point $P(x, y)$ with parameter $\phi_s \in [0, \pi)$ and slide to P_L is independent of the starting position and find this time.

5. Let x_1 and x_2 be the roots of the quadratic equation $x^2 - 2tx + 1 = 0$, where $-1 \leqslant t \leqslant 1$. Define $F_n(t) = \frac{1}{2}(x_1^n + x_2^n)$ for $n = 0, 1, 2, 3, \ldots$.

(a) Show that for $n \geqslant 1$,

$$F_{n+1}(t) = 2tF_n(t) - F_{n-1}(t) .$$

Hence, or otherwise, deduce that $F_n(t)$ is a polynomial in t, of degree n and with leading coefficient 2^{n-1} .

(b) Using induction or otherwise, show that $F_n(t) = \cos[n\cos^{-1} t]$.

Hence show that

$$\int_0^{\pi} F_m(\cos\theta)\, F_n(\cos\theta)\, d\theta = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n = 0, \\ \frac{\pi}{2} & \text{if } m = n > 0 . \end{cases}$$

6. Let $f_0 : \mathbf{R} \to \mathbf{R}$ be a continuous function.

Define $f_n(x) = \int_0^x f_{n-1}(t)\,dt$ for $x \geq 0$ and $n \geq 1$.

(a) If m and n are positive integers, show that

$$\int_0^x (x-t)^{m-1} f_n(t)\,dt = \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)\,dt \quad (x \geq 0).$$

(b) Using (a), or otherwise, show that

$$f_n(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f_0(t)\,dt \quad (x \geq 0)$$

for $n = 1, 2, 3, \ldots$.

(c) Assume $|f_0(t)| \leq M$ for $0 \leq t \leq 1$, where M is a positive constant. If $0 \leq x \leq 1$, show that

$$|f_n(x)| \leq \frac{M}{n!}.$$

Hence evaluate $\lim_{n\to\infty} f_n(x)$.

7. Let a, b and c be three given real numbers such that $a < b < c$. For any $x \in \mathbf{R}$, define

(a) $f(x) = |x-a|$,

(b) $g(x) = |x-a| + |x-b|$,

(c) $h(x) = |x-a| + |x-b| + |x-c|$.

For each of the above continuous functions,

(i) find its derivative wherever it exists and indicate where it does not,

(ii) determine the intervals in which the function is strictly increasing or strictly decreasing and hence find the minimum points,

(iii) sketch its graph.

[Note: Answers to (i) and (ii) may be given in table form]

8. Let r be a real number such that $0 < r < \frac{\pi}{4}$.

(a) Show that $\int_{\frac{\pi}{2}}^{\pi-2r} \ln\sin x\,dx = \int_{2r}^{\frac{\pi}{2}} \ln\sin x\,dx$.

Hence show that $\int_r^{\frac{\pi}{2}-r} \ln\sin 2x\,dx = \int_{2r}^{\frac{\pi}{2}} \ln\sin x\,dx$,

and deduce that

$$\int_r^{\frac{\pi}{2}-r} (\ln\sin x + \ln\cos x)\,dx = \int_{2r}^{\frac{\pi}{2}} \ln\sin x\,dx - \left(\frac{\pi}{2} - 2r\right)\ln 2 \;.........(*)$$

(b) Assume that $\lim_{r\to 0}\int_r^{\frac{\pi}{2}-r} \ln\sin x\,dx$ and $\lim_{r\to 0}\int_{2r}^{\frac{\pi}{2}} \ln\sin x\,dx$ exist and are both equal to A. Show that

$$\int_r^{\frac{\pi}{2}-r} \ln\sin x\,dx = \int_r^{\frac{\pi}{2}-r} \ln\cos x\,dx.$$

Hence use (*) to find the value of A.

9. (a) Let $f(x)$ and $g(x)$ be two functions continuous on the interval $[a, b]$. By considering the integral of the function $[\lambda f(x) + g(x)]^2$ on $[a, b]$, set up a quadratic inequality in the parameter λ. Hence show that

$$\left(\int_a^b f(x)\, g(x)\, dx\right)^2 \leq \left(\int_a^b [f(x)]^2\, dx\right)\left(\int_a^b [g(x)]^2\, dx\right).$$

(b) Let $f(x)$ be a non-constant function with continuous derivative on $[0, 1]$ satisfying $f(0) = 0$ and $f(1) = 0$.

(i) Show that

$$f(x) = \int_0^x f'(t)\, dt = -\int_x^1 f'(t)\, dt$$

for any $x \in [0, 1]$.

(ii) Use (i) and (a) to show that

$$[f(x)]^2 \leq x \int_0^{\frac{1}{2}} [f'(t)]^2\, dt \quad \text{if } x \in [0, \tfrac{1}{2}]$$

and $$[f(x)]^2 \leq (1 - x) \int_{\frac{1}{2}}^1 [f'(t)]^2\, dt \quad \text{if } x \in [\tfrac{1}{2}, 1].$$

(iii) Use (ii) to show that $\int_0^1 [f(x)]^2\, dx \leq \frac{1}{8} \int_0^1 [f'(x)]^2\, dx$.

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1986

純數學 試卷一

PURE MATHEMATICS PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.