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一九八四年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1984

PURE MATHEMATICS (PAPER I)

MARKING SCHEME

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SOLUTION

MARK

(a) $f(x) = \begin{vmatrix} a-x & 0 & 1 \\ 0 & b-x & 0 \\ 1 & 0 & c-x \end{vmatrix}$
 $= -x^3 + (a+b+c)x^2 - (ab+bc+ca-1)x + (abc-b)$
 $= -x^3 - (ab+bc+ca-1)x + (abc-b)$ as $a+b+c = 0$

$A^3 = \begin{pmatrix} a^2+1 & 0 & a+c \\ 0 & b^2 & 0 \\ a+c & 0 & c^2+1 \end{pmatrix} \begin{pmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & c \end{pmatrix}$

$= \begin{pmatrix} a^3+2a+c & 0 & a^2+ac+c^2+1 \\ 0 & b^3 & 0 \\ a^2+ac+c^2+1 & 0 & a+2c+c^3 \end{pmatrix}$

$\therefore f(A) = -A^3 - (ab+bc+ca-1)A + (abc-b)I$

$= \begin{pmatrix} -a^3-a-b-c-a^2b-a^2c & 0 & -a^2-2ac-c^2-ab-bc \\ 0 & -b^3-ab^2-b^2c & 0 \\ -a^2-2ac-c^2-ab-bc & 0 & -a-c-c^3-bc^2-ac^2-b \end{pmatrix}$

$= \begin{pmatrix} -a^2(a+b+c) - (a+b+c) & 0 & -a(a+c+b) - c(a+c+b) \\ 0 & -b^2(b+a+c) & 0 \\ -a(a+c+b) - c(a+c+b) & 0 & -(a+c+b) - c^2(c+b+a) \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Since $-A^3 - (ab+bc+ca-1)A + (abc-b)I = 0$
 $A^3 = \lambda A + \mu I,$

where $\lambda = -(ab+bc+ca-1)$
 $\mu = abc-b$

For $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda = 2, \mu = 1.$

$A^9 = (\lambda A + \mu I)^3$
 $= \lambda^3 A^3 + 3\lambda^2 \mu A^2 + 3\lambda \mu^2 A + \mu^3 I$
 $= \lambda^3 (\lambda A + \mu I) + 3\lambda^2 \mu A^2 + 3\lambda \mu^2 A + \mu^3 I$
 $= 3\lambda^2 \mu A^2 + (\lambda^4 + 3\lambda \mu^2) A + (\lambda^3 \mu + \mu^3) I$

$\therefore A^9 = 12A^2 + 22A + 9I$

$= \begin{pmatrix} 55 & 0 & 34 \\ 0 & -1 & 0 \\ 34 & 0 & 21 \end{pmatrix}$

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} may sub. A either of these steps

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-1 for each wrong entr.

7

SOLUTION

MARK

REMARK

2. (a) For any $\alpha \in \mathbb{R}, u, v \in \mathbb{R}^3,$
 $g(\alpha u) = \alpha \cdot \frac{u}{\|u\|}$
 $= \alpha \frac{u}{\|u\|}$
 $= \alpha g(u)$

$g(u+v) = \frac{u+v}{\|u+v\|}$
 $= \frac{u}{\|u\|} + \frac{v}{\|v\|}$
 $= g(u) + g(v)$

$\therefore g$ is linear.

(b) Since h is linear, for $u = \alpha x + \beta y + \gamma z,$

$h(u) = h(\alpha x + \beta y + \gamma z)$
 $= h(\alpha x) + h(\beta y) + h(\gamma z)$
 $= \alpha h(x) + \beta h(y) + \gamma h(z)$

Let $\{e_1, e_2, e_3\}$ be the usual base of $\mathbb{R}^3.$ For any $u = (u_1, u_2, u_3) \in \mathbb{R}^3, u = u_1 e_1 + u_2 e_2 + u_3 e_3.$

Since h is linear, $h(u) = u_1 h(e_1) + u_2 h(e_2) + u_3 h(e_3).$

Define $\underline{b} = (h(e_1), h(e_2), h(e_3)),$ then

$h(u) = \underline{b} \cdot u$ for any $u \in \mathbb{R}^3.$

(c) Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a linear function. By (b), there exists $\underline{b} = (f(e_1), f(e_2), f(e_3)) \in \mathbb{R}^3$ s.t. $f(u) = \underline{b} \cdot u \forall u \in \mathbb{R}^3.$

Consider the set $H = \{u \in \mathbb{R}^3 : f(u) = 0\}$

$= \{u \in \mathbb{R}^3 : f(e_1)u_1 + f(e_2)u_2 + f(e_3)u_3 = 0\}$

Since f is not identically zero, $\underline{b} \neq 0$

H is therefore a plane through the origin.

Conversely if H is a plane through the origin, equation of H can be written as $Au_1 + Bu_2 + Cu_3 = 0,$ where A, B, C are not all zero.

Define $\underline{a} = (A, B, C),$ then by (a), the non-zero function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$f(u) = \underline{a} \cdot u \forall u = (u_1, u_2, u_3) \in \mathbb{R}^3$ is linear.

Further $H = \{u \in \mathbb{R}^3 : f(u) = 0\}$

2+1

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2

2

\underline{u} as a lin. comb. of bas. vectors.

3

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3

1

1

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7

SOLUTION

(a) When n is even,

$$A_n = (n-3) + (n-5) + \dots + 1$$

$$= \frac{1}{2} \left(\frac{n-2}{2} \right) (n-3+1)$$

$$= \left(\frac{n-2}{2} \right)^2$$

when n is odd,

$$A_n = (n-3) + (n-1) + \dots + 2$$

$$= \frac{1}{2} \left(\frac{n-5}{2} + 1 \right) (n-3+2)$$

$$= \left(\frac{n-1}{2} \right) \left(\frac{n-3}{2} \right)$$

(b) (1) To form a non-degenerate triangle with the longest side n, we have $y < x < t = n$, $x + y > t$. The number of such triangles is equal to the number of integral points in (a), i.e. A_n .

(ii) The number of possible triangles formed is

$$B_{2k} = \sum_{i=4}^{2k} A_i$$

$$= \sum_{i=2}^k A_{2i} + \sum_{i=2}^{k-1} A_{2i+1}$$

$$= \sum_{i=2}^k (i-1)^2 + \sum_{i=2}^{k-1} i(i-1)$$

$$= \sum_{i=1}^{k-1} i^2 + \sum_{i=2}^{k-1} i^2 - \sum_{i=2}^{k-1} i$$

$$= 2 \times \frac{1}{6} (k-1)(k)(2k-1) - 1 - \frac{(k-2)(k+1)}{2}$$

$$= \frac{k(k-1)(4k-5)}{6}$$

(c) $p(2k) = \frac{B_{2k}}{C_1^{2k}}$

$$= \frac{\frac{1}{6} k(k-1)(4k-5)}{\frac{(2k)(2k-1)(2k-2)}{3 \cdot 2}}$$

$$= \frac{4k-5}{4(2k-1)}$$

$\therefore \lim_{k \rightarrow \infty} p(2k) = \lim_{k \rightarrow \infty} \frac{4k-5}{4(2k-1)} = \frac{1}{2}$

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SOLUTION

(a) $z^m = 1$ iff $(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n})^m = 1$

iff $\cos \frac{2\pi m}{n} + i \sin \frac{2\pi m}{n} = 1$

iff $n \mid m$

(i) If $n \mid m$, $z^m = 1$.

$$\sum_{r=0}^{n-1} z^{mr} = \sum_{r=0}^{n-1} 1 = n$$

(ii) If $n \nmid m$, $\sum_{r=0}^{n-1} z^{mr} = \frac{1-z^{mn}}{1-z^m} = 0$ as $z^{mn} = 1$

(b) $\sum_{r=0}^{n-1} f(z^r) z^{(n-j)r} = \sum_{r=0}^{n-1} \left[\sum_{k=0}^{n-1} a_k (z^r)^k \right] z^{(n-j)r}$

$$= \sum_{r=0}^{n-1} \sum_{k=0}^{n-1} a_k z^{(n+k-j)r}$$

$$= \sum_{k=0}^{n-1} \left[a_k \sum_{r=0}^{n-1} z^{(n+k-j)r} \right]$$

* As $0 \leq k, j \leq n-1$, $0 < n+k-j < 2n$.

$$\frac{n}{n+k-j} \text{ iff } k=j$$

By (a), $\sum_{r=0}^{n-1} z^{(n+k-j)r} = \begin{cases} n & \text{if } k=j \\ 0 & \text{if } k \neq j \end{cases}$

$\therefore \sum_{r=0}^{n-1} f(z^r) z^{(n-j)r} = na_j$

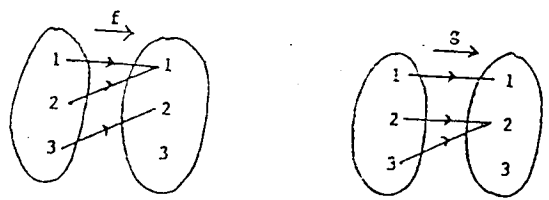
MARK	REMARK
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SOLUTION

	MARK	REMARK
4. (c) Let $f(x) = \sum_{j=0}^{n-1} a_j x^j$.		
By (b), $a_j = \frac{1}{n} \sum_{r=0}^{n-1} f(z^r) z^{(n-j)r}$		
$\therefore f(x) = \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{r=0}^{n-1} f(z^r) z^{(n-j)r} \right] x^j$	1	
$= \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{r=0}^{n-1} (z^{nr} - (z^{nr-1})h(z^r)) z^{(n-j)r} \right] x^j$	2	
$= \frac{1}{n} \sum_{j=0}^{n-1} \left[\sum_{r=0}^{n-1} z^{nr} z^{(n-j)r} \right] x^j$ since $z^{nr-1} = 0$.	1	
	4	

5. (a) (i) For any $x, y, z \in A$,		
1. $f(x) = f(x)$ and $g(x) = g(x)$		
$\therefore xRx$.	1	
2. If xRy , then $f(x) = f(y)$ and $g(x) = g(y)$		
$\therefore f(y) = f(x)$, $g(y) = g(x)$ and yRx .	1	
3. If xRy and yRz , then $f(x) = f(y)$, $g(x) = g(y)$		
and $f(y) = f(z)$, $g(y) = g(z)$		
$\therefore f(x) = f(z)$ and $g(x) = g(z)$		
and xRz .	1	
Thus R is an equivalence relation.		

(11) Let $A = B = \{1, 2, 3\}$. The following example of f and g does not define an equivalence relation S :
Here $1S2$ and $2S3$ but $1 \not S 3$.



SOLUTION

	MARK	REMARK
5. (b) For any $a/R \in A/R$, define $h(a/R) = f(a)$.	2	
$a/R = a'/R \Rightarrow aRa' \Rightarrow f(a) = f(a')$ $\therefore h$ is a well-defined mapping.	1	
Furthermore, for any $a \in A$, $h \circ u(a) = h(a/R)$ $= f(a)$	1	
$\therefore h \circ u = f$.	1	
Let $h' : A/R \rightarrow B$ be any mapping such that $h' \circ u = f$. Since u is surjective, any element, e.g. a/R , of A/R is the image of some element, e.g. a , in A .		
$\therefore h(a/R) = h' \circ u(a)$ $= h' \circ u(a)$ $= h'(a/R)$ for any $a/R \in A/R$.		
Hence $h = h'$, i.e. h is unique.	3	
[If g is a constant mapping and if f is surjective, obviously h must be surjective. Furthermore, let $a/R, a'/R \in A/R$ such that $h(a/R) = h(a'/R)$.	1	
Then $h \circ u(a) = h \circ u(a')$ $\Rightarrow f(a) = f(a')$.		
As $g(a) = g(a')$, $aRa' \Rightarrow a/R = a'/R$.	2	
$\therefore h$ is bijective	10	

(a) (i) $\bar{\alpha}\beta = 1 \Rightarrow \bar{\alpha}\beta\bar{\alpha}\bar{\beta} = 1$ $\Rightarrow \alpha = \frac{1}{ \beta }$		
As $ \alpha , \beta \leq 1$, the above is possible only if $ \alpha = \beta = 1$.		
$\therefore \bar{\alpha}\bar{\alpha} = 1$ and $\bar{\alpha}\beta = 1 \Rightarrow \bar{\alpha}\bar{\beta} = 1$ $\Rightarrow \alpha = \beta$		
(ii) By (i), $1 - \bar{\alpha}\beta \neq 0$.		
$\frac{ \alpha - \beta }{ 1 - \bar{\alpha}\beta } \leq 1 \Leftrightarrow \alpha - \beta \leq 1 - \bar{\alpha}\beta $ $\Leftrightarrow (\alpha - \beta)(\bar{\alpha} - \bar{\beta}) \leq (1 - \bar{\alpha}\beta)(1 - \alpha\bar{\beta})$ $\Leftrightarrow \alpha\bar{\alpha} + \beta\bar{\beta} - \alpha\bar{\beta} - \bar{\alpha}\alpha \leq 1 + \alpha\bar{\alpha}\beta\bar{\beta} - \alpha\bar{\beta} - \bar{\alpha}\alpha$ $\Leftrightarrow (1 - \alpha\bar{\alpha})(1 - \beta\bar{\beta}) \geq 0$		
which is true as $ \alpha , \beta \leq 1$.	3	
If $ \alpha < 1$, the equality holds iff $1 - \beta\bar{\beta} = 0$, i.e. $ \beta = 1$.	1	
	7	

(b) (i) $|f(1)| = |f(-1)| = 1$

$\Rightarrow \left| \frac{1-a}{b-1} \right| = 1$ and $\left| \frac{-1-a}{-b-1} \right| = 1$

$\Rightarrow |a-1| = |b-1|$ and $|a+1| = |b+1|$

$\therefore a, b$ are equidistant from 1 and from -1.

Hence $b = a$ or \bar{a}

Alternatively:

$|f(1)| = |f(-1)| = 1$

$\Rightarrow \frac{a\bar{a} - a - \bar{a} + 1}{a\bar{a} + a + \bar{a} + 1} = \frac{b\bar{b} - b - \bar{b} + 1}{b\bar{b} + b + \bar{b} + 1}$

$\therefore \frac{a + \bar{a}}{a\bar{a}} = \frac{b + \bar{b}}{b\bar{b}}$

i.e. $\operatorname{Re}(a) = \operatorname{Re}(b)$
and $|a| = |b|$
Hence $b = a$ or \bar{a}

$|f(i)| = 1 \Rightarrow \left| \frac{i-a}{ib-1} \right| = 1$
 $\Rightarrow |i-a| = |i+b|$

Together with the above result, this implies $b = \bar{a}$

Writing $f(z) = \frac{z-a}{az-1}$

If $|z| = 1$,

$|f(z)|^2 = \frac{(z-a)(\bar{z}-\bar{a})}{(az-1)(a\bar{z}-1)}$
 $= \frac{z\bar{z} + a\bar{a} - a\bar{z} - \bar{a}z}{a\bar{a}z\bar{z} - a\bar{z} - \bar{a}z + 1}$
 $= \frac{1 + a\bar{a} - a\bar{z} - \bar{a}z}{a\bar{a} - a\bar{z} - \bar{a}z + 1}$
 $= 1$

(ii) If $|a| = 1$,

$f(z) = \frac{z-a}{az-1}$
 $= \frac{1}{a} \left(\frac{z-a}{z-a} \right)$
 $= \frac{1}{a} (=a)$ which is constant.

(a) $(x_1 \ x_2 \ x_3) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \ a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \ a_{13}x_1 + a_{23}x_2 + a_{33}x_3)$

$\therefore x'_1 + x'_2 + x'_3 = (a_{11}+a_{21}+a_{31})x_1 + (a_{21}+a_{22}+a_{23})x_2 + (a_{31}+a_{32}+a_{33})x_3$
 $= x_1 + x_2 + x_3 \ (\because A \in \mathcal{D} \Rightarrow \sum_{j=1}^3 a_{ij} = 1)$

Similarly $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \end{pmatrix}$
 $y'_1 + y'_2 + y'_3 = (a_{11}+a_{21}+a_{31})y_1 + (a_{12}+a_{22}+a_{32})y_2 + (a_{13}+a_{23}+a_{33})y_3$
 $= y_1 + y_2 + y_3 \ (\because \sum_{i=1}^3 a_{ij} = 1)$

(b) Let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in \mathcal{D}$

(i) The ij -entry of $JB = \frac{1}{3} \sum_{i=1}^3 b_{ij}$
 $= \frac{1}{3}$

The ij -entry of $BJ = \frac{1}{3} \sum_{j=1}^3 b_{ij}$
 $= \frac{1}{3}$

$\therefore JB = J = BJ \ \forall B \in \mathcal{D}$

(ii) By (i) $BJ = J \ \forall B \in \mathcal{D}$

$\therefore J \in S(B) \Rightarrow S(B) \neq \emptyset$

(iii) By (i) $JB = J \ \forall B \in \mathcal{D}$

$\therefore \mathcal{D} \subset S(J)$
 $\subset \mathcal{D}$ (By definition)
 $\Rightarrow S(J) = \mathcal{D}$

SOLUTION

(c) If A is invertible, $A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Let $(x_1 \ x_2 \ x_3)$ be the first row of A^{-1} . By (a),
 $x_1 + x_2 + x_3 = 1 + 0 + 0 = 1$.

Similarly for other rows and columns. $\therefore A^{-1} \in \mathcal{D}$

$X \in S(A) \Rightarrow AX = J$
 $\Rightarrow X = A^{-1}J$
 $= J$ By (b)(i)

Since $J \in S(A)$ by (b)(ii), $\therefore S(A) = \{J\}$

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(d) If A is singular, the system of equations

$A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has a non-zero solution $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

with $b_1 + b_2 + b_3 = 0$ by (a)

Let $C = \begin{pmatrix} b_1 & b_1 & -2b_1 \\ b_2 & b_2 & -2b_2 \\ b_3 & b_3 & -2b_3 \end{pmatrix}$, then $C \neq 0$ but C has zero row and column sums and $AC = 0$.

Further $J + C \in \mathcal{D}$ and $A(J + C) = AJ + AC$
 $= J \Rightarrow J + C \in S(A)$
 $\therefore S(A) \neq \{J\}$

SOLUTION

(a) Since a non-zero polynomial has only a finite number of zeroes, $g(n) = a_n f(n) = 0$ and hence $a_n = 0$ for only a finite number of n .

Assume, for contradiction, that $\deg f(x) > \deg g(x)$.

Then if n is sufficiently large, $g(n), a_n, f(n) \neq 0$ and $|f(n)| > |g(n)|$.

This contradicts the fact the $g(n) = a_n f(n)$.
Hence $\deg f(x) \leq \deg g(x)$.

(b) As $\deg f(x) \leq \deg g(x)$, by the Euclidean algorithm, there exist polynomials $h(x)$ and $r(x)$ with $\deg r(x) < \deg f(x)$ such that $g(x) = h(x)f(x) + r(x)$.

It is obvious that the coefficients of $h(x)$ and $r(x)$ are rational.

Since for any $n \in \mathbb{N}$, $\exists a_n \in \mathbb{Z}$ such that $g(n) = a_n f(n)$,

$r(n) = a_n f(n) - h(n)f(n)$
 $= [a_n - h(n)]f(n) \dots \dots \dots (*)$

Let M be a common multiple of the denominators of coefficients of $r(x)$ and $h(x)$, then the polynomials $Mr(x)$ and $Mh(x)$ have integral coefficients.

From (*), for any $n \in \mathbb{N}$, $\exists b_n = [Ma_n - Mh(n)] \in \mathbb{Z}$ such that $Mr(n) = [Ma_n - Mh(n)]f(n)$

By (a), $\deg f(x) \leq \deg r(x)$ which contradicts the definition of $r(x)$ unless $r(x) \equiv 0$, i.e. $g(x) = f(x)h(x)$

(c) If $\deg f(x) = \deg g(x)$, since $g(x) = f(x)h(x)$, $h(x)$ is constant.

By (a), there is an n such that $g(n) = a_n f(n)$ and $a_n, g(n), f(n)$ are non-zero integers.

Then $h(n) = \frac{g(n)}{f(n)}$
 $= a_n$

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PURE MATHEMATICS (PAPER II)
MARKING SCHEME

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SOLUTION

$$\begin{aligned}
 (a) \quad u_{k+2} &= \int_0^{\pi} \frac{\sin(k+2)x}{\sin x} dx \\
 &= \int_0^{\pi} \frac{\sin kx \cos 2x + \cos kx \sin 2x}{\sin x} dx \\
 &= \int_0^{\pi} \frac{\sin kx (1 - 2\sin^2 x) + 2\cos kx \sin x \cos x}{\sin x} dx \\
 &= \int_0^{\pi} \frac{\sin kx}{\sin x} dx + \int_0^{\pi} (-2\sin kx \sin x + 2\cos kx \cos x) dx \\
 &= u_k + 2 \int_0^{\pi} \cos(k+1)x dx \\
 &= u_k + \frac{2}{k+1} [\sin(k+1)x]_0^{\pi} \\
 &= u_k
 \end{aligned}$$

$$\begin{aligned}
 \therefore u_k &= u_{k-2} \\
 &= \text{etc}
 \end{aligned}$$

$$= \begin{cases} u_0 & \text{if } k \text{ is even} \\ u_1 & \text{if } k \text{ is odd.} \end{cases}$$

$$= \begin{cases} 0 & \text{if } k \text{ is even} \\ \pi & \text{if } k \text{ is odd.} \end{cases}$$

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SOLUTION

MARK

MARK

(b) (i) Put $u = \cos^{m-1} \theta$, $dv = \cos \theta \sin^n \theta d\theta$,

then $du = -(m-1)\cos^{m-2} \theta \cdot \sin \theta d\theta$, $v = \frac{1}{n+1} \sin^{n+1} \theta$

$$I(m, n) = \left[\frac{1}{n+1} \cos^{m-1} \theta \sin^{n+1} \theta \right]_0^{\frac{\pi}{2}} + \frac{m-1}{n+1} \int_0^{\frac{\pi}{2}} \cos^{m-2} \theta \sin^{n+2} \theta d\theta$$

$$= \left(\frac{m-1}{n+1} \right) I(m-2, n+2), \quad m \geq 2$$

(ii) For $n \geq 0$, $I(1, n) = \int_0^{\frac{\pi}{2}} \cos \theta \sin^n \theta d\theta$

$$= \frac{1}{n+1} \sin^{n+1} \theta \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{n+1}$$

(iii) Put $u = \sin^{n-1} \theta$, $dv = \sin \theta d\theta$,

then $du = (n-1)\sin^{n-2} \theta \cos \theta d\theta$, $v = -\cos \theta$.

For $n \geq 2$, $I(0, n) = \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta$

$$= \left[-\sin^{n-1} \theta \cos \theta \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \cos^2 \theta d\theta$$

$$= (n-1) \left[\int_0^{\frac{\pi}{2}} \sin^{n-2} \theta d\theta - \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta \right]$$

$$\therefore I(0, n) = \left(\frac{n-1}{n} \right) I(0, n-2).$$

(iv) $I(6, 4) = \frac{5}{5} I(4, 6)$ by (i)

$$= \frac{5 \cdot 3 \cdot 1}{5 \cdot 7 \cdot 9} I(0, 10)$$

$$= \frac{5 \cdot 3 \cdot 1}{5 \cdot 7 \cdot 9} \cdot \frac{9}{10} I(0, 8) \quad \text{by (iii)}$$

$$= \frac{5 \cdot 3 \cdot 1}{5 \cdot 7 \cdot 9} \cdot \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} I(0, 0)$$

$$= \frac{3\pi}{512}$$

RESTRICTED 内部文件

1

11

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II

SOLUTION

MARK

REMARK

(a) Integrating by parts,

Let $u = \frac{h^n - x^n}{n}$, $dv = e^{-x} dx$

$$du = -\frac{(n-1)h^{n-1} - nx^{n-1}}{n} dx, \quad v = -e^{-x}$$

$$I_n = \frac{1}{(n-1)!} \int_0^h (h-x)^{n-1} e^{-x} dx - \frac{1}{(n-1)!} \left[(h-x)^{n-1} e^{-x} \right]_0^h$$

$$= \frac{1}{(n-1)!} \int_0^h (h-x)^{n-2} e^{-x} dx - \frac{1}{(n-1)!} \left[(h-x)^{n-2} e^{-x} \right]_0^h$$

$$I_1 = \int_0^h e^{-x} dx = e^{-0} - e^{-h}$$

$$I_2 = I_1 - h e^{-h}$$

$$= e^{-0} - e^{-h} - h e^{-h}$$

$$I_n = I_{n-1} - \frac{h^{n-1}}{(n-1)!} e^{-h}$$

$$= I_{n-2} - \frac{h^{n-2}}{(n-2)!} e^{-h} - \frac{h^{n-1}}{(n-1)!} e^{-h}$$

= etc.

$$= I_1 - h e^{-h} - \frac{h^2}{2!} e^{-h} - \dots - \frac{h^{n-1}}{(n-1)!} e^{-h}$$

Subst. $I_n = e^{-h} - e^{-h}$, we have

$$e^{-h} = e^{-0} + h e^{-0} + \frac{h^2}{2!} e^{-0} + \dots + \frac{h^{n-1}}{(n-1)!} e^{-0} + R_n$$

3

6

SOLUTION

2. (c) Putting $f(x) = \ln(1+x)$ which is infinitely differentiable on $(-1, 1)$,

$$f'(x) = \frac{1}{1+x}$$

$$f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{(1+x)^n}$$

By (b), for any $0 < h < 1$, $\ln(1+h) = \ln 1 + h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + R_5$

$$\text{Since } R_5 = \frac{1}{4!} \int_0^h (h-t)^4 \cdot \frac{(-1)^4 (4!)}{(1+t)^5} dt$$

> 0 as $(h-t)^4, (1+t)^5 > 0$ for $t \in (0, h)$.

$$\therefore \ln(1+h) - h + \frac{h^2}{2} - \frac{h^3}{3} + \frac{h^4}{4} = R_5 > 0$$

$$\text{Similarly, } R_6 = \frac{1}{5!} \int_0^h \frac{(h-t)^5 (-1)^5 (5!)}{(1+t)^6} dt$$

< 0

$$\therefore \ln(1+h) - h + \frac{h^2}{2} - \frac{h^3}{3} + \frac{h^4}{4} \approx R_6 + \frac{h^5}{5} > \frac{h^5}{5}$$

MARK
REMARK

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7

SOLUTION

(a) Since $a\theta = b\phi$,

if $P = (x, y)$,

$$x = (a-b)\cos\theta + b\sin\left[\frac{\pi}{2} - \theta - \left(\frac{\pi}{2} - \theta\right)\right]$$

$$= (a-b)\cos\theta + b\sin\left[\frac{\pi}{2} - (\theta - \theta)\right]$$

$$= (a-b)\cos\theta + b\cos(\theta - \theta)$$

$$= (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

$$y = (a-b)\sin\theta - b\cos\left(\frac{\pi}{2} - (\theta - \theta)\right)$$

$$= (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

(b) If $b = \frac{a}{4}$,

$$x = \frac{3}{4}a\cos\theta + \frac{a}{4}a\cos 3\theta$$

$$= \frac{3}{4}a\cos\theta + \frac{a}{4}[4\cos^3\theta - 3\cos\theta]$$

$$= a\cos^3\theta$$

$$y = \frac{3}{4}a\sin\theta - \frac{a}{4}a\sin 3\theta$$

$$= \frac{3}{4}a\sin\theta - \frac{a}{4}[3\sin\theta - 4\sin^3\theta]$$

$$= a\sin^3\theta$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\cos^2\theta + a^{\frac{2}{3}}\sin^2\theta$$

$$= a^{\frac{2}{3}}$$

(c) Length of hypocycloid

$$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2} d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} 3a\sin\theta\cos\theta d\theta$$

$$= 6a \left[\sin^2\theta \right]_0^{\frac{\pi}{2}}$$

$$= 6a$$

MARK

REMARK

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SOLUTION

(a) Equation of tangent at (x', y') is $\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1$.

Comparing coefficients with $lx + my = 1$, $l = \frac{x'}{a^2}$, $m = \frac{y'}{b^2}$

$\therefore lx + my = 1$ is tangent to (E) iff

$$\begin{aligned} a^2l^2 + b^2m^2 &= a^2\frac{x'^2}{a^4} + b^2\frac{y'^2}{b^4} \\ &= \frac{x'^2}{a^2} + \frac{y'^2}{b^2} \\ &= 1 \end{aligned}$$

since (x', y') lies on (E).

Alternatively

Let $l \neq 0$, substituting $x = \frac{1-my}{l}$ in (E),

$$\frac{(1-my)^2}{a^2l^2} + \frac{y^2}{b^2} = 1$$

$$\text{or } (a^2l^2 + b^2m^2)y^2 - 2mb^2y + (b^2 - a^2b^2l^2) = 0$$

$$lx+my = 1 \text{ is tangent iff } 4m^2b^4 - 4(a^2l^2 + b^2m^2)(b^2 - a^2b^2l^2) = 0$$

$$\text{iff } a^2l^2(a^2l^2 + b^2m^2 - 1) = 0$$

$$\text{iff } a^2l^2 + b^2m^2 = 1$$

It is noted that this still holds if $l = 0$.

b) Let $lx + my = 1$ be a common tangent to (E) and (F).

$$\text{By (a) } a^2l^2 + b^2m^2 = 1 \text{ and}$$

$$b^2l^2 + a^2m^2 = 1$$

$$\therefore l^2 = m^2 = \frac{1}{a^2 + b^2}$$

Equations of common tangents to (E) and (F) are

$$x \pm y = \pm \sqrt{a^2 + b^2}$$

MARK
REMARK

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SOLUTION

4. (c) Equation of tangent at $S(s_1, s_2)$ is

$$\frac{s_1x}{a^2} + \frac{s_2y}{b^2} = 1$$

If $R(h, k)$ lies on the tangent,

$$\frac{s_1h}{a^2} + \frac{s_2k}{b^2} = 1$$

Similarly, for the tangent at $T(t_1, t_2)$, we have

$$\frac{t_1h}{a^2} + \frac{t_2k}{b^2} = 1$$

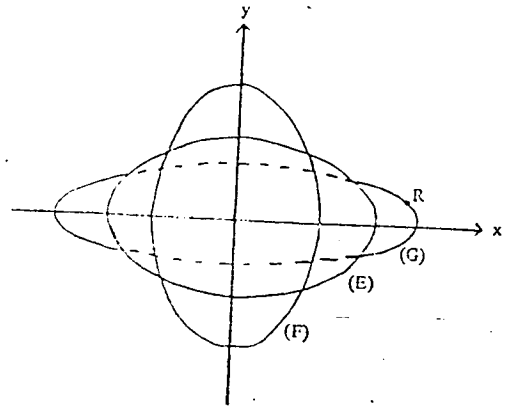
\therefore the straight line $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ passes through S and T.

(d) By (a), the chord ST is tangent to (F) iff

$$\frac{h^2b^2}{a^4} + \frac{k^2a^2}{b^4} = 1, \text{ where } (h, k) \text{ lies outside (E)}$$

\therefore the locus of R consists of the parts of the ellipse

$$(G): \frac{x^2}{(\frac{a^2}{b})^2} + \frac{y^2}{(\frac{b^2}{a})^2} = 1 \text{ which lie outside (E).}$$



(G), locus of R

MARK
REMARK

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SOLUTION

5. (a) If $\frac{a}{N+1} \geq 0$, let N be an integer such that $N > a$.
 $\frac{a}{(N+1)!} = \frac{a}{N+1} \cdot \frac{1}{N!} < \frac{N}{N+1} \cdot \frac{1}{N!}$
 $\frac{a^{N+2}}{(N+2)!} = \frac{a}{(N+2)} \cdot \frac{a}{(N+1)} \cdot \frac{a^N}{N!} < \left(\frac{N}{N+1}\right)^2 \frac{a^N}{N!}$, etc.
 $0 < \frac{a^n}{n!} < \left(\frac{N}{N+1}\right)^{n-N} \frac{a^N}{N!}$ for every $n > N$.

$\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$ as $0 < \frac{N}{N+1} < 1$.
 If $a < 0$, the same result follows from the inequalities
 $-\frac{|a|^n}{n!} \leq \frac{a^n}{n!} \leq \frac{|a|^n}{n!}$

(b) (i) $f(1-x) = (1-x)^n (1-(1-x))^n$
 $= (1-x)^n x^n$
 $= f(x) \quad \forall x \in \mathbb{R}$
 $f'(x) = -f'(1-x)$
 $f''(x) = f''(1-x)$
 etc
 $f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$

(ii) $f(x) = x^n \sum_{i=0}^n C_i^n (-1)^i x^i$
 $= C_0^n x^n - C_1^n x^{n+1} + \dots + C_n^n (-1)^n x^{2n}$

For $k > 2n$, $f^{(k)}(x) = 0$
 For $0 \leq k < n$, $f^{(k)}(0) = 0$
 For $n \leq k \leq 2n$, $f^{(k)}(x) = \sum_{i=0}^n C_i^n (-1)^i \frac{d^k}{dx^k} (x^{n+i})$
 At $x = 0$, $f^{(k)}(0) = C_{k-n}^n (-1)^{k-n} \frac{d^k}{dx^k} (x^k)$
 $= (-1)^{k-n} C_{k-n}^n \cdot k!$
 $= (-1)^{k-n} C_{2n-k}^n \cdot k!$

Now $\frac{k!n!}{(k-n)!(2n-k)!} = k! C_{k-n}^n \in \mathbb{Q}$
 and $C_{2n-k}^n = \frac{k!}{(2n-k)!(2k-2n)!} = \frac{k!}{(2n-k)! [2(k-n)]!} \in \mathbb{Q}$

It follows that $\frac{k!n!}{(k-n)!(2n-k)!}$ is an integer and hence $f^{(k)}(0)$ is divisible by $n!$.
 That $f^{(k)}(1)$ is also divisible by $n!$ follows from the fact that $f^{(k)}(1-x) = (-1)^k f^{(k)}(x)$.

MARK	REMARK
6	
1	
7	
1	
1	
1	
1	
5	
1	
10	

SOLUTION

6. (a) That $e^{-x^2} > 0 \quad \forall x$ is trivial.
 Since the exponential function is strictly increasing,
 $e^{x^2} \geq e^0 = 1$
 $\frac{1}{e^{x^2}} \leq 1$
 $\therefore 0 < f(x) \leq 1 \quad \forall x \in \mathbb{R}$.

ALTERNATIVELY,

Since $\frac{d}{dx} e^{-x^2} = -2xe^{-x^2}$
 $\begin{cases} 0 \geq & \text{if } x \leq 0 \\ 0 < & \text{if } x > 0 \end{cases}$

$f(x)$ has an absolute maximum at $x = 0$ and $f(0) = 1$.

Since $0 < [f(x)]^n \leq 1$,
 $I_n = \int_{-1}^1 [f(x)]^n dx \leq \int_{-1}^1 1 dx = 2$
 $= 2 \frac{1}{n}$

(b) $r \leq e^{-x^2}$
 $\Leftrightarrow \ln r \leq -x^2$
 $\Leftrightarrow \ln \frac{1}{r} \geq x^2$
 $\Leftrightarrow -\alpha \leq x \leq \alpha$, where $\alpha = \sqrt{\ln(\frac{1}{r})}$

For $\frac{1}{e} < r < 1$, $0 < \alpha < 1$
 $0 < r \leq f(x)$
 $0 < r^n \leq [f(x)]^n$

$\therefore \int_{-\alpha}^{\alpha} r^n dx \leq \int_{-\alpha}^{\alpha} [f(x)]^n dx$
 $\leq \int_{-1}^1 [f(x)]^n dx$

Hence $2r^n \alpha \leq \int_{-1}^1 [f(x)]^n dx$
 $\Rightarrow r [2 \sqrt{\ln(\frac{1}{r})}]^{\frac{1}{n}} \leq I_n$

MARK	REMARK
1+2	
2	
5	
2	
3	
2	
7	

(c) By (a) and (b)

$$r [2 \sqrt{\ln(\frac{1}{r})}]^{\frac{1}{n}} \leq I_n \leq 2^{\frac{1}{n}} \dots \frac{1}{e} < r < 1.$$

$$\text{Now } \lim_{n \rightarrow \infty} 2^{\frac{1}{n}} = 1 \text{ and } \lim_{n \rightarrow \infty} [2 \sqrt{\ln(\frac{1}{r})}]^{\frac{1}{n}} = 1.$$

As $\lim_{n \rightarrow \infty} I_n$ exists

$$r \leq \lim_{n \rightarrow \infty} I_n \leq 1$$

Since r is an arbitrary number between $\frac{1}{e}$ and 1 ,

$$\lim_{n \rightarrow \infty} I_n = 1.$$

MARK
REMARK

1
1+1
1
1
5

(a) (1) $f(-x) = (-x)^{\frac{2}{3}} - ((-x)^2 - 1)^{\frac{1}{3}} = f(x).$

$\therefore f$ is even.

$$f(x) = x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}} > 0$$

$$\Leftrightarrow x^{\frac{2}{3}} > (x^2 - 1)^{\frac{1}{3}}$$

$$\Leftrightarrow x^2 > x^2 - 1, \text{ which is true for all } x.$$

$$\therefore f(x) > 0 \text{ for all } x.$$

(11) Put $x^2 = \frac{1}{y}$, then $x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}} = (\frac{1}{y})^{\frac{1}{3}} - (\frac{1}{y} - 1)^{\frac{1}{3}}$
$$= \frac{1 - (1 - y)^{\frac{1}{3}}}{y^{\frac{1}{3}}}$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{y \rightarrow 0^+} \frac{1 - (1 - y)^{\frac{1}{3}}}{y^{\frac{1}{3}}}$$

$$= \lim_{y \rightarrow 0^+} \frac{-\frac{1}{3}(1 - y)^{-\frac{2}{3}}(-1)}{\frac{1}{3}y^{-\frac{2}{3}}} \quad (\text{by L'Hopital's rule})$$

$$= \lim_{y \rightarrow 0^+} \left(\frac{1}{(1 - y)^{\frac{2}{3}}} \right) = 1$$

2
3
5
2

(1) For $x \neq 0, 1$ or -1 ,

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}(x^2 - 1)^{-\frac{2}{3}}(2x)$$

$$= \frac{2}{3} \cdot \frac{(x^2 - 1)^{\frac{2}{3}} - x^{\frac{2}{3}}}{x^{\frac{2}{3}}(x^2 - 1)^{\frac{2}{3}}}$$

SOLUTION

MARK
REMARK

(b) (11) For $x > 0, x \neq \pm 1$, the denominator $x^{\frac{1}{3}}(x^2 - 1)^{\frac{2}{3}} > 0$.

$$(x^2 - 1)^{\frac{1}{3}} - x^{\frac{1}{3}} < 0 \text{ according as } (x^2 - 1)^2 > x^4$$

$$-2x^2 + 1 > 0$$

$$\frac{1}{\sqrt{2}} > x \quad (x > 0)$$

\therefore the required sets are

$$\{x : x = \frac{1}{\sqrt{2}}\}$$

$$\{x : x > \frac{1}{\sqrt{2}}\}$$

$$\{x : 0 < x < \frac{1}{\sqrt{2}}\}$$

2
1
1

x	x = 0	0 < x < 1/√2	x = 1/√2	1/√2 < x < 1	x = 1	1 < x
f'	does not exist	+	0	-	does not exist	-
f	min.	↗	max.	↘		↘

At $x = 0, 1$ and $-1, f(x) = 1$

$$\text{At } x = \pm \frac{1}{\sqrt{2}}, f(x) = 4^{\frac{1}{3}} \quad (\approx 1.5876)$$

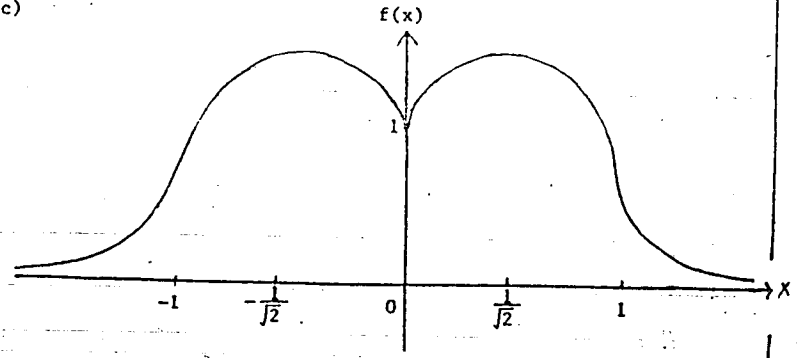
$$(\approx \pm 0.7071)$$

(0, 1) is a minimum

$(-\frac{1}{\sqrt{2}}, 4^{\frac{1}{3}}), (\frac{1}{\sqrt{2}}, 4^{\frac{1}{3}})$ are maxima.

1
1+1
9

(c)



3

SOLUTION

$g(a) = g'(t) - b - f(t)$

Since f is strictly increasing,

$g'(t) \geq 0$ according as $t \leq f^{-1}(b)$

As g is continuously differentiable on $[0, \infty)$, g attains its greatest value at $f^{-1}(b)$.

(b) (i) Integrating by parts,

$\int_0^{f^{-1}(b)} xf'(x) dx = xf(x) \Big|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} f(x) dx = g(f^{-1}(b))$

(ii) Putting $y = f(x)$, $dy = f'(x) dx$.

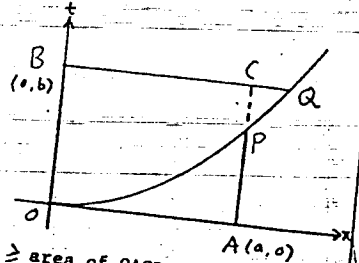
$x = f^{-1}(y)$ as f is strictly increasing.

$\int_0^{f^{-1}(b)} xf'(x) dx = \int_0^b f^{-1}(y) dy = \int_0^b f^{-1}(x) dx$

By (b), $\int_0^b f^{-1}(x) dx = \int_0^{f^{-1}(b)} xf'(x) dx = g(f^{-1}(b)) \geq bt - \int_0^t f(x) dx \quad \forall t \in [0, \infty)$ by (a)

Hence $\int_0^b f^{-1}(x) dx \geq ba - \int_0^a f(x) dx$, i.e. $\int_0^a f(x) dx + \int_0^b f^{-1}(x) dx \geq ab$

Geometrically,



Sum of areas of the two regions \geq area of OACB. Equality holds when $f(a) = b$.

MARK	REMARK
3	
2	
3	
5	
1	
1	
1	

SOLUTION

(d) Consider the function $f(x) = x^{p-1}$ ($x \in [0, \infty)$, $p > 2$) which satisfies the given conditions.

Since $\frac{r}{p} + \frac{1}{q} = 1$,

$f^{-1}(x) = x^{\frac{1}{p-1}} = x^{q-1}$

By (c), $\int_0^a x^{p-1} dx + \int_0^b x^{q-1} dx \geq ab$

$\therefore \frac{1}{p} a^p + \frac{1}{q} b^q \geq ab$

MARK	REMARK
1	
1	
1	
3	