1. (a) $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = ab^2c^3 + a^2b^3c + a^3bc^2 - ab^3c^2 - a^2bc^3 - a^3b^2c$ 1

$= abc(bc^2 + ab^2 + a^2c - b^2c - ac^2 - a^2b)$

$= abc(a - b)(b - c)(c - a)$ [There are various methods.] 2

If a, b, c are all distinct and non-zero, $\Delta \neq 0$. 1

$\therefore$ the system has a unique solution. 1

The solution is given by

$$x_0 = \frac{kbc(k - b)(b - c)(c - k)}{abc(a - b)(b - c)(c - a)}$$ a → k

$$= \frac{k(k - b)(c - k)}{a(a - b)(c - a)}$$

$$y_0 = \frac{k(a - k)(k - c)}{b(a - b)(b - c)}$$

$$z_0 = \frac{k(b - k)(k - a)}{c(b - c)(c - a)}$$

2

Since $x_0 = 0 \Rightarrow k = 0$ or b or c 1

$\Rightarrow y_0 = 0$ or $z_0 = 0$ 1

Similarly $y_0 = 0 \Rightarrow x_0 = 0$ or $z_0 = 0$;

$z_0 = 0 \Rightarrow x_0 = 0$ or $y_0 = 0$. It is impossible for one of x_0, y_0, z_0 to be zero. 1

10

(b) Since $\begin{vmatrix} -1 & 2 & -1 \\ 1 & 4 & 1 \\ -1 & 8 & -1 \end{vmatrix} = 0$, the system does not have a unique solution. 1

Putting $z = t$, the first two equations become 1

$$-x + 2y = d + t$$
$$x + 4y = d^2 - t$$

whose solution is $y = \frac{1}{6}(d^2 + d)$

$x = \frac{1}{3}d^2 - \frac{2}{3}d - t$. 1

Substituting in the third equation

$$-(\tfrac{1}{3}d^2 - \tfrac{2}{3}d - t) + \tfrac{4}{3}(d^2 + d) - t = d^3$$

$$d^3 - d^2 - 2d = 0$$

$$d(d^2 - d - 2) = 0$$

$$d = 0, -1 \text{ or } 2 .$$ 2

As the solution is $x = \frac{1}{3}d^2 - \frac{2}{3}d - t$

$y = \frac{1}{6}(d^2 + d)$

$z = t$.

Putting $d = 0, -1, 2$ in the above yields the solutions for each case. 2

$\{(-t, 0, t), (1 - t, 0, t), (-t, 1, t), t \in R\}$ 7

2. (a) $\det(M) = p^3 + q^3 + r^3 - 3pqr$.

Since $p+q+r = 1$, $(p+q+r)^3 = p^3 + q^3 + r^3 + 3(p^2q + p^2r + q^2p + q^2r + r^2p + r^2q) + 6pqr$

$$\det(M) = (p+q+r)^3 - 3(p^2q + p^2r + q^2p + q^2r + r^2p + r^2q) - 9pqr$$
$$= 1 - 3[(p^2q + q^2p + pqr) + (q^2r + r^2q + pqr) + (r^2p + p^2r + pqr)]$$
$$= 1 - 3[pq(p+q+r) + qr(r+q+p) + rp(r+p+q)]$$
$$= 1 - 3(pq + qr + rp) \quad \text{as } p+q+r = 1 .$$ 2

Next $\frac{1}{2}[(p-q)^2 + (q-r)^2 + (r-p)^2] = \frac{1}{2}[p^2 - 2pq + q^2 + q^2 - 2qr + r^2 + r^2 - 2pr + p^2]$
$$= p^2 + q^2 + r^2 - (pq + qr + rp)$$
$$= (p+q+r)^2 - 2(pq + qr + rp) - (pq + qr + rp)$$
$$= 1 - 3(pq + qr + rp)$$ 2

As $0 \leq p, q, r \leq 1$, $1 - 3(pq + qr + rp) \leq 1$

and $\frac{1}{2}[(p-q)^2 + (q-r)^2 + (r-p)^2] \geq 0$

$\therefore 0 \leq \det(M) \leq 1$ 1

5

(b) The case where $n = 1$ is given.

Assume that M^k satisfies the required conditions for some $k \geq 1$.

Expanding MM^k, we obtain a matrix in the required form where

$$p_{k+1} = pp_k + qr_k + rq_k$$
$$q_{k+1} = pq_k + qp_k + rr_k$$
$$r_{k+1} = pr_k + qq_k + rp_k \ .$$ 2

Obviously, $p_{k+1}, q_{k+1}, r_{k+1} \geq 0$. 1

Further, $p_{k+1} + q_{k+1} + r_{k+1} = pp_k + qr_k + rq_k + pq_k + qp_k + rr_k + pr_k + qq_k + rp_k$
$$= p(p_k + q_k + r_k) + q(r_k + p_k + q_k) + r(q_k + r_k + p_k)$$
$$= p + q + r \quad \text{(by induction assumption)}$$
$$= 1$$ 1

4

2. (c) (i) If at least two of p, q, r are non-zero, by (a)

$$0 \leq \det(M) = 1 - 3(pq + qr + rp)$$
$$< 1$$ 1

$$\therefore \lim_{n \to \infty} \det(M^n) = \lim_{n \to \infty} [\det(M)]^n$$ 1
$$= 0$$ 1

3

(ii) $\because \det(M^n) = \frac{1}{2}[(p_n - q_n)^2 + (q_n - r_n)^2 + (r_n - p_n)^2]$
$$\geq \frac{1}{2}(p_n - q_n)^2$$
$$\geq 0$$ 1

and $\lim_{n \to \infty} \det(M^n) = 0$,

$$\lim_{n \to \infty} \frac{1}{2}(p_n - q_n)^2 = 0$$

$$\therefore \lim_{n \to \infty} (p_n - q_n) = 0 .$$

Similarly, $\lim_{n \to \infty} (q_n - r_n) = 0$ and $\lim_{n \to \infty} (r_n - p_n) = 0$.

Now $3p_n - (p_n + q_n + r_n) = (p_n - q_n) + (p_n - r_n)$

As $\lim_{n \to \infty} (p_n - q_n)$, $\lim_{n \to \infty} (q_n - r_n) = 0$.

$$\lim_{n \to \infty} [3p_n - (p_n + q_n + r_n)] = 0 .$$ 1

Further, as $\lim_{n \to \infty} (p_n + q_n + r_n) = 1$,

$$3p_n = 3p_n - (p_n + q_n + r_n) + (p_n + q_n + r_n)$$

$$\therefore \lim_{n \to \infty} 3p_n = 0 + 1 = 1 .$$

Hence $\lim_{n \to \infty} p_n = \frac{1}{3}$ 2

5

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3. (a) $|w - i| = |w + i|$

$\Leftrightarrow \quad (w - i)(\bar{w} - \bar{i}) = (w + i)(\bar{w} + \bar{i})$

$\Leftrightarrow \quad w\bar{w} + i\bar{i} - i\bar{w} - \bar{i}w = w\bar{w} + i\bar{i} + i\bar{w} + \bar{i}w$ 2

$\Leftrightarrow \quad 2i\bar{w} = -2\bar{i}w$

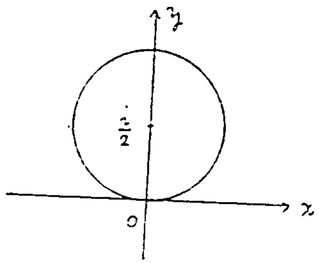
$\Leftrightarrow \quad \bar{w} = w$

$\Leftrightarrow \quad \bar{w}$ is real. 2

4

(b) $|2u - i| = 1 \Leftrightarrow \left|u - \frac{i}{2}\right| = \frac{1}{2}$ 2

This is the equation of a circle with centre $\frac{i}{2}$ and radius $\frac{1}{2}$. 1



2

5

(c) By (a), v is real

$\Leftrightarrow |v - i| = |v + i|$ 2

$\Leftrightarrow \left|\frac{iu}{i - u} - i\right| = \left|\frac{iu}{i - u} + i\right| \quad (u \neq i)$

$\Leftrightarrow \left|\frac{i(2u - i)}{i - u}\right| = \left|\frac{i^2}{i - u}\right|$

$\Leftrightarrow |2u - i| = 1.$ 2

Now $\frac{u - i}{v - i} = \frac{(u - i)(\bar{v} + i)}{(v - i)(\bar{v} + i)}$

$= \frac{(u - i)(v + i)}{(v - i)(v + i)}$ ($\because$ v is real)

$= \frac{(u - i)\left(\frac{iu + i^2 - iu}{i - u}\right)}{v^2 + 1}$

$= \frac{1}{v^2 + 1}.$ 2

Since $\frac{u - i}{v - i}$ is real, the points representing u, v, i are collinear. 2

8

3. ALTERNATIVELY

(a) Let $w = x + yi$

$|w - i| = |w + i|$

iff $x^2 + (y - 1)^2 = x^2 + (y + 1)^2$ "Only if"

iff $y = 0$ "if" 3

iff w is real. 1

4

(b) $u = x + yi$

$(2x)^2 + (2y - 1)^2 = 1.$

$x^2 + y^2 - y = 0$ 2

etc.

(c) Put $u = x + iy$.

$v = \frac{i(x + iy)}{i - (x + iy)}$

$= \frac{x - i(x^2 + y^2 - y)}{x^2 + (1 - y)^2}$ 2

$= \frac{x}{1 - y}$, as $x^2 + y^2 - y = 0$. 2

Slope joining u, i $= \frac{y - 1}{x}$ 2

Slope joining v, i $= \frac{-1}{\frac{x}{1 - y}}$

$= \frac{y - 1}{x}.$ 2

$\therefore$ u, v, i are collinear.

8

4\. (a) We shall show $a_{n+2} - a_n = \frac{(-1)^n}{2^n}(a_1 - a_2)$ by induction. The case is trivial when $n = 1$.

Assume that $a_{k+2} - a_k = \frac{(-1)^k}{2^k}(a_1 - a_2)$ for some $k \geqslant 1$,

$$\begin{aligned} a_{k+3} - a_{k+1} &= \frac{1}{2}(a_{k+2} + a_{k+1}) - a_{k+1} \\ &= \frac{1}{2}(a_{k+2} - a_{k+1}) \\ &= \frac{1}{2}[a_{k+2} - (2a_{k+2} - a_k)] \\ &= \frac{(-1)^{k+1}}{2^{k+1}}(a_1 - a_2). \end{aligned}$$

1

3

Hence the equality holds for all $n \geqslant 1$.

Since $a_1 > a_2$, $a_{n+2} - a_n \lessgtr 0$ according as n is odd or even.

$\therefore \{a_1, a_3, a_5, \ldots\}$ is strictly decreasing and

$\{a_2, a_4, a_6, \ldots\}$ is strictly increasing.

2

6

(b) For any $m, n \geqslant 1$, consider the following 3 cases:

(i) Let $m = n$.

$$\begin{aligned} 2a_{2m} &= a_{2m-1} + a_{2m-2} \\ &< a_{2m-1} + a_{2m} \quad \text{by (a)} \end{aligned}$$

$\therefore a_{2m} < a_{2m-1}$. 2

(ii) Let $m < n$.

By (a) $a_{2m} < a_{2m+2} < \cdots < a_{2n}$

$< a_{2n-1}$ by (b)(i). 2

(iii) Let $m > n$.

$a_{2n-1} > a_{2n+1} > \cdots > a_{2m-1}$

$> a_{2m}$ by (b)(i). 2

In all cases, $a_{2m} < a_{2n-1}$ for $m, n \geqslant 1$.

6

(c) By (a) and (b) $\{a_1, a_3, a_5, \ldots\}$ is decreasing and bounded below, e.g. by a_2

$\{a_2, a_4, a_6, \ldots\}$ is increasing and bounded above, e.g. by a_1

$\therefore$ both sequences converge. 2

Let $\lim_{n\to\infty} a_{2n-1} = \ell_1$, $\lim_{n\to\infty} a_{2n} = \ell_2$

Since $a_{2n+2} = \frac{1}{2}(a_{2n+1} + a_{2n})$

$$\lim_{n\to\infty} a_{2n+2} = \frac{1}{2}\left[\lim_{n\to\infty} a_{2n+1} + \lim_{n\to\infty} a_{2n}\right]$$

$$\ell_2 = \frac{1}{2}(\ell_1 + \ell_2)$$

$$\ell_1 = \ell_2.$$

1

5

4. ALTERNATIVELY

(a)

$$a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$$

$$\Rightarrow a_{n+2} - a_{n+1} = -\frac{1}{2}(a_{n+1} - a_n)$$

$$= \text{etc.}$$

$$= \left(-\frac{1}{2}\right)^n (a_2 - a_1) \qquad 2$$

$$a_{n+1} - a_n = \text{etc.}$$

$$= \left(-\frac{1}{2}\right)^{n-1} (a_2 - a_1) \qquad 1$$

$$\therefore a_{n+2} - a_n = \frac{(-1)^n}{2^n}(a_1 - a_2) \qquad 1$$

etc.

$a_{n+2} = a_n$

$= \frac{1}{2}(a_{n+1} + a_n) - a_n = \frac{1}{2}(a_{n+1} - a_n)$

$= \frac{1}{2}[a_{n+1} - 2a_{n-1} + a_{n-1}]$

$(\because 2a_{n+1} = a_n + a_{n-1})$

$= \frac{1}{2}(a_{n-1} - a_{n+1})$

$\Rightarrow a_n - a_{n+2} = (-1)\frac{1}{2}(a_{n-1} - a_{n+1})$

(b)

$$a_{2m} - a_{2m-2} = \frac{(-1)^{2m-2}}{2^{2m-2}}(a_1 - a_2)$$

$$a_{2m-2} - a_{2m-4} = \frac{(-1)^{2m-4}}{2^{2m-4}}(a_1 - a_2)$$

etc.

$$a_4 - a_2 = \frac{(-1)^2}{2^2}(a_1 - a_2) .$$

$$\therefore a_{2m} - a_2 = (a_1 - a_2)\left(\frac{1}{2^2} + \frac{1}{2^4} + \dots + \frac{1}{2^{2m-2}}\right) \qquad 2$$

$$a_{2m-1} - a_1 = -(a_1 - a_2)\left(\frac{1}{2} + \frac{1}{2^3} + \dots + \frac{1}{2^{2m-1}}\right) \qquad 1$$

$$a_{2m-1} - a_{2m} = (a_1 - a_2) \cdot$$

$$-(a_1 - a_2)\left[\left(\frac{1}{2} + \frac{1}{2^3} + \dots + \frac{1}{2^{2m-1}}\right) + \left(\frac{1}{2^2} + \frac{1}{2^4} + \dots + \frac{1}{2^{2m-2}}\right)\right] \qquad 1$$

$$a_{2m-1} - a_{2m} = -(a_1 - a_2)\left[\frac{\frac{1}{2}\left(1 - \frac{1}{4^m}\right)}{1 - \frac{1}{4}} + \frac{\frac{1}{2^2}\left(1 - \frac{1}{4^{m-1}}\right)}{1 - \frac{1}{4}} - 1\right]$$

$$= -(a_1 - a_2)\left(\frac{2}{3} + \frac{1}{3} - \frac{1}{2}\cdot\frac{1}{4^m} - \frac{1}{4^m} - 1\right) \qquad 2$$

$$> 0 .$$

6

5. (a) Let $E(n, k)$ be the number of simple events that exactly k out of n men end up with the right umbrellas. $F(n)$ be the number of ways that n men all end up with wrong umbrellas.

$$E(n+1, k+1) = \text{number of way of selecting } k+1 \text{ out of } n+1 \text{ men} \times F((n+1) - (k+1))$$

$$= {}_{n+1}C_{k+1}\, F(n-k)$$

$$E(n, k) = {}_nC_k\, F(n-k)$$

$$\therefore \frac{E(n+1, k+1)}{E(n, k)} = \frac{{}_{n+1}C_{k+1}}{{}_nC_k}$$

$$= \frac{n+1}{k+1}$$

$$\frac{k+1}{n+1}\cdot\frac{E(n+1, k+1)}{n!} = \frac{E(n, k)}{n!}$$

$$\therefore (k+1)\, P_{n+1,\, k+1} = P_{n,\, k}$$

6

(b) (i)

$$\frac{d}{dx}F_{n+1}(x) = \frac{d}{dx}\left[\sum_{k=0}^{n+1} P_{n+1,\, k}\, x^k\right].$$

$$= \frac{d}{dx}\sum_{k=0}^{n} P_{n+1,\, k+1}\, x^{k+1}$$

$$= \sum_{k=0}^{n} (k+1)\, P_{n+1,\, k+1}\, x^k$$

$$= F_n(x) \qquad 2$$

(ii) By (i)

$$F_n^{(k)}(1) = F_{n-1}^{(k-1)}(1) \qquad 1$$

$$= F_{n-2}^{(k-2)}(1)$$

$$= \text{etc.}$$

$$= F_{n-k}(1)$$

$$= \sum_{r=0}^{n-k} P_{n-k,\, r} \qquad 1$$

$$= 1 \qquad 1$$

5

. ALTERNATIVELY

Solution 1

Consider a particular event that exactly $(k+1)$ out of $(n+1)$ men end up with the right umbrellas. Let A be one of the $(k+1)$ men. If we disregard A, there corresponds one event that exactly k out of n men end up with the right umbrellas.

As there are ${}_nC_k$, ${}_{n+1}C_{k+1}$ ways of selecting k and $k+1$ men out of n and $n+1$ men respectively,

$$E(n+1,\ k+1) = \frac{{}_{n+1}C_{k+1}}{{}_nC_k} E(n,\ k)$$

etc.

Solution 2

Consider a particular man X out of $(n+1)$ men. Let A be the event that exactly $(k+1)$ men end up with the right umbrellas, B be the event that X ends up with the right umbrella.

$$P(A|B)\,P(B) = P(A \cap B) = P(B|A)\,P(A)\ .$$

$$P_{n,k}\left(\frac{1}{n+1}\right) = \left(\frac{{}_nC_k}{{}_{n+1}C_{k+1}}\right) P_{n+1,k+1}$$

$$= \left(\frac{k+1}{n+1}\right) P_{n+1,\ k+1}$$

$$\therefore\ P_{n,k} = (k+1)\,P_{n+1,\ k+1}\ .$$

6/6

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3\. (c) Putting $n = 0, 1$ in the given expansion, we have

$$F_n(x) = \sum_{k=0}^{n} \frac{1}{k!} F_n^{(k)}(0)\, x^k \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$F_n(x) = \sum_{k=0}^{n} \frac{1}{k!} F_n^{(k)}(1)\,(x-1)^k = \sum_{k=0}^{n} \frac{1}{k!} (x-1)^k \ \ldots\ldots\ldots (2)$$

But by definition, $F_n(x) = \sum_{k=0}^{n} P_{n,k} x^k$,

$$\therefore\ P_{n,k} = \frac{1}{k!} F_n^{(k)}(0)$$ 2

From (2), $F_n^{(k)}(x) = \sum_{j=0}^{n} \frac{1}{j!} j(j-1)\ldots(j-k+1)(x-1)^{j-k}$, $0 \leqslant k \leqslant j$. 1

$$= \sum_{j=0}^{n} \frac{1}{(j-k)!}(x-1)^{j-k}\ ,\quad 0 \leqslant k \leqslant j.$$

$$= \sum_{j=k}^{n} \frac{1}{(j-k)!}(x-1)^{j-k}\ ,$$ 2

$$\therefore\ P_{n,k} = \frac{1}{k!} F_n^{(k)}(0)$$

$$= \frac{1}{k!} \sum_{j=k}^{n} \frac{(-1)^{j-k}}{(j-k)!}$$ 1

6

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6. (a) $X_1 \subset X_2 \Rightarrow f[X_1] \subset f[X_2]$ 2

$\Rightarrow B \setminus f[X_1] \supset B \setminus f[X_2]$

$\Rightarrow g[B \setminus f[X_1]] \supset g[B \setminus f[X_2]]$

$\Rightarrow \bar{\Phi}(X_1) \supset \bar{\Phi}(X_2)$. 3

5

(b) By (a), $X_1 \subset X_2 \subset A \Rightarrow \bar{\Phi}(X_1) \supset \bar{\Phi}(X_2)$

$$\therefore \Psi(X_1) = A \setminus \bar{\Phi}(X_1) \subset A \setminus \bar{\Phi}(X_2) = \Psi(X_2)$$ 2

Next, since $S \subset X \ \forall \ X \in \mathcal{F}$

$\Psi(S) \subset \Psi(X)$ 2

$\subset X \quad \forall X \in \mathcal{F}$ 1

$\therefore \Psi(S) \subset S$ 1

i.e. $S \in \mathcal{F}$

6

(c) $\therefore \Psi(S) \subset S \Rightarrow \Psi(\Psi(S)) \subset \Psi(S)$

i.e. $\Psi(S) \in \mathcal{F}$. 2

By definition of S, $\Psi(S) \in \mathcal{F} \Rightarrow S \subset \Psi(S)$.

But $\Psi(S) \subset S$.

Hence $S = \Psi(S)$ 2

$A \setminus S = A \setminus \Psi(S)$

$= A \setminus (A \setminus \bar{\Phi}(S))$

$= \bar{\Phi}(S)$. 2

6

ALTERNATIVELY,

The second part of (b).

$\forall y, \quad y \in \Psi(S) \Rightarrow y \in A$ and $y \notin \bar{\Phi}(S)$

$\Rightarrow y \in A$ and $y \notin \bar{\Phi}(X) \ \forall X \in \mathcal{F}$ 2

$\Rightarrow y \in A \setminus \bar{\Phi}(X) \quad \forall X \in \mathcal{F}$

$\Rightarrow y \in \Psi(X) \subset X \quad \forall X \in \mathcal{F}$

$\therefore y \in S$. 2

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7. (a) $(1 + \frac{1}{n})^n = \sum_{r=0}^{n} C_r^n (\frac{1}{n})^r$

$= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{1}{n^r}$

$= 1 + \sum_{r=1}^{n} \frac{1}{r!} \frac{n(n-1)\ldots(n-r+1)}{n^r}$

$= 1 + \sum_{r=1}^{n} \left[\frac{1}{r!} \prod_{k=0}^{r-1} (1 - \frac{k}{n})\right]$. 2

For $n \geq 2$, $(1 + \frac{1}{n})^n = 1 + 1 + \sum_{r=2}^{n} \left[\frac{1}{r!} \prod_{k=0}^{r-1} (1 - \frac{k}{n})\right]$

> 2 as the last term is positive. 1

Next $0 < 1 - \frac{k}{n} \leq 1$ for $0 \leq k < n$,

$0 < \prod_{k=0}^{r-1} (1 - \frac{k}{n}) < 1$ for $2 \leq r \leq n$.

$(1 + \frac{1}{n})^n = 2 + \sum_{r=2}^{n} \left[\frac{1}{r!} \prod_{k=0}^{r-1} (1 - \frac{k}{n})\right]$

$< 2 + \sum_{r=2}^{n} \frac{1}{r!}$

$\leq 2 + \sum_{r=2}^{n} \frac{1}{2^{r-1}}$

$= 3 - \frac{1}{2^{n-1}}$

< 3 3

Now $(1 + \frac{1}{n+1})^{n+1} = 1 + \sum_{r=1}^{n+1} \left[\frac{1}{r!} \prod_{k=0}^{r-1} (1 - \frac{k}{n+1})\right]$

$> 1 + \sum_{r=1}^{n} \left[\frac{1}{r!} \prod_{k=0}^{r-1} (1 - \frac{k}{n+1})\right]$

$> 1 + \sum_{r=1}^{n} \left[\frac{1}{r!} \prod_{k=0}^{r-1} (1 - \frac{k}{n})\right]$ as $1 - \frac{k}{n+1} > 1 - \frac{k}{n} > 0$

$= (1 + \frac{1}{n})^n$

$\therefore (1 + \frac{1}{n})^n$ is increasing. 3

Since it is also bounded above, it is a convergent sequence. 9

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7. (b) Putting $y = 1 - x$,

$$\sum_{i=0}^{n-1}(1-x)^i = \sum_{i=0}^{n-1} y^i$$

$$= \frac{1-y^n}{1-y}$$

$$= \frac{1-(1-x)^n}{x}$$

$$= \frac{1-\sum_{k=0}^{n} C_k^n (-1)^k x^k}{x}$$

$$= \sum_{k=1}^{n} C_k^n (-1)^{k-1} x^{k-1} . \qquad 3$$

It can be checked that the identity also holds for $x = 0, 1$. 1

Integrating both sides of the identity,

$$\text{L.S.} = \sum_{k=1}^{n}\left[C_k^n (-1)^{k-1}\int_0^1 x^{k-1}\,dx\right]$$

$$= \sum_{k=1}^{n}\left[\frac{1}{k}C_k^n (-1)^{k-1} x^k\right]_0^1$$

$$= C_1^n - \frac{1}{2}C_2^n + \dots + (-1)^{n-1}\frac{1}{n}C_n^n . \qquad 2$$

$$\text{R.S.} = \sum_{i=0}^{n-1}\int_0^1 (1-x)^i\,dx$$

$$= \sum_{i=0}^{n-1}\int_0^1 y^i\,dy$$

$$= \sum_{i=0}^{n-1}\left[\frac{1}{i+1}y^{i+1}\right]_0^1$$

$$= \sum_{i=0}^{n-1}\frac{1}{i+1} \qquad 2$$

8

8. (a) $S = \sum_{i=1}^{n} a_i(B_i - B_{i-1})$ $(B_0 = 0)$ 2

$$= \sum_{i=1}^{n} a_i B_i - \sum_{i=1}^{n} a_i B_{i-1}$$

$$= a_n B_n + \sum_{i=1}^{n-1} a_i B_i - \sum_{i=1}^{n-1} a_{i+1} B_i - a_1 B_0$$

$$= a_n B_n + \sum_{i=1}^{n-1}(a_i - a_{i+1})B_i \qquad 2$$

4

(b) $$|S| = \left|a_n B_n + \sum_{i=1}^{n-1}(a_i - a_{i+1})B_i\right|$$

$$\leqslant |a_n B_n| + \left|\sum_{i=1}^{n-1}(a_i - a_{i+1})B_i\right|$$

$$\leqslant K\left\{|a_n| + \sum_{i=1}^{n-1}|a_i - a_{i+1}|\right\}. \qquad 2$$

As $\{a_i\}$ is monotonic, $\sum_{i=1}^{n-1}|a_i - a_{i+1}|$

$$= \left|\sum_{i=1}^{n-1}(a_i - a_{i+1})\right| \qquad 2$$

$$= |a_1 - a_n|$$

$$\leqslant |a_1| + |a_n| \qquad 1$$

$|S| \leqslant K(|a_1| + 2|a_n|)$

5

8. (c) Consider the function $x^{\frac{1}{x}}$, $x \geqslant 3$.

$$\frac{d}{dx} \ln x^{\frac{1}{x}} = \frac{d}{dx}\left[\frac{1}{x} \ln x\right] \qquad 1$$

$$= \frac{1}{x^2}(1 - \ln x)$$

$$< 0 \quad \text{as} \quad \ln x \geqslant \ln 3 > 1 .$$

$\therefore$ $\ln x^{\frac{1}{x}}$ is monotonic decreasing and hence $\frac{1}{x^{\frac{1}{x}}}$ is monotonic increasing. 2

Let $a_k = \frac{1}{\sqrt[k]{k}}$, $b_k = (-1)^k$, $k \geqslant 3$, 1

then $\left| \sum_{k=n}^{n+p} (-1)^k \right| \leqslant 1$. 1

By (b), we have

$$\left| \sum_{k=n}^{n+p} \frac{(-1)^k}{\sqrt[k]{k}} \right| \leqslant \left(\frac{1}{\sqrt[n]{n}} + \frac{2}{\sqrt[n+p]{n+p}} \right) \qquad 2$$
$$\leqslant 3$$

as $\frac{1}{\sqrt[n]{n}} \leqslant 1 \quad \forall n \geqslant 1$. 1

8

<u>LTERNATIVELY</u>

(c)

$$\frac{d}{dx}\left(\frac{1}{x}\right)^{\frac{1}{x}} = \left(\frac{1}{x}\right)^{\frac{1}{x}}\left(-\frac{1}{x^2} - \frac{1}{x^2}\ln\frac{1}{x}\right)$$

$$= -\left(\frac{1}{x}\right)^{\frac{1}{x}} \frac{1}{x^2}\left(1 + \ln\frac{1}{x}\right)$$

$$= \left(\frac{1}{x}\right)^{\frac{1}{x}} \frac{1}{x^2} (\ln x - 1) > 0 \qquad 3$$

HONG KONG EXAMINATIONS AUTHORITY

Hong Kong Advanced Level Examination 1983

Pure Mathematics II

Marking Scheme

(a) $\displaystyle\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \int \frac{dx}{\sqrt{\left(x+\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}}$ 2

$\displaystyle = \ln\left|\left(x+\frac{a+b}{2}\right) + \sqrt{\left(x+\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}\right| + c$ 3

(using $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u + \sqrt{u^2-a^2}\right| + c$)

$\displaystyle = \ln\left|2x+a+b+2\sqrt{(x+a)(x+b)}\right| + c$

-1 if c missing 5

ALTERNATIVELY,

Putting $t = \sqrt{\dfrac{x+a}{x+b}}$, $x = \dfrac{bt^2-a}{1-t^2}$

$dx = \dfrac{2(b-a)t}{(1-t^2)^2}\,dt$

$(x+a)(x+b) = \dfrac{(b-a)^2 t^2}{(1-t^2)^2}$ 1

$\displaystyle\int \frac{dx}{\sqrt{(x+a)(x+b)}} = 2\int \frac{dt}{1-t^2}$ 1

$\displaystyle = \ln\left|\frac{1+t}{1-t}\right| + c$

$\displaystyle = \ln\left|\frac{\sqrt{x+b}+\sqrt{x+a}}{\sqrt{x+b}-\sqrt{x+a}}\right| + c$ (if $a \neq b$) 2

If $a = b$, the integral is $\ln|x+a| + c$. 1

5

(b) Putting $u = \frac{\pi}{4} - x$, $du = -dx$

$x = 0 \Rightarrow u = \frac{\pi}{4}$

$x = \frac{\pi}{4} \Rightarrow u = 0$.

$\displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = -\int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)du$ 1

$\displaystyle = \int_0^{\frac{\pi}{4}} \ln\left[1 + \frac{\tan\frac{\pi}{4} - \tan u}{1+\tan\frac{\pi}{4}\tan u}\right]du$

$\displaystyle = \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan u}\,du$ 2

$\displaystyle = \int_0^{\frac{\pi}{4}} \ln 2\,du - \int_0^{\frac{\pi}{4}} \ln(1+\tan u)\,du$ 1

$\displaystyle\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln 2\,du$ 1

$\displaystyle = \frac{\pi}{8}\ln 2$ (= 0.2722) 1

6

(c) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left[\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \dots + \cos\frac{(n-1)\pi}{n}\right]$.

$\displaystyle = \lim_{n\to\infty}\left\{\frac{1}{n}\left[\cos 0 + \cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \dots + \cos\frac{(n-1)\pi}{n}\right] - \frac{1}{n}\right\}$ 2

$\displaystyle = \int_0^1 \cos\pi x\,dx - 0$

$= 0$ (maximum 4 marks if omit $\cos 0$, $-\frac{1}{n}$.) 2

-2 if not distinguishing between n odd and n even in summing $\sum_{r=1}^{n-1}\cos\frac{r\pi}{n}$ by grouping terms 2

6

(a) $\frac{df}{dx} = e^{-x}(3x^2 - 4x) - e^{-x}x^2(x - 2)$

$= -(x^3 - 5x^2 + 4x)e^{-x}$

$= -x(x - 1)(x - 4)e^{-x}$.

$\frac{df}{dx} = 0 \iff x = 0,\ 1$ or 4 . —— (at least two correct) 1

$\frac{d^2f}{dx^2} = -(3x^2 - 10x + 4)e^{-x} + e^{-x}(x^3 - 5x^2 + 4x)$

$= (x^3 - 8x^2 + 14x - 4)e^{-e}$ (or sign of f' changes from + to − at $x = 0$,)

$= (x - 2)(x^2 - 6x + 2)e^{-x}$.

At $x = 0$, $\frac{d^2f}{dx^2} < 0$ (at least one max. or min. verified) 1

∴ (0, 0) is a maximum point.

At $x = 1$, $\frac{d^2f}{dx^2} > 0$

$(1, -\frac{1}{e})$ is a minimum point (or (1, −0.3679).)

At $x = 4$, $\frac{d^2f}{dx^2} < 0$

$(4, \frac{32}{e^4})$ is a maximum point (or (4, 0.5861)) 0+1+1

3* Since f is continuous in R, the graph of f has no vertical asymptote. 1

Let $y = ax + b$ be an asymptote.

$a = \lim_{x \to \infty} \frac{x^2(x - 2)e^{-x}}{x} = \lim_{x \to \infty} \frac{x^2 - 2x}{e^x} = \lim_{x \to \infty} \frac{2x-2}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0$.

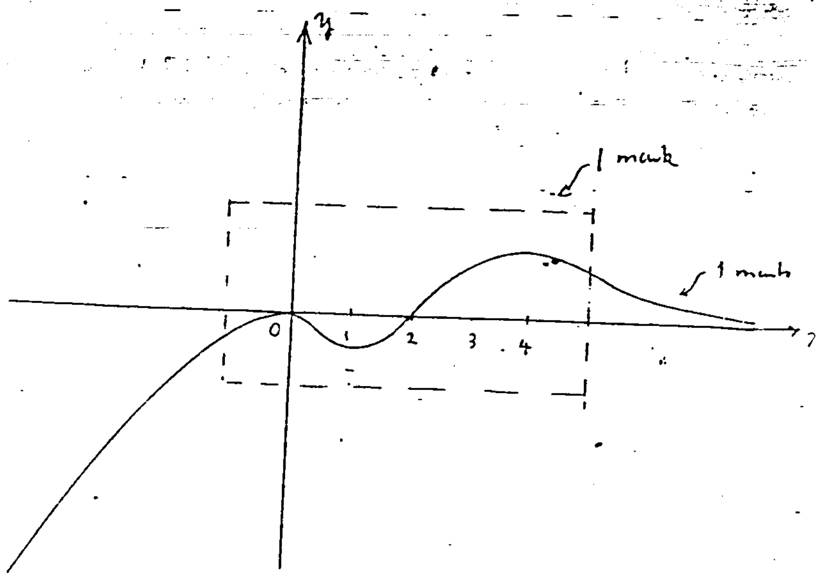
∴ $b = \lim_{x \to \infty} \frac{x^2(x - 2)}{e^{x}} =$ etc. $= 0$. (For "$y \to 0$ as $x \to \infty$", award 1 mark only) 2

Further, 3* $\lim_{x \to -\infty} \frac{x^2(x - 2)e^{-x}}{x}$ does not exist . 1

∴ the x-axis is the only asymptote.

The curve intersects the axes at (0. 0), (2, 0). 1

2. (a)



1

(b) Integrating by parts,

$\int (x^3 - 2x^2)e^{-x}dx = -(x^3 - 2x^2)e^{-x} + \int (3x^2 - 4x)e^{-x}dx$

$= -(x^3 - 2x^2)e^{-x} - (3x^2 - 4x)e^{-x} + \int (6x - 4)e^{-x}dx$

$= -(x^3 - 2x^2)e^{-x} - (3x^2 - 4x)e^{-x} - (6x - 4)e^{-x} + \int 6e^{-x}dx$

$= -e^{-x}(x^3 + x^2 + 2x + 2) + C$. 2

∴ for $k > 2$, $A_k = -\int_0^2 (x^3 - 2x^2)e^{-x}dx + \int_2^k (x^3 - 2x^2)e^{-x}dx$ 1

$= -2 + \frac{36}{e^2} - e^{-k}(k^3 + k^2 + 2k + 2)$ 1

As $\lim_{k \to \infty} e^{-k}(k^3 + k^2 + 2k + 2) = \lim_{k \to \infty} \frac{3k^2 + 2k + 2}{e^k} =$ etc. $= 0$ (by L'Hopital's Rule)

$\lim_{k \to \infty} A_k = \frac{36}{e^2} - 2$ (= 2.872) 1

6

(a) Putting $x = \ell t + x_0$, $y = mt + y_0$, $z = nt + z_0$ in

we obtain $(A\ell + Bm + Cn)t + Ax_0 + By_0 + Cz_0 + D = 0$ 2

The plane contains L iff the above equation is satisfied for all t, i.e. iff

$A\ell + Bm + Cn = 0$

and $Ax_0 + By_0 + Cz_0 + D = 0$ 2

4

(b) (i) Any plane passing through the point (x_1, y_1, z_1) can be written as

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 .$$

If this plane contains L_1 and L_2, by (a),

$$A\ell_1 + Bm_1 + Cn_1 = 0$$
$$A\ell_2 + Bm_2 + Cn_2 = 0 .$$ 3

The condition for this system of equations in A, B, C to have a non-trivial solution is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 ,$$

which is a linear equation in x, y, z. 3

~~Since L_1 and L_2 are not parallel, the direction ratios $\ell_1 : m_1 : n_1$, $\ell_2 : m_2 : n_2$ are not equal.~~

~~The determinant is therefore not identically zero.~~ 2

Hence it is the plane passing through L_1 and L_2.

8

ERNATIVELY

Let (x, y, z) be any point on the plane.

$(\ell_1, m_1, n_1) \times (\ell_2, m_2, n_2)$ is a vector $\perp$ to the plane.

$\therefore$ $(x - x_1, y - y_1, z - z_1) \cdot [(\ell_1, m_1, n_1) \times (\ell_2, m_2, n_2)] = 0$

i.e.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

3. (b) (ii) Any plane containing L_1 satisfies the conditions

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$
$$A\ell_1 + Bm_1 + Cn_1 = 0 .$$

If this plane passes through (x_2, y_2, z_2),

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 .$$

The required plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell_1 & m_1 & n_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \end{vmatrix} = 0 .$$ 4

The fact that L_1 and L_2 are parallel and distinct guarantees that

* ~~$\ell_1 : m_1 : n_1 \neq x_2 - x_1 : y_2 - y_1 : z_2 - z_1$~~ .

1

5

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(b) (ii) Putting $\ell_2 : m_2 : n_2 = x_2 - x_1 : y_2 - y_1 : z_2 - z_1$ in (i), etc.

(a) (i) $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = \dfrac{-g'(x)}{g^2(x)}$

$= \dfrac{f(x)}{g(x)}$

(ii) $\dfrac{d}{dx}\left(\dfrac{1}{g^2(x)}\right) = \dfrac{-2}{g^3(x)}g'(x)$

$= \dfrac{2f(x)}{g^2(x)}$ 2+1

3

(b) $\dfrac{d}{dx}(1 + f^2(x)) = 2f(x)\ f'(x)$ 1

$= \dfrac{2f(x)}{g^2(x)}$ 1

$= \dfrac{d}{dx}\left(\dfrac{1}{g^2(x)}\right)$ by (a)(ii)

$1 + f^2(x) = \dfrac{1}{g^2(x)} + c$ 1

Putting $x = 0$,

$1 + 0 = 1 + c \Rightarrow c = 0$ 1

$\therefore\ 1 + f^2(x) = \dfrac{1}{g^2(x)}$

4

(c) Differentiating w.r.t. x, 1

R.S. $= f'(x)g(x)g(a-x) + f(x)g'(x)g(a-x) - f(x)g(x)g'(a-x)$

$- f'(a-x)g(a-x)g(x) - f(a-x)g'(a-x)g(x) + f(a-x)g(a-x)g'(x)$

$= \dfrac{g(a-x)}{g(x)} - f^2(x)g(x)g(a-x) + f(x)g(x)f(a-x)g(a-x)$

$- \dfrac{g(x)}{g(a-x)} + f^2(a-x)g(a-x)g(x) - f(a-x)g(a-x)f(x)g(x)$ 1

$= \dfrac{g^2(a-x) - f^2(x)g^2(x)g^2(a-x) - g^2(x) + f^2(a-x)g^2(a-x)g^2(x)}{g(x)g(a-x)}$

$= \dfrac{g^2(a-x)[1 - f^2(x)g^2(x)] - g^2(x)[1 - f^2(a-x)g^2(a-x)]}{g(x)g(a-x)}$

$= 0$ since $1 - f^2(x)g^2(x) = g^2(x)$ by (b). 2

$\therefore\ f(x)g(x)g(a-x) + f(a-x)g(a-x)g(x) = c$. 1

Putting $x = a$, $c = f(a)g(a)$. 1

6

(d) (i) Putting $a = x + y$ in (c),

$f(x+y)g(x+y) = f(x)g(x)g(y) + f(y)g(y)g(x)$ 1

$= g(x)g(y)[f(x) + f(y)]$. 1

(ii) Putting $a = 0$ in (c),

$0 = f(x)g(x)g(-x) + f(-x)g(-x)g(x)$

$[f(x) + f(-x)]g(x)g(-x) = 0$ 1

Since $g(x)g(-x) > 0$ by definition,

$f(-x) = -f(x)$. 1

4

(a). Since f is non-constant, $\exists\, x_1$ s.t. $f(x_1) \neq 0$, 1

$f(x_1) = f(x_1 + 0)$ 1

$= f(x_1)\, f(0)$. 1

Since $f(x_1) \neq 0$, $f(0) = 1$. 1

-2 if fail to consider the case $f(x)=0$ in $f(x+x) = (f(x))^2$

Let $f(x_2) = 0$ for some $x_2 \in \mathbb{R}$. 1

Then $f(x_1) = f(x_2 + (x_1 - x_2))$

$= f(x_2)\, f(x_1 - x_2)$

$= 0$, which is false. 1

$\therefore f(x) \neq 0 \quad \forall x \in \mathbb{R}$.

6

(b) Since f is differentiable at x_0,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0)(f(h) - 1)}{h} \text{ exists.}$$ 1

As $f(x_0) \neq 0$, $\lim\limits_{h\to 0} \frac{f(h) - 1}{h}$ exists and equals $\frac{f'(x_0)}{f(x_0)}$ 1

$$\therefore f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - 1}{h}$$

$$= \frac{f'(x_0)}{f(x_0)}$$ 1

Next $\lim\limits_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim\limits_{h\to 0} \frac{f(x)(f(h) - 1)}{h}$ exists as $f(x) \neq 0 \ \forall x \in \mathbb{R}$. 1

$\therefore f'(x)$ exists and equals $\frac{f'(x_0)}{f(x_0)} f(x)$. 1

5

(c) By (b) $f'(x)$ exists.

$$\frac{d}{dx} e^{-\alpha x} f(x) = -\alpha e^{-\alpha x} f(x) + e^{-\alpha x} f'(x)$$ 1

$$= e^{-\alpha x} f(x) \left[-\alpha + \frac{f'(x_0)}{f(x_0)}\right]$$ 1

Putting $\alpha = \frac{f'(x_0)}{f(x_0)}$ which is non-zero, we have 1

$$\frac{d}{dx}\left(e^{-\alpha x} f(x)\right) = 0,$$

or $$\frac{f(x)}{e^{\alpha x}} = c,$$ 1

where $$c = \frac{f(0)}{e^0}$$

$$= 1.$$ 1

i.e. $f(x) \equiv e^{\alpha x}$, where $\alpha = \frac{f'(x_0)}{f(x_0)}$. 1

6

(a) Differentiating both sides w.r.t. x,

R.S. $= p\cos^{p-1}x\sin x\cos qx + q\cos^{p}x\sin qx$.

L.S. $= (p+q)\cos^{p}x\sin qx - p\cos^{p-1}x\sin(q-1)x$

$= (p+q)\cos^{p}x\sin qx - p\cos^{p-1}x[\sin qx\cos x - \cos qx\sin x]$

$= p\cos^{p-1}x\cos qx\sin x + q\cos^{p}x\sin qx$ 2

R.S. = L.S. *No mark if proved by integration*

$\Rightarrow (p+q)F_{p,q}(x) - pF_{p-1,q-1}(x) = -\cos^{p}x\cos qx + C$. 1

Putting $x = 0$,
$0 = -1 + C$ 1
$C = 1$. 1

6

(b) $(p+q)F_{p,q}(\pi) = pF_{p-1,q-1}(\pi) - \cos^{p}\pi\cos q\pi + 1$

$= pF_{p-1,q-1}(\pi) - (-1)^{p+q} + 1$

$F_{p,q}(\pi) = \left(\frac{p}{p+q}\right)F_{p-1,q-1}(\pi)$ if p, q are both even or both odd. 2

(i) Let $p \geq q$,

$F_{p,q}(\pi) = \left(\frac{p}{p+q}\right)F_{p-1,q-1}(\pi)$

$= \left(\frac{p}{p+q}\right)\left(\frac{p-1}{p+q-2}\right)F_{p-2,q-2}(\pi)$

= etc.

$= \left(\frac{p}{p+q}\right)\left(\frac{p-1}{p+q-2}\right)\cdots\left(\frac{p-q+1}{p+q-2q+2}\right)F_{p-q,0}(\pi)$ 2

But $F_{p-q,0}(\pi) = \int_0^{\pi}\cos^{p-q}t\sin 0\,dt = 0$ 1

$\therefore F_{p,q}(\pi) = 0$.

(ii) Let $p < q$,

$F_{p,q}(\pi) = \left(\frac{p}{p+q}\right)\left(\frac{p-1}{p+q-2}\right)\cdots\left(\frac{1}{p+q-2p+2}\right)F_{0,q-p}(\pi)$ 1

But $F_{0,q-p}(\pi) = \int_0^{\pi}\sin(q-p)t\,dt$

$= -\frac{1}{q-p}\cos(q-p)t\Big|_0^{\pi}$ 1

$= 0$ since $q - p$ is even when p, q are both odd or both even

$\therefore F_{p,q}(\pi) = 0$.

7

6. (c) $\int_0^{\frac{\pi}{2}}\sin^2x\sin 3x\,dx = \int_0^{\frac{\pi}{2}}(1-\cos^2x)\sin 3x\,dx$

$= \int_0^{\frac{\pi}{2}}\sin 3x\,dx - \int_0^{\frac{\pi}{2}}\cos^2x\sin 3x\,dx$ 1

Now $(p+q)F_{p,q}\left(\frac{\pi}{2}\right) - pF_{p-1,q-1}\left(\frac{\pi}{2}\right) = 0 + 1$

$F_{p,q}\left(\frac{\pi}{2}\right) = \frac{1}{p+q}\left(pF_{p-1,q-1}\left(\frac{\pi}{2}\right) + 1\right)$ 1

$\int_0^{\frac{\pi}{2}}\cos^2x\sin 3x\,dx = F_{2,3}\left(\frac{\pi}{2}\right)$

$= \frac{1}{5}\left(2F_{1,2}\left(\frac{\pi}{2}\right) + 1\right)$

$= \frac{1}{5}\left[2\times\frac{1}{3}\left(F_{0,1}\left(\frac{\pi}{2}\right) + 1\right) + 1\right]$

$= \frac{2}{15}\int_0^{\frac{\pi}{2}}\sin x\,dx + \frac{1}{3}$

$= \frac{7}{15}$

$\therefore \int_0^{\frac{\pi}{2}}\sin^2x\sin 3x = \int_0^{\frac{\pi}{2}}\sin 3x\,dx - \int_0^{\frac{\pi}{2}}\cos^2x\sin 3x\,dx$ 1

$= \frac{1}{3} - \frac{7}{15} = \frac{-2}{15}$ 1

4

7. (a) $\frac{1}{r+1} \leqslant \int_r^{r+1} \frac{1}{x}\,dx \leqslant \frac{1}{r}$ ($\frac{1}{x}$ is decreasing) 2

$$\sum_{r=1}^{k-1} \frac{1}{r+1} \leqslant \sum_{r=1}^{k-1} \int_r^{r+1} \frac{1}{x}\,dx \leqslant \sum_{r=1}^{k-1} \frac{1}{r}$$

But $\sum_{r=1}^{k-1} \int_r^{r+1} \frac{1}{x}\,dx = \int_1^k \frac{1}{x}\,dx$

$= \ln k$

$H_k - 1 \leqslant \ln k \leqslant H_k - \frac{1}{k}$ 1

$H_k \leqslant 1 + \ln k$

$\ln k \leqslant H_k - \frac{1}{k} \leqslant H_k$. 1

i.e. $\ln k \leqslant H_k \leqslant 1 + \ln k$.

Dividing throughout by $\ln k$ (if $k > 1$) ,

$$1 \leqslant \frac{H_k}{\ln k} \leqslant 1 + \frac{1}{\ln k} .$$

As $\lim_{k\to\infty} 1 = 1$ and $\lim_{k\to\infty} 1 + \frac{1}{\ln k} = 1$. 1

$\therefore \lim_{k\to\infty} \frac{H_k}{\ln k} = 1$ 1

7

(b) From (a) $H_k - \ln k \geqslant 0$. 1

$\therefore \gamma_k$ is bounded below by zero.

$\gamma_k - \gamma_{k+1} = (H_k - \ln k) - (H_{k+1} - \ln(k+1))$

$= (H_k - H_{k+1}) + (\ln(k+1) - \ln k)$ 1

$= \frac{-1}{k+1} + \int_k^{k+1} \frac{1}{x}\,dx$ 1

$\geqslant \frac{-1}{k+1} + \int_k^{k+1} \frac{1}{k+1}\,dx$ 1

$= -\frac{1}{k+1} + \frac{1}{k+1}$

$= 0$.

$\therefore \gamma_k$ is monotonic decreasing and hence $\lim_{k\to\infty} \gamma_k$ exists.

4

(c) Area of $\triangle$ SQR $\leqslant A_r \leqslant$ Area of PQRS 1

$\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) \leqslant A_r \leqslant \frac{1}{r} - \frac{1}{r+1}$ 1

But $\sum_{r=1}^{k-1} A_r = H_{k-1} - \ln k$ 1

$\therefore \frac{1}{2} \sum_{r=1}^{k-1}\left(\frac{1}{r} - \frac{1}{r+1}\right) \leqslant H_{k-1} - \ln k \leqslant \sum_{r=1}^{k-1}\left(\frac{1}{r} - \frac{1}{r+1}\right)$

$\frac{1}{2}(1 - \frac{1}{k}) \leqslant H_{k-1} - \ln k \leqslant 1 - \frac{1}{k}$

$\therefore \frac{1}{2}(1 + \frac{1}{k}) \leqslant H_k - \ln k \leqslant 1$. 1

Since $\lim_{k\to\infty} (H_k - \ln k)$ exists 1

$\lim_{k\to\infty} \frac{1}{2}(1 - \frac{1}{k}) \leqslant \lim_{k\to\infty}\left(H_k - \ln k\right) \leqslant \lim_{k\to\infty} 1$

$\frac{1}{2} \leqslant \lim_{k\to\infty}\left(H_k - \ln k\right) \leqslant 1$ 1

6

(a) Let $P_1(x_1, y_1) = L_2 \cap L_3$.

We have $a_2x_1 + b_2y_1 + c_2 = 0$

$a_3x_1 + b_3y_1 + c_3 = 0$.

For any non-zero $\lambda_1, \lambda_2, \lambda_3$, 2

$$\lambda_3(a_1x_1 + b_1y_1 + c_1)(a_2x_1 + b_2y_1 + c_2) + \lambda_1(a_2x_1 + b_2y_1 + c_2)(a_3x_1 + b_3y_1 + c_3) + \lambda_2(a_3x_1 + b_3y_1 + c_3)(a_1x_1 + b_1y_1 + c_1) = 0 + 0 + 0$$

$\therefore (x_1, y_1)$ lies in C. 1

Similarly, the points P_2 and P_3 also lie in C.

Further, C is an equation of the second degree, it therefore represents a conic 1

through P_1, P_2, P_3.

Differentiating C w.r.t. x, 1

$$\lambda_3[(a_1 + b_1y')(a_2x + b_2y + c_2) + (a_1x + b_1y + c_1)(a_2 + b_2y')]$$ 1
$$+ \lambda_1[(a_2 + b_2y')(a_3x + b_3y + c_3) + (a_2x + b_2y + c_2)(a_3 + b_3y')]$$
$$+ \lambda_2[(a_3 + b_3y')(a_1x + b_1y + c_1) + (a_3x + b_3y + c_3)(a_1 + b_1y')] = 0$$

The slope of the tangent to C at (x_1, y_1) is therefore given by

$$\lambda_3(a_2 + b_2y')(a_1x_1 + b_1y_1 + c_1) + \lambda_2(a_1x_1 + b_1y_1 + c_1)(a_3 + b_3y') = 0$$

or $y' = -\dfrac{\lambda_3a_2 + \lambda_2a_3}{\lambda_3b_2 + \lambda_2b_3}$ $(a_1x_1 + b_1y_1 + c_1 \neq 0)$ 2

But the slope of $T_1 = -\dfrac{\lambda_3a_2 + \lambda_2a_3}{\lambda_3b_2 + \lambda_2b_3}$. 1

T_1 is tangent to C at P_1.

Similary it can be shown that T_2, T_3 are tangent to C at P_2, P_3 respectively. 1

10

(b) C is given by

$$\lambda_3(x + y - 2)(x - y + 2) + \lambda_1(x - y + 2)(2x - y) + \lambda_2(2x - y)(x + y - 2) = 0$$

The coefficient of the xy-term $= -3\lambda_1 + \lambda_2$.

Since the axes of C are parallel to the coordinate axes,

$\therefore \lambda_2 = 3\lambda_1$, 2

T_1 : $\lambda_3(x - y + 2) + \lambda_2(2x - y) = 0$.

Putting $\lambda_2 = 3\lambda_1$, $\lambda_3(x - y + 2) + 3\lambda_1(2x - y) = 0$ 1

or $\dfrac{\lambda_3}{3\lambda_1}(x - y + 2) + (2x - y) = 0$. .... (*)

T_2 : $\lambda_1(2x - y) + \lambda_3(x + y - 2) = 0$

or $(2x - y) + \dfrac{\lambda_3}{\lambda_1}(x + y - 2) = 0$. .... (**)

Eliminating $\dfrac{\lambda_3}{\lambda_1}$ from (*) and (**) 2

$$\frac{-(2x - y)}{x + y - 2} = \frac{-3(2x - y)}{x - y + 2}$$

As $(x, y) \notin L_3 \Rightarrow 2x - y \neq 0$.

$\therefore$ $x + 2y - 4 = 0$ is the required locus. 2

7